Definition System and Criteria Related to the Static Stability of Airplanes (Second Report) —Static Pitch Stability—

By Osamu KOBAYASHI

Department of Aeronautics and Astronautics, Tokai University, Hiratsuka, Japan

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This paper discusses the static stability of airplanes. The conventional definitions of static stability are neither clear nor systematic. On the basis of these situations, in the first report of this paper a system of definitions of six static stability concepts was proposed. “Static flight speed stability” and “static angle of attack stability” were discussed in detail, and their stability criteria were obtained in terms of the aerodynamic characteristics of an airframe. In this second report, the definitions and the criteria on static pitch stability are discussed in detail, and besides “ $d\delta_e/d\theta < 0$,” the derived form of definition “ $d\delta_e/dU > 0$ ” is deduced from the fundamental definition “ $dM/d\theta < 0$.” The general criterion is also obtained from the fundamental definition. This criterion gives “ $C_{m\alpha} < 0$ ” as the aerodynamic requirement in a normal low subsonic flight region and in each condition in which the phenomena of “flight path divergence” and “tuck under” arise.

Key Words: Aircraft, Stability and Control, Static Stability

Nomenclature

A : aspect ratio of wing
 $\bar{c}$: mean aerodynamic chord of wing
 C_D, C_L, C_m : coefficients of drag, lift, and pitching moment
 C_{Di}, C_{Dmin} : coefficients of induced drag and parasite drag
 $C_{D\alpha}, C_{L\alpha}, C_{m\alpha}$: $\partial C_D/\partial\alpha, \partial C_L/\partial\alpha, \partial C_m/\partial\alpha$
 C_{Du}, C_{Lu}, C_{mu} : $\partial C_D/\partial(\Delta U/U), \partial C_L/\partial(\Delta U/U), \partial C_m/\partial(\Delta U/U)$
 $C_{D\delta_e}, C_{L\delta_e}, C_{m\delta_e}$: $\partial C_D/\partial\delta_e, \partial C_L/\partial\delta_e, \partial C_m/\partial\delta_e$
 C_ℓ, C_n, C_Y : coefficients of rolling moment, yawing moment, and side force
 $C_{\ell\beta}, C_{n\beta}, C_{Y\beta}$: $\partial C_\ell/\partial\beta, \partial C_n/\partial\beta, \partial C_Y/\partial\beta$
 D, D_{net} : aerodynamic drag, net drag
 $(= D - T)$
DLC, DSFC : direct lift control, direct side force control
 e : Oswald’s efficiency coefficient
FC.0 : initial trimmed flight condition
FC.1 : steady flight condition after a flight variable of interest is changed from FC.0
FC.2 : nonequilibrium flight condition just after the change in primary control in FC.1 is returned quickly to zero
 Fe : elevator control force
 L : aerodynamic lift
 L_u, L_α : $\partial L/\partial U, \partial L/\partial\alpha$
 M : aerodynamic pitching moment

M_u, M_α : $\partial M/\partial U, \partial M/\partial\alpha$
 N : aerodynamic yawing moment
 R : aerodynamic rolling moment
 S : wing area
 T : thrust produced by propulsion system
 U : flight speed
 W : weight of airplane
 Y : aerodynamic side force
 α, β, γ : angle of attack, sideslip angle, and flight path angle
 δ_a : aileron deflection (positive: right aileron down and left aileron up)
 δ_e : elevator deflection (positive: elevator trailing edge down)
 δ_r : rudder deflection (positive: rudder trailing edge left)
 δ_{DLC} : DLC deflection (positive: lift L increases)
 δ_{DSFC} : DSFC deflection (positive: side force Y increases)
 ρ : air density
 Θ, Φ, Ψ : pitch, roll, and yaw attitudes
 Δ : small increment
Subscript 0, 1, 2 : FC.0, FC.1, FC.2

1. Introduction

The definition of static stability appears not to be systematically discussed. For example, although “ $C_{m\alpha} < 0$,” “ $C_{\ell\beta} < 0$,” and “ $C_{n\beta} > 0$ ” are merely the aerodynamic criteria corresponding to each static stability, they are used just like the definition of static stability.^{1–6)} Moreover, “flight

Table 1. Definition system of static stability.

