Flutter of Fixed-Free Elastic Strip in Uniform Flow

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The dynamic instability of a slender front-fixed-aft-free elastic strip in a uniform airflow is studied analytically. The analysis is performed by using the unsteady slender body aerodynamic theory and the beam theory. The analysis reveals that flutter may occur as a result of coupling between the first and second bending oscillation modes as a result of aerodynamic force when the flow speed overcomes the critical speed. One approximate method is derived, and the flutter speed, mode, and frequency are examined numerically. These approximate results are compared with other approximate ones derived by the two-term Galerkin method. And when the virtual-air-mass to strip mass ratio is sufficiently small, the flutter speed by the present method coincides with that by the two-term Galerkin method.

Key Words: Aeroelasticity, Flutter, Slender-Body-Aerodynamics

Nomenclature

- A_1, A_2, A_3, A_4 : integration constants
 b : width of elastic strip
 C_1, C_2, C_3, C_4 : integration constants
 $\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4$: see Eq. (15)
 $e = (1 + v^2)^{3/2}$
 EI : flexural rigidity
 $F(\xi), \bar{F}(x)$: function of ξ only, function of x only
 $G(\eta), \bar{G}(t)$: function of η only, function of t only
 $i = \sqrt{-1}$
 $I(x) = |\text{Im}[\bar{F}(x)]|$
 l : length of strip
 M : mass of elastic strip per unit length
 M_{air} : virtual-air-mass per unit length, $M_{\text{air}} = (\pi/4)\rho_{\text{air}}b^2$ in linear theory
 $p = |\text{Re}(s)|$
 $R(x) = |\text{Re}[\bar{F}(x)]|$
 s : root of Eq. (16)
 $\text{sh}(\cdot), \text{ch}(\cdot)$: hyperbolic function, $\text{sh}(\cdot) = \sinh(\cdot)$,
 $\text{ch}(\cdot) = \cosh(\cdot)$
 T : time
 $t = T/(l^2\sqrt{M/EI})$: nondimensional time
 V : free stream velocity
 $\bar{V} = Vl\sqrt{M/EI}$: nondimensional free stream velocity
 V_f : flutter speed
 X, Y : coordinates
 $x = X/l, y = Y/l$: nondimensional coordinates
 α, β : see Eq. (13)
 $\bar{\omega} = |\text{Im}(s)|$
 ω_f : circular frequency at flutter

1. Introduction

Flutter may occur for a cantilever slender elastic strip in a parallel flow with a fixed leading edge. In the analysis that follows, the width and thickness of the strip are chosen uniformly for simplicity. This study was motivated by some aeroelastic problems of stopped rotor V/STOL aircraft in the stopped rotor mode flight.¹⁾ Namely, a blade with a fixed forward end might flutter, whereas with a fixed aft end, static aeroelastic divergence might occur. These problems are simple aeroelastic problems because aerodynamic force may be evaluated by the slender body aerodynamic theory^{2,3)} and elastic force by the beam theory.

The writer studied this flutter problem through analysis and low-speed wind tunnel tests.⁴⁾ The wind speed increased over the flutter start velocity, and the oscillation appeared. When the wind speed decreased under the flutter stop velocity to suppress the flutter, the oscillation disappeared. The flutter start velocity was higher than the flutter stop velocity. If the elastic strip was pushed by the wooden bar in the wind speed slightly higher than the flutter stop velocity, the small amplitude oscillation of the strip appeared. The flutter stop velocity was the danger speed in the practice.

In the analysis, the lift by the unsteady slender body theory and the elastic force by the beam theory were taken into account, but the fluid friction force and structural damping were assumed to be zero for the sake of simplicity. The equation of motion obtained was a fourth order linear partial differential equation with constant coefficients; it was solved numerically by the two-term Galerkin method,⁵⁾ using a high-speed digital computer. The flutter speed obtained there was

$$V_f = 4.46(1/l)\sqrt{EI/M_{\text{air}}}$$

In the present study, the same assumptions of Ref. 4 are made and the numerical analysis is performed. A separation

of variables can be adapted by using independent variable transformations, and the characteristic equation and the approximate mode equation are obtained analytically. These equations are examined numerically by using a high-speed digital computer.

