Instability of Compressible Three-Dimensional Boundary Layers to Stationary Disturbances*1

By Masahito Asai,*2 Naoto Saitoh*2 and Nobutake Itoh*3

Key Words: Compressible Flows, Three-Dimensional Boundary Layers, Cross-Flow Instability, Cross-Flow Vortices

Abstract

The stability of compressible three-dimensional boundary layers to stationary disturbances is examined on the basis of the linear stability theory. Comparisons of stability characteristics are made between the subsonic and supersonic boundary layers at the edge Mach numbers 0.2 and 2.0, respectively. The result shows that the boundary layer becomes unstable to stationary three-dimensional modes when the cross-flow velocity exceeds a rather small threshold of less than 1% of the external flow velocity. Important to note that the critical Reynolds number for stationary modes does not strongly depend on the Mach number. It is also found that the wavelength of the most amplified stationary three-dimensional mode is four or five times the boundary-layer thickness, not depending on the magnitude of cross-flow velocity both for the subsonic and supersonic flows.

1. Introduction

The transition of three-dimensional boundary layers on a highly swept wing is initially governed by a cross-flow instability occurring under a favorable pressure gradient near the leading edge.1-7) According to the linear stability theory, the most amplified mode for the cross-flow instability is traveling waves with nonzero frequency. In actual boundary layers on a swept wing in low-turbulence environments such as flight conditions, however, more dangerous disturbances may be stationary instability modes because surface roughness is a more serious external disturbance capable of triggering the instability than free-stream turbulence. The onset of transition is controlled by the stability characteristics and the boundary-layer receptivity,8) so the relative importance of traveling and stationary modes crucially depends on disturbance conditions. Indeed, in laboratory experiments that use a low-turbulence wind tunnel, it has been reported that the transition of a three-dimensional boundary layer is initially dominated by the growth of stationary cross-flow vortices rather than by the growth of traveling instability waves.9) The stationary instability modes are thus important in actual three-dimensional boundary layers and have been studied by several authors.9-13) However, most results are limited to the case of incompressible boundary layers.

As for the supersonic three-dimensional boundary layer stability, Balakumar and Reed14) examined local characteristics of cross-flow instability at various edge Mach numbers $M_e = 1.5, 3.0, 5.0,$ and $8.0$ for the boundary layers on a sharp cone rotating about its axis in supersonic free-stream on the basis of the linear stability theory. Their results show that the maximum amplification rate of the instability mode (first mode) is markedly increased because of the presence of cross-flow, especially for low Mach numbers. The results also show that cross-flow instability becomes important when the cross-flow Reynolds number exceeds the order of 40 for $M_e = 1.5$ and 100 for $M_e = 5.0$ or when the magnitude of cross-flow velocity exceeds about 4% of the external flow velocity. The present authors15) examined the influence of compressibility on the stability characteristics in detail at Mach numbers up to 2.0 for the Falkner-Skan-Cooke flow,16) extended to compressible flows. They showed that the most amplified cross-flow instability wave (traveling wave mode) with a large oblique angle was not strongly affected by compressibility, at least at the Mach numbers examined. Furthermore, it was also shown clearly that the cross-flow instability became dominant for the cross-flow velocity greater than 4% of the external flow velocity both for the subsonic and supersonic flows.

The present study concerns the influence of compressibility on the stationary cross-flow instability modes. Comparisons of stability characteristics of stationary three-dimensional modes are made between low subsonic (incompressible) and supersonic flows at the Mach numbers of external flow $M_e = 0.2$ and 2.0, respectively, on the basis of the linear stability theory with parallel base flow assumption. A family of the Falkner-Skan-Cooke type solutions extended to compressible flows is used for the base flow profiles, as in our previous study.15) It is known that two-dimensional boundary layers without cross-flow velocity component are stable to stationary disturbances. So our particular attention is focused on the dependency of the critical Reynolds number on the Mach number and on the magnitude of cross-flow velocity.
2. Base Flow