	Fundamental definition	Derived form of definition	Control equipment	
			Primary control	Secondary control
Static flight speed stability	$\frac{dD_{net}}{dU} > 0$	$\frac{dT}{dU} > 0$	Propulsion system	Elevator (Holding flight path angle)
		$\frac{d\gamma}{dU} < 0$	Gravity force component or flight path angle	—
Static angle of attack stability	$\frac{dL}{d\alpha} > 0$	$\frac{d\delta_{DLC}}{d\alpha} < 0$	DLC system	Elevator (Holding flight path angle)
Static pitch stability	$\frac{dM}{d\Theta} < 0$	$\frac{d\delta_e}{d\Theta} < 0$	Elevator	—
Static sideslip stability	$\frac{dY}{d\beta} < 0$	$\frac{d\delta_{DSFC}}{d\beta} > 0$	DSFC system	Aileron/rudder (Holding roll attitude and straight flight)
		$\frac{d\Phi}{d\beta} > 0$	Gravity force component or roll attitude	Rudder (Holding straight flight)
Static roll stability	$\frac{dR}{d\Phi} < 0$	$\frac{d\delta_a}{d\Phi} < 0$	Aileron	Rudder (Holding straight flight)
Static yaw stability	$\frac{dN}{d\psi} < 0$	$\frac{d\delta_r}{d\psi} < 0$	Rudder	Aileron (Holding straight flight)

path stability” is well known as the static stability concept on a forward translational motion,^{7,8)} but two static stability concepts on the heaving and sliding motions of an airplane are not built up systematically in a clear form. And although “tuck under”^{5,9)} in transonic flights and “flight path divergence”¹⁰⁾ of aircraft with powered high-lift devices (PHLD) are also static stability problems, they are not treated systematically, but individually as special problems. So the first report¹¹⁾ of this paper proposes the definition system consisting of six static stability concepts on each degree of freedom motion of an airplane to discuss static stability problems systematically. The six concepts are “static flight speed stability,” “static angle of attack stability,” “static pitch stability,” “static sideslip stability,” “static roll stability,” and “static yaw stability.” The summary of their definitions is shown in Table 1.

In the first report of this paper, “static flight speed stability” and “static angle of attack stability” are discussed in detail, and their stability criteria in terms of the aerodynamic characteristics of an airframe are obtained.

This second report discusses “static pitch stability,” which is commonly called “longitudinal static stability” in much literature. At first, the general criterion on static pitch stability is deduced from the proposed definition mentioned above. Next, the relation between the slope of elevator deflection to flight speed “ $d\delta_e/dU > 0$ ” and the above definition of static pitch stability is explained. Moreover, “ $C_{m\alpha} < 0$ ” is deduced from the general criterion as the aerodynamic requirement in a normal low subsonic flight region, and the particular criteria on “tuck under” and “flight path divergence” are studied.

The details of “static sideslip stability,” “static roll stability,” and “static yaw stability” are then discussed in the third report of this paper.

2. Definition of Static Pitch Stability

The procedures in flight tests for judging static pitch stability are as follows. We presume as a starting point that an airplane flies in an initial trimmed flight condition (a condition called “FC.0”).

A small change in the pitch attitude in FC.0 requires that a pitching moment is applied as an external force to the airplane by use of its elevator (called “primary control”), and thus the new steady flight condition in which the pitch attitude Θ_1 is “ $\Theta_0 + \Delta\Theta$ ” is established (called “FC.1”).

The additive elevator deflection $\Delta\delta_e$ is then quickly returned to zero (a flight condition called “FC.2”). At FC.2, if the nonequilibrium pitching moment ΔM results to restore the increment in pitch attitude $\Delta\Theta$, the static pitch stability is stable, and vice versa. Therefore the condition for static pitch stability to be stable is given by

$$dM/d\Theta < 0 \quad (1)$$

This is the fundamental definition of static pitch stability.

The pitching moment applied by the elevator in FC.1 balances against ΔM (i.e., $[\rho U_1^2/2]S\bar{c}C_{m\delta_e}\Delta\delta_e = -\Delta M$). This means that static stability can be judged by checking the sense of primary control in FC.1. Therefore since $C_{m\delta_e} < 0$, the derived form of the definition of static pitch stability is given by

$$(\rho U_1^2/2)S\bar{c}C_{m\delta_e}(d\delta_e/d\Theta) > 0, \quad \text{or} \quad d\delta_e/d\Theta < 0 \quad (2)$$

It is assumed that the thrust force produced by the propulsion system acts on the line of aerodynamic drag and is independent for U and α (i.e., if the throttle position is constant, $T_0 = T_1 = T_2$). And as a general rule in this paper, the aerodynamic coefficients and derivatives without subscript 0, 1, or 2 show the values in FC.0.

3. General Criterion on Static Pitch Stability

By using the equilibrium equations of an airplane, the stability criterion expressed in terms of the aerodynamic characteristics of the airframe can be deduced as follows from the above definitions.

3.1. Equilibrium equation and restoring force

The equilibrium equations on the drag, lift, and pitching moment in FC.0 and FC.1 are respectively described as follows.