The writer has studied the static aeroelastic divergence of slender elastic strip in a uniform flow whose free end is the leading edge through both the analysis and the low speed wind tunnel tests.⁽⁶⁾ In the analysis, the slender body aerodynamic theory for aerodynamic force and the beam theory for elastic force were also used. The static aeroelastic divergence speed obtained there was

$$V_{\text{div}} = (\pi/2)(1/l)\sqrt{EI/M_{\text{air}}}$$

and the divergence mode

$$y_{\text{div}} = 1 - \sin[(\pi/2)x], \quad (0 \leq x \leq 1)$$

The low-speed wind tunnel tests showed that the divergence speed at a large initial angle of attack was lower than the theoretical divergence speed based on the linear lift because the cross-flow effects^(7,8) as a result of side-edge flow separations make the lift large at a high angle of attack. The divergence speed at a zero initial angle of attack coincided with the theoretical divergence speed based on the linear lift.

The danger speed in the practice might be lower than the flutter speed, based on the linear lift. The nonlinear lift resulting from the side-edge separation vortices effect,⁽⁹⁾ which was strong in the slender wing, acted on the strip in hard flutter tests in which the outer disturbance was added. In Ref. 4, the flutter stop speed was examined in the wind tunnel tests. However, because the linear lift might be worked on the strip at the flutter start critical condition at a zero initial angle of attack in the soft flutter test, the theoretical flutter speed resulting from linear lift should be compared with the flutter start speed. The flutter analysis based on the nonlinear lift would be more complex than the present analysis, which is the subject for a future study.

Datta and Gottenberg studied the flutter of a slender elastic strip hanging vertically in a downward flowing airstream both analytically and experimentally.⁽¹⁰⁾ That problem is similar to the present one, but the present problem differs from that of Ref. 10 because the equation of motion in that reference is the fourth order linear partial differential equation with the variable coefficient resulting from the gravity force. They used a two-term Galerkin method in the analysis. Nemat-Nasser, Parasad, and Herrman analytically studied the dynamic instability of a cantilevered continuous pipe conveying fluid at a constant velocity.⁽⁵⁾ In that study, not only was the two-term Galerkin method used, but also a more exact one. The aeroelastic problems of a pipe containing a fluid flow⁽¹¹⁾ are similar to those of a slender elastic strip in a fluid flow, except for the free leading-edge cases, because the fluid mass in the pipe corresponds to the virtual fluid mass of the elastic strip.

2. Equation of Motion

The slender front-fixed-aft-free elastic strip in a uniform flow is considered. For the aerodynamic force, the lift force by unsteady slender body aerodynamics is considered, whereas the fluid friction force is assumed to be zero for simplicity. Then the aerodynamic force w_{air} is

$$w_{\text{air}} = - \left(\frac{\partial}{\partial T} + V \frac{\partial}{\partial X} \right) \left\{ M_{\text{air}} \left(\frac{\partial}{\partial T} + V \frac{\partial}{\partial X} \right) \right\} Y_{\text{air}}$$

where

$$Y_{\text{air}} = Y - \alpha X$$

For the elastic force, the beam theory is applied; however, the structural damping force is ignored. The inertia force and elastic force act on the strip.

$$EI \partial^4 Y / \partial X^4 = -M (\partial^2 Y / \partial T^2) + w_{\text{air}}$$

For the sake of simplicity, a uniform width and the thickness of the strip were chosen. Then

$$M_{\text{air}} = \text{constant}$$

The equation of motion of the elastic strip is as follows:

$$\frac{\partial^4 y}{\partial x^4} + \nu^2 \bar{V}^2 \frac{\partial^2 y}{\partial x^2} + 2\nu^2 \bar{V} \frac{\partial^2 y}{\partial t \partial x} + (1 + \nu^2) \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

where ν^2 is the virtual-air-mass to strip-mass ratio.

$$\nu^2 = M_{\text{air}} / M \quad (2)$$

Equation (1) is in nondimensional form. It is noted that the initial angle of attack, α , is not present in the equation of motion, Eq. (1); i.e., the aeroelastic phenomena described here are independent of α in the present theory.

Because at the elastic strip leading edge the strip is supported rigidly, the boundary condition at $x = 0$ is as follows:

$$y = \partial y / \partial x = 0 \quad \text{at} \quad x = 0 \quad (3)$$

At the trailing edge of the strip, no aerodynamic force acts because the vortex wake is trailing along the flow.

$$\partial^2 y / \partial x^2 = \partial^3 y / \partial x^3 = 0 \quad \text{at} \quad x = 1 \quad (4)$$

The independent variable transformations are adapted.

$$\left. \begin{aligned} \xi &= -\nu^2 \bar{V} t + (1 + \nu^2)x \\ \eta &= \nu \bar{V} t \end{aligned} \right\} \quad (5)$$

From Eqs. (1) and (5), the following expression is obtained.