We consider compressible three-dimensional boundary layers of perfect gas on a swept wing, as illustrated in Fig. 1. The Prandtl number $Pr$ and the ratio of specific heat coefficients $\gamma$ are assumed to be 0.72 and 1.4, respectively, and the specific heat coefficient at constant pressure $C_p^*$ is given by $\gamma R/(\gamma - 1)$ where $R$ is the gas constant. We also assume that the molecular viscosity $\mu^*$ is linearly proportional to the gas temperature $T^*$, that is, $\mu^*/\mu_0^* = T^*/T_0^*$, where the subscript 0 denotes stagnation conditions in external isentropic flow. The base flow is expressed by a family of the Falkner-Skan-Cooke profiles\cite{10} extended to those in compressible flows with the Illingworth-Stewartson transformation.\cite{17} Here the adiabatic condition is imposed on the wall boundary. The solution procedure was presented in detail in our previous paper.\cite{13} The velocity components in the streamwise ($x_1$) and cross-flow ($x_2$) directions of the external flow are denoted by $U_1$ and $U_2$, which are nondimensionalized with the velocity of external flow at the boundary-layer edge $Q_e^*$. The temperature $T$ is nondimensionalized with the edge temperature $T_e^*$. Figures 2 and 3 illustrate the $\zeta$-distributions of $U_1$, $U_2$, and $T$ at the edge Mach number $M_e = 2.0$ and 0.2, respectively, for various magnitudes of cross-flow velocity $U_{2m}$. Here, $\zeta$ is the normal-to-wall coordinate nondimensionalized with the boundary-layer thickness $\delta_{ov}$, at which $U_1$ approaches 95% of the boundary-layer edge velocity, and $U_{2m}$ is defined as the maximum value ($\zeta$-maximum) of the cross-flow velocity $U_2(\zeta)$. The six different boundary-layer profiles in each figure have the cross-flow velocity $U_{2m}$ of 0.01, 0.02, 0.04, 0.06, 0.08, and 0.10, respectively. The temperature, therefore the density $\rho = 1/\gamma T$, is almost constant across the boundary layer at the low Mach number $M_e = 0.2$, and the temperature increases up to 1.7 times the edge temperature across the boundary layer at $M_e = 2.0$. In the following, we examine the linear stability of these boundary-layer profiles.

3. Disturbance Equations

In the present stability analysis of compressible three-dimensional boundary layers, the infinitesimal disturbances superposed to the base flow are $u$, $v$, $w$, $\pi$, $\theta$, and $r$, which are perturbations to the base flow components, pressure, temperature, and density, respectively. Furthermore, the disturbances to viscosity $\mu$ nondimensionalized with the boundary-edge value $\mu_0^*$ and the nondimensional thermal conductivity $\kappa$ ($= \kappa^*C_p^*\mu_0^*$), $\mu'$, and $\kappa'$, are expressed by using the temperature fluctuation $\theta$, as

$$
\mu' = \frac{du}{dT} \theta, \quad \kappa' = \frac{d\kappa}{dT}, \quad \frac{1}{Pr} \frac{d\mu}{dT} \theta \quad (1)
$$

Under parallel flow assumption, we assume that these disturbances take the form

$$q(\xi, \eta, \zeta, \tau) = \hat{q}(\zeta)e^{i(\alpha \xi + \beta \eta - \omega \tau)} \quad (2)$$

where $q$ is each of the nondimensional disturbance quantities ($u$, $v$, $w$, $\pi$, $\theta$, $r$) and the nondimensional coordinates $(\xi, \eta, \zeta, \tau) = (x_1/\delta_{ov}, x_2/\delta_{ov}, \tau \delta_{ov}/Q_e^*)$. Here, $\alpha$, $\beta$, and $\omega$ are the nondimensional wave numbers in the streamwise ($x_1$) and cross-flow ($x_2$) directions and the nondimensional angular frequency, respectively. The Reynolds number $Re$ is defined by $Re = \rho_0^*Q_e^*\delta_{ov}/\mu_0^*$. In the present study, we consider spatially growing stationary modes only, and therefore $\omega$ is set equal to zero. As far as the local instability is concerned, the direction of the spatial growth can be chosen arbitrarily. In the case of swept-wing boundary layers, it is reasonable to assume that localized disturbances grow in the external flow ($x_1$) direction, and spanwise-periodic disturbances are amplified in the chordwise ($x_2$) direction. Note that the neutral stability curves are independent of the direction of the disturbance amplification. In the present study, it is assumed that the disturbances are amplified only in the external streamwise ($x_1$) direction, so $\beta = (\beta_2) = 0$ and $\alpha = \alpha_1 + i\alpha_2$ is complex, where $-\alpha_1$ denotes the spatial growth rate.