“In FC.0”:

Net drag: $D_{\text{net } 0}(= D_0 - T_0)$

$$= \frac{1}{2} \rho U_0^2 S C_D - T_0 = -W \sin \gamma_0 \quad (3a)$$

$$\text{Lift: } L_0 = \frac{1}{2} \rho U_0^2 S C_L = W \cos \gamma_0 \quad (3b)$$

$$\text{Pitching moment: } M_0 = \frac{1}{2} \rho U_0^2 S \bar{c} C_m = 0, \quad \text{or } C_m = 0 \quad (3c)$$

“In FC.1”:

Net drag: $D_{\text{net } 1}(= D_1 - T_1) = -W \sin(\gamma_0 + \Delta\gamma)$,

$$\text{or } \frac{1}{2} \rho (U_0 + \Delta U)^2 S \left(C_D + C_{Du} \frac{\Delta U}{U_0} + C_{D\alpha} \Delta\alpha \right) + \Delta D_{\delta_e} - T_0 = -W \sin(\gamma_0 + \Delta\gamma),$$

$$\text{or } \frac{1}{2} \rho (U_0 + \Delta U)^2 S \left(C_D + C_{Du} \frac{\Delta U}{U_0} + C_{D\alpha} \Delta\alpha + C_{D\delta_e} \Delta\delta_e \right) - T_0 = -W \sin(\gamma_0 + \Delta\gamma) \quad (4a)$$

Lift: $L_1 = W \cos(\gamma_0 + \Delta\gamma)$,

$$\text{or } \frac{1}{2} \rho (U_0 + \Delta U)^2 S \left(C_L + C_{Lu} \frac{\Delta U}{U_0} + C_{L\alpha} \Delta\alpha \right) + \Delta L_{\delta_e} = W \cos(\gamma_0 + \Delta\gamma),$$

$$\text{or } \frac{1}{2} \rho (U_0 + \Delta U)^2 S \left(C_L + C_{Lu} \frac{\Delta U}{U_0} + C_{L\alpha} \Delta\alpha + C_{L\delta_e} \Delta\delta_e \right) = W \cos(\gamma_0 + \Delta\gamma) \quad (4b)$$

$$\text{Pitching moment: } M_1 = \frac{1}{2} \rho (U_0 + \Delta U)^2 S \bar{c} \left(C_{mu} \frac{\Delta U}{U_0} + C_{m\alpha} \Delta\alpha \right) + \Delta M_{\delta_e} = 0,$$

$$\text{or } \frac{1}{2} \rho (U_0 + \Delta U)^2 S \bar{c} \left(C_{mu} \frac{\Delta U}{U_0} + C_{m\alpha} \Delta\alpha + C_{m\delta_e} \Delta\delta_e \right) = 0 \quad (4c)$$

The nonequilibrium pitching moment ΔM at FC.2 when the elevator deflection in FC.1 is returned to zero can be obtained by subtracting ΔM_{δ_e} caused by the elevator from the equilibrium condition “ $M_1 = 0$ ” in FC.1. It is shown by

$$\begin{aligned} \Delta M &= M_1 - \Delta M_{\delta_e} \\ &= \frac{1}{2} \rho (U_0 + \Delta U)^2 S \bar{c} \left(C_{mu} \frac{\Delta U}{U_0} + C_{m\alpha} \Delta\alpha \right) \end{aligned} \quad (5a)$$

After ΔM_{δ_e} is obtained from Eq. (3c) and Eq. (4c), it is reasonably shown to balance against ΔM (i.e., $\Delta M_{\delta_e} = -\Delta M$); thus the definitions Eq. (1) and Eq. (2) are equivalent.

Once Eq. (5a) is expanded and the insignificantly small products and squares of the changes in variables are ignored, we can have this equation:

$$\Delta M = \frac{1}{2} \rho U_0^2 S \bar{c} \left(C_{mu} \frac{\Delta U}{U_0} + C_{m\alpha} \Delta\alpha \right) \quad (5b)$$

This linearized equation shows that ΔM is a function of ΔU and $\Delta\alpha$.

3.2. Criteria for static flight speed stability

Total differential dz in the vicinity of point $(x, y, \dots)$ of “ $z = f(x, y, \dots)$ ” has the following relation.

$$dz = (\partial z / \partial x) dx + (\partial z / \partial y) dy + \dots \quad (6)$$

Therefore the definition Eq. (1) can be described by

$$\frac{dM}{d\Theta} = \frac{\partial M}{\partial U} \cdot \frac{dU}{d\Theta} + \frac{\partial M}{\partial \alpha} \cdot \frac{d\alpha}{d\Theta} > 0 \quad (7)$$

The partial differentials in this equation are given by differentiating Eq. (5b) as follows:

$$\partial M / \partial U = \rho U_0 S \bar{c} C_{mu} / 2 \quad (8a)$$

$$\partial M / \partial \alpha = \rho U_0^2 S \bar{c} C_{m\alpha} / 2 \quad (8b)$$