$$\partial^4 y / \partial \xi^4 + (\gamma^2 / e^2) (\partial^2 y / \partial \xi^2 + \partial^2 y / \partial \eta^2) = 0 \quad (6)$$

where

$$\gamma^2 = \nu^2 \bar{V}^2, \quad e^2 = (1 + \nu^2)^3$$

The boundary conditions of Eqs. (3) and (4) are transformed to the following expressions by Eq. (5).

$$\left. \begin{aligned} y = \partial y / \partial \xi &= 0 & \text{at} & \quad \xi = -\nu \bar{V}^2 t \\ \partial^2 y / \partial \xi^2 = \partial^3 y / \partial \xi^3 &= 0 & \text{at} & \quad \xi = -\nu \bar{V}^2 t + (1 + \nu^2) \end{aligned} \right\} \quad (7)$$

Fig. 1. Notations and coordinates.

3. Separation of Variables

The solution in the form of a product of two functions is sought.

$$y = F(\xi) \cdot G(\eta) \quad (8)$$

Thus Eq. (6) is equivalent to

$$[F'''' + (\gamma^2/e^2)F'']/F = -(s/e)^2 = -(\gamma^2/e^2)G''/G$$

or a pair of equations

$$G''(\eta) - (s^2/\gamma^2)G(\eta) = 0 \quad (9)$$

$$F''''(\xi) + (\gamma^2/e^2)F''(\xi) + (s^2/e^2)F(\xi) = 0 \quad (10)$$

where (s^2/e^2) is some constant. The general solution of Eq. (9) is obtained easily. When t instead of η is used by Eq. (5), the solution is as follows.

$$\bar{G}(t) = A_1 \exp(st) + A_2 \exp(-st) \quad (11)$$

where A_1 and A_2 are the constants of integration.

On the other hand, the general solution of Eq. (10) is

$$F(\xi) = C_1 e^{\alpha\xi} + C_2 e^{-\alpha\xi} + C_3 e^{\beta\xi} + C_4 e^{-\beta\xi} \quad (12)$$

where

$$\left. \begin{aligned} \alpha &= (1/\sqrt{2})(\gamma/e)(-1 + \sqrt{1 - (4e^2 s^2/\gamma^4)})^{1/2} \\ \beta &= (1/\sqrt{2})(\gamma/e)(-1 - \sqrt{1 - (4e^2 s^2/\gamma^4)})^{1/2} \end{aligned} \right\} \quad (13)$$

From Eqs. (12) and (7), the following expressions are obtained after lengthy calculations.

$$\left. \begin{aligned} \bar{C}_1 + \bar{C}_2 + \bar{C}_3 + \bar{C}_4 &= 0 \\ \alpha \bar{C}_1 - \alpha \bar{C}_2 + \beta \bar{C}_3 - \beta \bar{C}_4 &= 0 \\ \alpha^2 e^{\alpha(1+v^2)} \bar{C}_1 + \alpha^2 e^{-\alpha(1+v^2)} \bar{C}_2 + \beta^2 e^{\beta(1+v^2)} \bar{C}_3 \\ + \beta^2 e^{-\beta(1+v^2)} \bar{C}_4 &= 0 \\ \alpha^3 e^{\alpha(1+v^2)} \bar{C}_1 - \alpha^3 e^{-\alpha(1+v^2)} \bar{C}_2 + \beta^3 e^{\beta(1+v^2)} \bar{C}_3 \\ - \beta^3 e^{-\beta(1+v^2)} \bar{C}_4 &= 0 \end{aligned} \right\} \quad (14)$$

Where

$$\left. \begin{aligned} \bar{C}_1 &= e^{-v^2 \bar{V} \alpha t} C_1, & \bar{C}_2 &= e^{v^2 \bar{V} \alpha t} C_2 \\ \bar{C}_3 &= e^{-v^2 \bar{V} \beta t} C_3, & \bar{C}_4 &= e^{v^2 \bar{V} \beta t} C_4 \end{aligned} \right\} \quad (15)$$

The system (14) has nontrivial solutions if, and only if, the determinant of the coefficients is identically zero; i.e., the characteristic equation is

$$\begin{aligned} \Delta &\equiv 2\alpha\beta[2(\alpha^4 + \beta^4) + \alpha\beta(\alpha - \beta)^2 \text{ch}\{(1 + v^2)(\alpha + \beta)\} \\ &- \alpha\beta(\alpha + \beta)^2 \text{ch}\{(1 + v^2)(\alpha - \beta)\}] = 0. \end{aligned} \quad (16)$$

The s values are determined by this equation, which is the frequency equation of the present problem. This will be clear after calculation of flutter modes. The solution of $F(\xi)$, which satisfies the boundary conditions, is analyzed, this is described with x instead of ξ by Eqs. (5), (12), (14), and (15).