The disturbance equations are written as follows:

$$i(\alpha U_1 + \beta U_2 - \omega) \left( \frac{\hat{u}}{T} - \frac{\hat{\theta}}{T} \right) + \left( i\alpha \hat{u} + i\beta \hat{\theta} + \frac{d\hat{w}}{d\zeta} \right) + \frac{1}{T} \frac{dT}{d\zeta} \hat{w} = 0 \quad (3a)$$

$$\frac{1}{T} \left( -i\omega \hat{u} + i\alpha U_1 \hat{u} + i\beta U_2 \hat{u} + \frac{dU_1}{d\zeta} \hat{w} \right) = -\frac{i\alpha}{\gamma M_e^2} \hat{\pi} + \frac{1}{Re} \left( \frac{d^2 \hat{u}}{d\zeta^2} - \alpha_1^2 \hat{u} - \beta_2^2 \hat{u} \right) + \frac{d\mu}{dT} \frac{d\hat{u}}{d\zeta} \hat{w} + \frac{i\alpha\mu}{3} \left( i\alpha \hat{u} + i\beta \hat{\theta} + \frac{d\hat{w}}{d\zeta} \right) + \alpha \frac{d\mu}{dT} \frac{d\hat{\theta}}{d\zeta} \hat{w} + \frac{d\mu}{dT} \frac{d^2 U_1}{d\zeta^2} + \frac{i\mu}{3} \left( i\alpha \hat{u} + i\beta \hat{\theta} + \frac{d\hat{w}}{d\zeta} \right) + \frac{dU_1}{dT} \frac{d\hat{u}}{d\zeta} \hat{w} + \frac{dU_1}{dT} \frac{d\hat{\theta}}{d\zeta} \hat{w} + \frac{d\hat{w}}{dT} \frac{d\hat{\theta}}{d\zeta} \hat{w} \quad (3b)$$

Fig. 1. Geometry and coordinate system.
\begin{align}
\frac{1}{T} \left( -i\omega \hat{\nu} + i\alpha U \hat{\nu} + i\beta U \hat{\nu} + \frac{dU}{dz} \hat{\nu} \right) & \\
= -i\frac{\beta}{\gamma M_{\infty}^2} \hat{\pi} + \frac{1}{Re} \left[ \mu \left( \frac{d^2 \hat{\nu}}{dz^2} - \alpha^2 \hat{\nu} - \beta^2 \hat{\nu} \right) \right] + \frac{d\mu}{dt} \frac{d\nu}{dz} + \frac{i\beta}{3} \left( i\alpha \hat{\nu} + i\beta \hat{\nu} + \frac{d\hat{\nu}}{dz} \right) + i\beta \frac{d\mu}{dt} \frac{d\nu}{dz} \\
+ \left( \frac{d\mu}{dz^2} + \frac{dU_2}{dz} \right) \left( \frac{d\mu}{dz} + \frac{d\theta}{dz} \right) & \\
\ldots \ldots (3c)
\end{align}

\begin{align}
\frac{1}{T} \left( -i\omega \hat{\nu} + i\alpha U \hat{\nu} + i\beta U \hat{\nu} \right) & \\
= -\frac{1}{\gamma M_{\infty}^2} \frac{d\pi}{dz} + \frac{1}{Re} \left[ \mu \left( \frac{d^2 \hat{\nu}}{dz^2} - \alpha^2 \hat{\nu} - \beta^2 \hat{\nu} \right) \right] + \frac{d\mu}{dt} \frac{d\nu}{dz} + \frac{i\alpha}{3} \left( i\alpha \hat{\nu} + i\beta \hat{\nu} + \frac{d\hat{\nu}}{dz} \right) + \frac{2 \mu}{3} \frac{d\nu}{dz} \frac{d\hat{\nu}}{dz} \\
+ i \left( \alpha \frac{dU_1}{dt} \frac{d\nu}{dz} + \beta \frac{dU_2}{dt} \frac{d\nu}{dz} \right) & \\
\ldots \ldots (3d)
\end{align}

\begin{align}
\frac{1}{T} \left( -i\omega \hat{\theta} + i\alpha U \hat{\theta} + i\beta U \hat{\theta} \right) + \frac{1}{T} \frac{d\nu}{dz} \hat{\theta} & \\
= -(\gamma - 1) \left( i\alpha \hat{\nu} + i\beta \hat{\nu} + \frac{d\nu}{dz} \right) + 2\gamma(\gamma - 1) \frac{M_{\infty}^2 \mu}{Re} \left( i \left( \alpha \frac{dU_1}{dz} + \beta \frac{dU_2}{dz} \right) \hat{\nu} \right) \\
+ \frac{dU_1}{dz} \frac{d\nu}{dz} + \frac{dU_2}{dz} \frac{d\nu}{dz} & \\
+ \frac{\gamma}{Pr Re} \mu \left( \frac{d^2 \hat{\theta}}{dz^2} - \alpha^2 \hat{\theta} - \beta^2 \hat{\theta} \right) + \frac{d\mu}{dt} \left( 2 \frac{d\theta}{dz} + \frac{d^2 \theta}{dz^2} \right) \\
+ \frac{d^2 \mu}{dt^2} \left( \frac{d\theta}{dz} \right) + \frac{\gamma(\gamma - 1) M_{\infty}^2}{Re} \frac{d\mu}{dt} \left( \frac{dU_1}{dz} \right)^2 + \left( \frac{dU_2}{dz} \right)^2 & \\
\ldots \ldots (3e)
\end{align}