The total differentials $dU/d\Theta$ and $d\alpha/d\Theta$ in Eq. (7) are deduced in this way. By subtracting Eq. (3a) from Eq. (4a) and ignoring the insignificantly small products and squares of the changes in variables, we obtain the following linearized equation:

$$\begin{aligned} & \rho U_0^2 S \{ C_{Du} (\Delta U / U_0) + C_{D\alpha} \Delta\alpha + C_{D\delta_e} \Delta\delta_e \} / 2 \\ & + \rho U_0 S C_D \Delta U \\ & = -W \cos \gamma_0 \cdot \Delta\gamma \end{aligned} \quad (9a)$$

Since $W \cos \gamma_0 = \rho U_0^2 S C_L / 2$ as shown in Eq. (3b), Eq. (9a) may be written

$$\begin{aligned} & (2C_D + C_{Du}) (\Delta U / U_0) + C_{D\alpha} \Delta\alpha \\ & + C_{D\delta_e} \Delta\delta_e + C_L \cdot \Delta\gamma = 0 \end{aligned} \quad (9b)$$

In a similar way, the following equations are obtained from the pair of Eqs. (3b) and (4b) and the pair of Eqs. (3c) and (4c).

$$(2C_L + C_{Lu})(\Delta U/U_0) + C_{L\alpha}\Delta\alpha + C_{L\delta_e}\Delta\delta_e + C_L \tan \gamma_0 \cdot \Delta\gamma = 0 \quad (10)$$

$$C_{mu}(\Delta U/U_0) + C_{m\alpha}\Delta\alpha + C_{m\delta_e} \cdot \Delta\delta_e = 0 \quad (11)$$

Next, by solving the simultaneous equations of Eq. (9b), (10), and (11), we obtain the ratios of ΔU , $\Delta\alpha$, and $\Delta\gamma$ to $\Delta\delta_e$ as follows:

$$\frac{\Delta U/U_0}{\Delta\delta_e} = \frac{C_L \{C_{D\delta_e}C_{m\alpha} \tan \gamma_0 - C_{L\delta_e}C_{m\alpha} + C_{m\delta_e}(C_{L\alpha} - C_{D\alpha} \tan \gamma_0)\}}{A} \quad (12)$$

$$\frac{\Delta\alpha}{\Delta\delta_e} = \frac{C_L}{A} [-C_{D\delta_e}C_{mu} \tan \gamma_0 + C_{L\delta_e}C_{mu} + C_{m\delta_e} \{(2C_D + C_{Du}) \tan \gamma_0 - (2C_L + C_{Lu})\}] \quad (13)$$

$$\frac{\Delta\gamma}{\Delta\delta_e} = \frac{1}{A} [C_{D\delta_e} \{C_{mu}C_{L\alpha} - (2C_L + C_{Lu})C_{m\alpha}\} + C_{L\delta_e} \{(2C_D + C_{Du})C_{m\alpha} - C_{mu}C_{D\alpha}\} + C_{m\delta_e} \{(2C_L + C_{Lu})C_{D\alpha} - (2C_D - C_{Du})C_{L\alpha}\}] \quad (14)$$

where

$$A = C_L \{(2C_L + C_{Lu})C_{m\alpha} + C_{mu}C_{D\alpha} \tan \gamma_0\} - C_L \{(2C_D + C_{Du})C_{m\alpha} \tan \gamma_0 + C_{mu}C_{L\alpha}\} \quad (15)$$

And by applying the following relation to Eq. (13) and Eq. (14);

$$\Theta = \gamma + \alpha, \quad \text{or} \quad \Delta\Theta = \Delta\gamma + \Delta\alpha \quad (16)$$

We find the ratio of $\Delta\Theta$ to $\Delta\delta_e$ given by

$$\Delta\Theta/\Delta\delta_e = \Delta\gamma/\Delta\delta_e + \Delta\alpha/\Delta\delta_e = B/A \quad (17)$$

where

$$B = C_{D\delta_e} \{C_{mu}(C_{L\alpha} - C_L \tan \gamma_0) - (2C_L + C_{Lu})C_{m\alpha}\} + C_{L\delta_e} \{(2C_D + C_{Du})C_{m\alpha} + C_{mu}(C_L - C_{D\alpha})\} + C_{m\delta_e} \{(2C_L + C_{Lu})(2C_{D\alpha} - C_L) + (2C_D + C_{Du})(C_L \tan \gamma_0 - C_{L\alpha})\} \quad (18)$$