$$\begin{aligned} \bar{F}(x) &= \text{constant}(t)[\text{ch}\{(1 + v^2)\alpha x\} - \text{ch}\{(1 + v^2)\beta x\} \\ &- \{(\alpha^2 \text{ch}\{(1 + v^2)\alpha\} - \beta^2 \text{ch}\{(1 + v^2)\beta\}) \\ &/(\alpha^2 \text{sh}\{(1 + v^2)\alpha\} - \alpha\beta \text{sh}\{(1 + v^2)\beta\})\} \\ &\times (\text{sh}\{(1 + v^2)\alpha x\} - (\alpha/\beta) \text{sh}\{(1 + v^2)\beta x\})] \end{aligned} \quad (17)$$

Equation (17) is an approximate solution imposed with the following conditions;

$$\bar{C}_1 \cong C_1, \quad \bar{C}_2 \cong C_2, \quad \bar{C}_3 \cong C_3 \quad \text{and} \quad \bar{C}_4 \cong C_4, \quad \text{when } v^2 \cong 0$$

and constant(t) in Eq. (17) corresponds to constant $\cdot \bar{G}(t)$.

4. Numerical Calculations

The roots s of the characteristic Eq. (16) are sought by using the high-speed digital computer in application of the Newton-Raphson method. In the calculation of root s , $\alpha\beta = 0$, which gives $s = 0$ and $y = \text{constant} = 0$, is eliminated. The root locus is given in Fig. 2 for the nondimensional mass ratio $v^2 = 0.001$. In this figure, the roots of the first and second bending oscillations are shown. It is seen that some critical speed exists, that is, $\bar{V} = 141.5$ for $v^2 = 0.001$. And when the flow speed overcomes this speed, the system becomes unstable as roots appear in the right half-plane.

The oscillation mode $\bar{F}(x)$ is also calculated. When the flow speed is below the critical speed, there is the real mode shape for either the first or the second bending oscillation solution. When the flow speed overcomes the critical speed, however, $\bar{F}(x)$ becomes the complex function. The real mode $R(x)/R(1)$ and the imaginary mode $I(x)/R(1)$ are shown in Fig. 3 for the mass ratio $v^2 = 0.001$ and, $\bar{V} = 142.8$, which is slightly larger than the critical speed. There are the four roots of s , i.e. $\bar{p} \pm i\bar{\omega}$ and $-\bar{p} \pm i\bar{\omega}$, when the flow speed overcomes the critical speed. The following modes correspond to these four roots.

$$\left. \begin{aligned} \bar{F}(x) &= -R(x) - iI(x) & \text{to} & \quad s = \bar{p} + i\bar{\omega} \\ \bar{F}(x) &= -R(x) + iI(x) & \text{to} & \quad s = \bar{p} - i\bar{\omega} \\ \bar{F}(x) &= -R(x) + iI(x) & \text{to} & \quad s = -\bar{p} + i\bar{\omega} \\ \bar{F}(x) &= -R(x) - iI(x) & \text{to} & \quad s = -\bar{p} - i\bar{\omega} \end{aligned} \right\} \quad (18)$$

First, for the first and second sets of Eq. (18), $\bar{G}(t)$ and $\bar{F}(x)$ are combined.

Fig. 2. Root locus of characteristic equation.

Fig. 3. Vibration modes at flutter.

$$y_1 = e^{\bar{p}t} [(A_1 + iA_2)e^{i\bar{\omega}t} \{-R(x) - iI(x)\} + (A_3 + iA_4)e^{-i\bar{\omega}t} \{-R(x) + iI(x)\}]$$

Because the displacement y_1 must be the real function, the following expression is obtained.

$$y_1 = (\text{constant})e^{\bar{p}t} [-R(x) \cos(\bar{\omega}t + \phi) + I(x) \sin(\bar{\omega}t + \phi)] \quad (19)$$

where (constant) and ϕ are the integration constants.

Second, for the 3rd and 4th sets of Eq. (18), the following damping oscillation is obtained.

$$y_2 = (-\text{constant}') \times e^{-\bar{p}t} [R(x) \cos(\bar{\omega}t + \phi') + I(x) \sin(\bar{\omega}t + \phi')] \quad (20)$$

The postflutter deformation in one cycle can be calculated from the displacement solution y_1 , Eq. (19). These deformation are shown in Fig. 4. Because the $y_1 = 0$ point, except $x = 0$, moves in one cycle, it is seen that there exists not a standing wave, but a traveling wave.