\[ \hat{u} = \hat{v} = \hat{w} = \frac{d\hat{\theta}}{dz} = 0 \text{ at } \zeta = 0 \text{ and } \hat{u} = \hat{v} = \hat{w} = \hat{\theta} = 0 \text{ as } \zeta \to \infty \ldots \ldots (4) \]

where the adiabatic wall condition is used for the temperature fluctuation \( \hat{\theta} \). The system of these homogeneous equations gives an eigenvalue problem. For the given \( Re \) and the base-flow profiles, the dispersion relation \( \omega = \omega (a, \beta, Re) \) is determined numerically. In the present study, we focus only on the stationary modes with zero frequency \( \omega = 0 \). The numerical integration of these equations is done by means of the fourth-order Runge-Kutta method with the Gram-Schmidt orthogonalization procedure, as done by Mack.18,19

4. Results and Discussion

First the stability diagrams are obtained for the boundary-layer profiles given in Figs. 2 and 3. Figure 4(a) illustrates the neutral stability curves for the stationary states at \( M_{\infty} = 2 \), where the ordinate \( k = \sqrt{\alpha^2 + \beta^2} \) is the magnitude of the wave number vector. In the case of the smallest cross-flow velocity examined, \( U_{2m} = 0.01 \), the unstable domain appears only beyond \( Re = 5,600 \). When the cross-flow velocity \( U_{2m} \) is increased up to 0.02, the critical Reynolds number \( Re_{cr} \) is decreased down to about 2,800, half the value for \( U_{2m} = 0.01 \). For the still larger cross-flow velocity, \( U_{2m} \geq 0.04 \), \( Re_{cr} \) decreases more to about 740 and 610 for \( U_{2m} = 0.08 \) and 0.10, respectively. Besides, we can see that the unstable wave number domain is almost independent of the Reynolds number for \( U_{2m} \geq 0.04 \), except near the critical Reynolds number \( Re_{cr} \). According to our first report,25 the cross-flow instability becomes dominant for \( U_{2m} \geq 0.04 \). So the weak Reynolds number dependency observed for \( U_{2m} \geq 0.04 \) is no doubt due to the cross-flow instability being essentially of the involutional type. This is also so for the low subsonic (nearly incompressible) boundary layers at \( M_{\infty} = 0.2 \), as illustrated in Fig. 4(b). In the subsonic case, the critical Reynolds number \( Re_{cr} \) decreases from about 4,500 to about 550 as \( U_{2m} \) increases from 0.01 to 0.10. In these figures, we also see that the wave number at the critical Reynolds number, which is marked by an open circle on each neutral stability curve, is almost constant over the range of \( U_{2m} \) examined; that is, 1.3 to 1.35 and 1.2 to 1.25 at \( M_{\infty} = 2.0 \) and 0.2, respectively.

The critical Reynolds numbers are more directly compared between the subsonic and supersonic flows in Fig. 5, which plots \( Re_{cr} \) against \( U_{2m} \) for the two Mach numbers \( M_{\infty} = 2.0 \) and 0.2. In both cases, \( Re_{cr} \) is increased rapidly with decreasing \( U_{2m} \) for small \( U_{2m} \), of less than 0.04 and tends toward infinity as \( U_{2m} \) becomes slightly less than 0.01. Indeed, for \( U_{2m} = 0.01 \), the critical Reynolds number \( Re_{cr} \) has already attained 5,620 and 4,480 at \( M_{\infty} = 2.0 \) and 0.2, respectively; it should be noted that in the two-dimensional boundary layer without cross-flow velocity, stationary modes cannot be amplified. It is important to note that, \( Re_{cr} \)
at $M_e=2.0$ is only 10% higher than that of the low subsonic boundary layer at $M_e=0.2$, for the large cross-flow velocity $U_{2m}=0.08$ and 0.10. Figure 6 illustrates $\chi_{cr}$, the oblique angle of the stationary mode at the critical Reynolds number $Re_{cr}$ against $U_{2m}$. Here the oblique angle $\chi=\tan^{-1}(\beta/\alpha_t)$ is defined as the angle between the wave number vector and the external streamline direction (the $x_l$-direction). We can see that $\chi_{cr}$ is very close to 90° for $U_{2m}=0.01$ and decreases only slightly when $U_{2m}$ increases to 0.10, that is, $\chi_{cr}=85°$ and 87° for $M_e=2$ and 0.2, respectively. Therefore, the axis of stationary cross-flow vortices resulting from growth of the stationary instability mode is aligned almost parallel to the external streamline.