Thus “ $dU/d\Theta$ ” and “ $d\alpha/d\Theta$ ” are obtained as the following:

$$\begin{aligned} dU/d\Theta &= U_0 \{(\Delta U/U_0)/\Delta\delta_e\} / (\Delta\Theta/\Delta\delta_e) \\ &= U_0 C_L \{C_{D\delta_e}C_{m\alpha} \tan \gamma_0 - C_{L\delta_e}C_{m\alpha} + C_{m\delta_e}(C_{L\alpha} - C_{D\alpha} \tan \gamma_0)\} / B \end{aligned} \quad (19a)$$

$$\begin{aligned} d\alpha/d\Theta &= (\Delta\alpha/\Delta\delta_e) / (\Delta\Theta/\Delta\delta_e) \\ &= \frac{C_L}{B} [-C_{D\delta_e}C_{mu} \tan \gamma_0 + C_{L\delta_e}C_{mu} + C_{m\delta_e} \{(2C_D + C_{Du}) \tan \gamma_0 - (2C_L + C_{Lu})\}] \end{aligned} \quad (19b)$$

The application of Eqs. (8a), (8b), (19a), and (19b) for definition Eq. (7) gives the following stability criterion in terms of the aerodynamic characteristics of an airplane.

$$\frac{C_L C_{m\delta_e}}{B} [C_{mu}(C_{L\alpha} - C_{D\alpha} \tan \gamma_0) + C_{m\alpha} \{(2C_D + C_{Du}) \tan \gamma_0 - (2C_L + C_{Lu})\}] < 0 \quad (20)$$

When $C_{L\delta_e}$ and $C_{D\delta_e}$ are small and may be ignored as they are in conventional airplanes, Eq. (20) is given as the following:

$$\frac{C_L [C_{mu}(C_{L\alpha} - C_{D\alpha} \tan \gamma_0) + C_{m\alpha} \{(2C_D + C_{Du}) \tan \gamma_0 - (2C_L + C_{Lu})\}]}{(2C_L + C_{Lu})(C_{D\alpha} - C_L) + (2C_D + C_{Du})(C_L \tan \gamma_0 - C_{L\alpha})} < 0 \quad (21)$$

This is the general form of criterion on the static pitch stability for conventional airplanes.

4. Criteria on Static Pitch Stability in Normal Low Subsonic Flight

In this chapter, the criterion in terms of the aerodynamic characteristics of an airplane in a normal low subsonic flight is obtained. In this flight region, it can be considered that the compressibility of flow and the aeroelastic effects of an airplane are negligible, that the flight-path angle is not so large, and that the lift slope is positive. That is,

$$C_{Lu} = C_{Du} = C_{mu} = 0, \quad \gamma_0 = 0, \quad C_{L\alpha} > 0, \quad C_L > 0 \quad (22)$$

Consequently, the criterion in this case is given from Eq. (21) by

$$C_{m\alpha} C_L^2 / (C_{L\alpha} C_D - C_{D\alpha} C_L + C_L^2) < 0 \quad (23)$$

The denominator of this inequality may be rewritten

$$C_L \{C_{L\alpha} (C_D / C_L - C_{D\alpha} / C_{L\alpha}) + C_L\} \quad (24)$$

The polynomial “ $C_D / C_L - C_{D\alpha} / C_{L\alpha}$ ” is the same as the polynomial of the criterion on static flight speed stability discussed in the first report of this paper and is positive in frontside flights. Moreover, since $C_L > 0$ and $C_{L\alpha} > 0$, the polynomial (24) is positive in frontside flights and would not be easily negative even in backside flights. This matter can be explained quantitatively as follows.

A drag coefficient C_D is the sum of a parasite drag coefficient C_{Dmin} and an induced drag coefficient C_{Di} ($= C_L^2 / \pi A e$) as follows:

$$C_D = C_{Dmin} + C_L^2 / \pi A e \quad (25)$$

Thus $C_{D\alpha}$ is given by

$$C_{D\alpha} = 2 C_L C_{L\alpha} / \pi A e \quad (26)$$

Consequently, the polynomial (24) is shown by

$$C_{L\alpha} (C_{Dmin} + C_L^2 / \pi A e) - 2 C_L C_{L\alpha}^2 / \pi A e + C_L^2 \quad (27a)$$

Dividing this polynomial by “ $C_{L\alpha} C_L^2 (> 0)$,”

$$(C_L^2 + C_{Dmin} C_{L\alpha}) / C_{L\alpha} C_L^2 - 1 / \pi A e \quad (27b)$$

Since $C_{Dmin} C_{L\alpha} > 0$, the first term of Eq. (27b) is then always larger than “ $1 / C_{L\alpha}$,” and since $C_{L\alpha} < 2\pi$ in conventional airplanes, Eq. (27b) is surely positive except when the following inequality is satisfied.

$$1 / 2\pi - 1 / \pi A e < 0, \quad \text{or} \quad A e < 2 \quad (28)$$

Because the effective aspect ratio “ $A e$ ” of a conventional airplane is usually much larger than 2, the denominator of Eq. (23) is usually positive.