Fig. 4. Deformation of strip in one cycle.

Fig. 5. Nondimensional flutter speed vs. mass ratio M_{air}/M .

From the above analyses, Eq. (16) is seen as the frequency equation and Eq. (17) is the mode shape equation of the present problem. And the critical speed is the flutter speed.

The nondimensional flutter speed $\bar{v}_f$ vs. mass ratio ν^2 is shown in Fig. 5. And the nondimensional flutter circular frequency $\bar{\omega}$ vs. ν^2 is shown in Fig. 6. In the Figs. 5 and 6, the approximate solutions by the two-term Galerkin method described in the appendix are shown by the dotted lines.

Fig. 6. Nondimensional flutter circular frequency vs. M_{air}/M .

Fig. 7. Root locus by the two-term Galerkin method.

5. Conclusion

The flutter of a slender front-fixed-aft-free elastic strip in a parallel flow with fixed leading edge is studied analytically. By use of the unsteady slender body theory for the aerodynamic force and the beam theory for the elastic force, the numerical approximate solutions of the flutter speed and the flutter mode are obtained. The flutter may occur as a result of coupling between the first and the second oscillation modes, when the flow speed overcomes the critical speed.

References

- 1) Celniker, L., Carlson, R. M. and Donham, R. E.: Stopped and Slow Rotor Aircraft Configuration, *Natinal Aeronautics and Space Engineering and Manufacturing Meeting*, 650805, Oct. 4–8, 1965, pp. 1–9.
- 2) Jones, R. T.: Properties of Low-Aspect-Ratio Pointed Wings at Speeds below and above the Speed of Sound, NACA Rep. 835, 1946, pp. 59–63.
- 3) Miles, J. W.: Slender Body Theory for Supersonic Unsteady Flow, *J.A.S.*, **19** (1952), pp. 280–281.
- 4) Yamamoto, Y.: Aeroelastic Problems of Fixed-Free Bar in Longitudinal Flow, *Memo. Gifu Tech. Coll.*, **13** (1978), pp. 7–14 (in Japanese).
- 5) Nemat-Nasser, S., Parasad, S. N. and Herrmann, G.: Destabilizing Effect of Velocity-Dependent Forces in Nonconservative Continuous Systems, *AIAA J.*, **4** (1966), pp. 1276–1280.
- 6) Yamamoto, Y.: Aeroelastic Divergence of Front-Free-Aft-Fixed Elastic Strip Parallel to Uniform Flow, *Trans. Japan Soc. Aero. Space Sci.*, **31** (1989), pp. 188–194.
- 7) Lamer, J. E. and Gloss, B. B.: Subsonic Aerodynamic Characteristics of Interacting Lifting Surface with Separated Flow around Sharp Edges Predicted by a Vortex-Lattice Method, NASA TN D-7921, 1975, pp. 1–89.
- 8) Katz, J. and Plotkin, A.: *Low-Speed Aerodynamics from Wing Theory to Panel Method*, McGraw-Hill, New York, 1991.
- 9) Stark, V. J. E.: Leading-Edge Vortex Effect on the Flutter Speed, *J. Aircraft*, **14** (1977), pp. 921–922.
- 10) Datta, S. K. and Gottenberg, W. G.: Instability of an Elastic Strip Hanging in Airstream, *J. Appl. Mech.*, **42** (1975), pp. 195–198.
- 11) Blevins, R. D.: *Vibrations of a Pipe Containing a Fluid Flow*, *Flow-Induced Vibration*, 2nd ed., Van Nostrand, New York, 1990, pp. 384–414.

Appendix: Analysis by the Two-Term Galerkin Method

The analysis of Ref. 4 is briefly explained here.

The equation of motion (1) with the boundary condition Eqs. (3) and (4) is approximately solved for a sufficiently small value of v^2 . When $\bar{V}^2$ and $\bar{V}$ terms are neglected, Eq. (1) is reduced to

$$\partial^4 y / \partial y^4 + (1 + v^2) \partial^2 y / \partial t^2 = 0$$

The solutions concerning x are the oscillation modes of the cantilever elastic bar. By use of the first and second bending oscillation modes, the approximate equation about t of Eq. (1) is derived by the Galerkin method with a two-term approximation.⁵⁾

The root locus of the characteristic equation calculated by the high-speed digital computer is shown in Fig. 7. The flutter speed obtained by that approximate method is as follows.

$$\bar{V}_f = 4.46(M/M_{air})^{0.495} \cong 4.46\sqrt{M/M_{air}}$$

or

$$V_f = 4.46(1/l)\sqrt{EI/M_{air}}$$