Next examined is the spatial growth rate ($-\alpha_t$) of the stationary mode. The neutral stability curves are almost symmetric with respect to the wavenumber (see Fig. 4), suggesting that the wavenumber of the most amplified mode is not strongly dependent on the Reynolds numbers beyond $Re_{cr}$. Figure 7(a) illustrates the spatial growth rate ($-\alpha_t$) of stationary mode against the wavenumber $k_t$ for $U_{2m}=0.04$ and 0.08 at a Reynolds number $Re=5000$. The results are also compared between $M_e=0.2$ and 2.0. Note that disturbances are assumed to grow in the direction of the external streamline (in the $x_l$ direction in Fig. 1). For the two cases $U_{2m}=0.04$ and 0.08, the maximum growth rates are obtained at and around $k_t=1.5$ and 1.3 at $M_e=2.0$ and 0.2, respectively, which are nearly the same as those at $Re_{cr}$. This wave number indicates that the most dangerous stationary instability mode has a wave-length of four or five times the boundary layer thickness $\delta_{lin}$. Furthermore, the growth rate of the most amplified mode is reduced only 10% when the Mach number increases to 2.0, from 0.2. Thus the influence of compressibility on the stationary cross-flow instability modes is very weak. Such a weak dependency of the stability characteristics on the Mach number is probably due to the cross-flow instability modes being highly oblique so that the flow speed in the direction of the wave number vector is low and subsonic even at $M_e=2.0$. Figure 7(b) illustrates the oblique angle ($\chi$) of stationary mode against the wave number $k_t$ for the two cases $U_{2m}=0.04$ and 0.08 corresponding to Fig. 7(a). In both, we see that the amplified stationary modes have an almost constant oblique angle close to a right angle except near the lower branch of neutral curve. That is, $\chi=86°$ to 88° for the subsonic flow at $M_e=0.2$ and 87° to 89° for the supersonic flow at $M_e=2.0$.

5. Concluding Remarks

The linear stability of compressible three-dimensional boundary layers to stationary disturbances was investigated theoretically at the edge Mach numbers $M_e=0.2$ and 2.0. The stability analysis was based on the local parallel flow theory, and a family of the Falkner-Skan-Cooke type solutions extended to compressible flows was used for the base
Fig. 4. Neutral stability curves for stationary modes. (a) $M_e=2.0$, (b) $M_e=0.2$. The open circle on each neutral curve denotes the critical Reynolds number.

Fig. 5. Critical Reynolds number for stationary modes $Re_c$ versus $U_{2m}$ at $M_e=2.0$ (solid circle) and 0.2 (open circle).

Fig. 6. The oblique angle of the stationary mode giving the critical Reynolds number $\chi_c$ at $M_e=2.0$ (solid circle) and 0.2 (open circle).

Fig. 7. Comparisons of (a) spatial growth rate $-\alpha_c$ and (b) oblique angle $\chi_c$ of the stationary mode between $M_e=2.0$ (solid curves) and 0.2 (broken curves) at $Re=5,000$.

Flow profiles. The instability appears both at $M_e=0.2$ and 2.0 when the magnitude of cross-flow velocity $U_{2m}$ exceeds a small threshold of less than 1% of the external flow velocity. The critical Reynolds number $Re_c$ (based on $\delta_{sys}$) to stationary disturbances decreases monotonously down to about 600 as the cross-flow velocity increases up to 10% of the external flow velocity. The wave number vector of the most amplified stationary mode takes nearly a right angle to
the external flow direction both for the subsonic and supersonic flows. It is interesting that the wave-length of the most amplified stationary mode (or the spacing of the most amplified cross-flow vortices) is almost constant, about four or five times the boundary layer thickness $\delta_{\text{ext}}$, not depending on the magnitude of cross-flow velocity. Besides, the growth rate of the most amplified stationary mode is decreased only by 10% as the Mach number $M_a$ is increased to 2.0, from 0.2. Thus the influence of compressibility on the stationary cross-flow modes is extremely weak, similar to the case of traveling cross-flow instability modes.

Acknowledgments

This work was in part supported by a Special Research Fund from the Tokyo Metropolitan Government and by a Grant-in-Aid for Scientific Research (No. 10650897) from the Japan Society for the Promotion of Science.

References