Moreover, since $C_L > 0$, Eq. (23) can be rewritten in final form as follows:

$$C_{m\alpha} < 0 \quad (29)$$

This is the criterion for static pitch stability in a normal low subsonic flight, and it is the same as the familiar form of criterion for the so-called longitudinal static stability.

5. Slope of Elevator Deflection to Flight Speed

It is well known that the slope of elevator deflection to flight speed “ $d\delta_e / dU$ ” or the slope of elevator control force to flight speed “ dF_e / dU ” in equilibrium flights is measured in flight tests to examine the longitudinal static stability of an airplane, as shown in MIL-F-8785B⁷⁾ 3.2.1.1. This criterion is shown by

$$d\delta_e / dU > 0 \quad (30)$$

In this chapter, it is shown that “ $d\delta_e / dU > 0$ ” is equivalent to the definition of static pitch stability mentioned above, at least in normal low subsonic flights.

By applying “ $C_{L\delta_e} = C_{D\delta_e} = 0$ ” for conventional airplanes to Eq. (12) and Eq. (15), we have this equation:

$$\begin{aligned} \frac{\Delta\delta_e}{\Delta U / U_0} = & [C_{mu} (C_{L\alpha} - C_{D\alpha} \tan \gamma_0) \\ & + C_{m\alpha} \{ (2C_D + C_{Du}) \tan \gamma_0 - (2C_L + C_{Lu}) \}] \\ & / C_{m\delta_e} (C_{D\alpha} \tan \gamma_0 - C_{L\alpha}) \end{aligned} \quad (31)$$

Its numerator is equal to the polynomial in parentheses in the numerator of the general criterion Eq. (21). Since $C_{Lu} = C_{Du} = C_{mu} = 0$ and $C_L > 0$ in normal low speed flight, Eq. (31) is simplified as follows:

$$\frac{\Delta\delta_e}{\Delta U / U_0} = \frac{2C_{m\alpha} (C_L - C_D \tan \gamma_0)}{C_{m\delta_e} (C_{L\alpha} - C_{D\alpha} \tan \gamma_0)} \quad (32)$$

The sign of “ $C_L - C_D \tan \gamma_0$ ” in the numerator of Eq. (32) is reasonably positive because it is quite improbable that the lift-drag ratio “ C_L / C_D ” is smaller than “ $\tan \gamma_0$.” For example, even if $\gamma_0 = 45^\circ$, the unrealistic condition “ $C_L / C_D < 1$ ” is required for “ $C_L - C_D \tan \gamma_0 < 0$.”

Similarly, the sign of “ $C_{L\alpha} - C_{D\alpha} \tan \gamma_0$ ” in the denominator of Eq. (32) is also reasonably positive, as follows. The application of Eq. (26) to “ $C_{L\alpha} - C_{D\alpha} \tan \gamma_0$ ” gives the following polynomial.

$$\begin{aligned} C_{L\alpha} - C_{D\alpha} \tan \gamma_0 &= C_{D\alpha} (C_{L\alpha} / C_{D\alpha} - \tan \gamma_0) \\ &= C_{D\alpha} (\pi A e / 2 C_L - \tan \gamma_0) \end{aligned} \quad (33)$$

It is quite improbable that “ $\pi A e / 2 C_L$ ” in this equation is smaller than “ $\tan \gamma_0$.” For example, even if $\gamma_0 = 45^\circ$ (i.e., $\tan \gamma_0 = 1$) and $C_L = 4$, the effective aspect ratio “ $A e < 2.6$ ” is required. This is unrealistic for a conventional airplane. Thus “ $C_{L\alpha} - C_{D\alpha} \tan \gamma_0$ ” is reasonably positive. Moreover, since $C_{m\delta_e}$ is negative, “ $d\delta_e / dU > 0$ ” is given by

$$C_{m\alpha} < 0 \quad (34)$$

This result is the same as the criterion Eq. (29) for static pitch stability in the previous chapter and shows that “ $d\delta_e / dU > 0$ ” can be used as the derived form of definition on static pitch stability for a conventional airplane.

Then “ $d\delta_e / dU > 0$ ” can be shown as the following.

$$\frac{d\delta_e}{dU} = \frac{d\delta_e}{d\Theta} \frac{d\Theta}{dU} > 0 \quad (35)$$

“ $d\delta_e/d\Theta < 0$ ” is the derived form of definition on static pitch stability, as shown in Eq. (2). Therefore it should be appreciated that if the unrealistic flight condition that “ $d\Theta/dU$ ” is positive exists, “ $d\delta_e/dU > 0$ ” is not satisfied even if the static pitch stability is stable.

6. Criteria on Flight Path Divergence Phenomenon

“Flight path divergence” is the static unstable phenomenon that is found out by research on the flying qualities of short takeoff and landing aircraft with powered high-lift devices (PHLD).¹⁰⁾ This phenomenon occurs in the higher angle of attack region than the angle of attack of the maximum lift coefficient of a PHLD airplane. This means that flight path divergence occurs in the flight condition in which the lift curve slope $C_{L\alpha}$ is negative, the drag curve slope $C_{D\alpha}$ is positive. This flight condition causes a small lift-drag ratio that in turn gives rise to a large negative flight path angle. The feature of this phenomenon is that the pitch attitude in the equilibrium flight condition decreases when the angle of attack is increased by the elevator control (i.e., $d\Theta/d\alpha < 0$), as shown in Fig. 1. This unique phenomenon brings about the divergence of the flight path angle and is called “flight path divergence.” The reason why this phenomenon occurs has not been explained clearly. In this chapter, it is shown that the criterion for “flight path divergence” phenomena can be deduced from the general criterion on static pitch stability.

It can be considered that the compressibility of flow and the aeroelastic effects of an airplane are negligible in the flight region of interest. That is,

$$C_{Lu} = C_{Du} = C_{mu} = 0, \quad C_L > 0 \quad (36)$$

Consequently, the criterion in this situation is given from Eq. (21) by

$$\frac{C_{m\alpha} \cdot C_L(C_L - C_D \tan \gamma_0)}{C_{L\alpha}C_D - C_{D\alpha}C_L + C_L(C_L - C_D \tan \gamma_0)} < 0 \quad (37)$$

Applying “ $C_{D\delta_e} = C_{L\delta_e} = 0$ ” to Eq. (18) and Eq. (19b), we may write “ $d\Theta/d\alpha$ ” as

$$\frac{d\Theta}{d\alpha} = \frac{C_{L\alpha}C_D - C_{D\alpha}C_L(C_L - C_D \tan \gamma_0)}{C_L(C_L - C_D \tan \gamma_0)} \quad (38)$$

Consequently, the criterion Eq. (37) is rewritten as follows:

$$C_{m\alpha}/(d\Theta/d\alpha) < 0 \quad (39)$$

This inequality shows that the static pitch stability depends not only on $C_{m\alpha}$, but also on $d\Theta/d\alpha$, and if $d\Theta/d\alpha < 0$, the static pitch stability is unstable even if “ $C_{m\alpha} < 0$ ” is satisfied.

The sign of the denominator of Eq. (38) is reasonably positive, as shown in the previous chapter.

Next, we examine the condition that the numerator of Eq. (38) becomes negative, as follows.

$$C_{L\alpha}C_D - C_{D\alpha}C_L + C_L(C_L - C_D \tan \gamma_0) < 0 \quad (40)$$

Fig. 1. Flight region of flight path divergence (Ref. 10).

The following equation is then obtained from the relation “ $\tan \gamma_0 = (T_0 - D_0)/L_0$.”

$$C_D/C_L = D_0/L_0 = -\tan \gamma_0 + T_0/L_0 \quad (41)$$

Applying Eq. (41) to Eq. (40) gives this inequality:

$$C_{D\alpha} > C_L + C_{L\alpha}(T_0/L_0) - (C_D + C_{L\alpha}) \tan \gamma_0 \quad (42)$$

Although the first term C_L in the right hand of Eq. (42) is positive, the second term $C_{L\alpha}(T_0/L_0)$ is negative when $C_{L\alpha} < 0$. Moreover, if $C_{L\alpha}$ is smaller than “ $-C_D$ ” and the flight path angle γ_0 is negative, the third term becomes negative.

Therefore when $C_{L\alpha} < -C_D$ and $\gamma_0 < 0$, the above $d\Theta/d\alpha$ tends easily to be negative, and the static pitch stability may be unstable even if $C_{m\alpha} < 0$. This tendency is identical to the features of the flight path divergence phenomenon pointed out above.

7. Criteria on Tuck under Phenomenon

“Tuck under” is the static unstable phenomenon that causes the pitch-down motion and the divergence of flight speed when the equilibrium flight speed is increased slightly in a transonic flight region. It is well known that its main causes are the rapid backward movement of the wing’s aerodynamic center and the rapid decrease of downwash at the horizontal tail.^{5,9)}

In the transonic flight conditions for studying a tuck under phenomenon, the compressibility of flow and the aeroelastic effects of an airplane play an important role, and the importance of the flight path angle and the angle of attack is small. That is,

$$\gamma_0 \approx 0, \quad C_{L\alpha} > 0 \quad (43)$$

The general criterion Eq. (21) is rewritten as follows:

$$\frac{C_{m\alpha}(2C_L + C_{Lu}) - C_{mu}C_{L\alpha}}{C_{L\alpha}(2C_D + C_{Du}) - C_{D\alpha}(2C_L + C_{Lu}) + C_L(2C_L + C_{Lu})} < 0 \quad (44)$$

This is the criterion on static pitch stability in high subsonic and transonic flight regions.

Table 2. Definitions and criteria of static pitch stability.

Definition	Criteria for conventional airplanes	Note
$\frac{dM}{d\Theta} < 0$	General form of stability criterion: $\frac{C_L [C_{mu}(C_{L\alpha} - C_{D\alpha} \tan \gamma_0) + C_{m\alpha} \{(2C_D + C_{Du}) \tan \gamma_0 - (2C_L + C_{Lu})\}]}{(2C_L + C_{Lu})(C_{D\alpha} - C_L) + (2C_D + C_{Du})(C_L \tan \gamma_0 - C_{L\alpha})} < 0$	$C_{D\delta_e} = C_{L\delta_e} = 0$
$\frac{d\delta_e}{d\Theta} < 0$	Stability criterion in normal and low subsonic region: $C_{m\alpha} < 0$	$C_{Du} = C_{Lu}$ $= C_{mu} = 0$ $\gamma_0 = 0$
$\frac{d\delta_e}{dU} > 0$	Instability criterion that flight path divergence occurs: $C_{L\alpha} C_D - C_{D\alpha} C_L + C_L (C_L - C_D \tan \gamma_0) < 0$	In low subsonic and high angle of attack region and $d\Theta/d\alpha < 0$
	Instability criterion that tuck under occurs: $\frac{C_{m\alpha}(2C_L + C_{Lu}) - C_{mu} C_{L\alpha}}{C_{L\alpha}(2C_D + C_{Du}) - C_{D\alpha}(2C_L + C_{Lu}) + C_L(2C_L + C_{Lu})} > 0$	In transonic region and $\gamma_0 = 0$

In the following discussion, we presume the simple case in which C_{Du} and C_{Lu} , except for C_{mu} , are small and may be ignored for study of the physical meaning of the above criterion. In this study, Eq. (44) is simplified as follows:

$$(2C_{m\alpha} C_L - C_{mu} C_{L\alpha}) / (C_{L\alpha} C_D - C_{D\alpha} C_L + C_L^2) < 0 \quad (45)$$

The denominator in this inequality is positive for conventional airplanes, as shown in chapter 4. Thus Eq. (45) is given by

$$C_{m\alpha}(2C_L) - C_{mu} C_{L\alpha} < 0 \quad (46)$$

By subtracting Eq. (3b) from Eq. (4b) and ignoring the insignificantly small products and squares of the changes in variables, we can then give the linearized equilibrium relation with respect of the increment of lift as follows:

$$\begin{aligned} \frac{1}{2} \rho U_0^2 S \left\{ (2C_L + C_{Lu}) \frac{\Delta U}{U_0} + C_{L\alpha} \Delta \alpha + C_{L\delta_e} \Delta \delta_e \right\} \\ = -W \sin \gamma_0 \cdot \Delta \gamma \end{aligned} \quad (47a)$$

When the above assumption of “ $C_{Lu} = 0$ ”, “ $C_{L\delta_e} = 0$ ”, and “ $\gamma_0 = 0$ ” is applied, Eq. (47a) is written as follows:

$$2C_L \frac{\Delta U}{U_0} + C_{L\alpha} \Delta \alpha = 0 \quad (47b)$$

Therefore Eq. (46) may be rewritten

$$C_{mu}(\Delta U/U_0) + C_{m\alpha} \Delta \alpha > 0 \quad (48)$$

This equation means that as long as the pitching moment component accompanied by the increment in flight speed does not cancel the pitching moment component caused by the change in angle of attack, the static pitch stability of the airplane is stable. However, if the former is smaller than the latter and Eq. (48) is not satisfied, the aircraft moves in a more nose-down attitude following the increase in flight speed, and this nose-down movement causes a further increase in flight speed even without pilot control. This unstable phenomenon agrees with the physical content of the tuck under phenomena.

Moreover, it is well known that speed instability in transonic flights (or tuck under phenomena) occurs when $M_\alpha L_u - M_u L_\alpha > 0$ in terms of dimensional derivatives.¹²⁾ This “ $M_\alpha L_u - M_u L_\alpha$ ” is equivalent to the numerator in the criterion Eq. (44) because the nondimensional representation of L_u is “ $2C_L + C_{Lu}$ ”. ($2C_L$ shows the increment in lift caused by the dynamic pressure effect, and C_{Lu} denotes the increment in lift caused by the compressibility and aeroelasticity effects.)

Consequently, it is concluded that tuck under phenomena occur when Eq. (44) is not satisfied, as follows:

$$\frac{C_{m\alpha}(2C_L + C_{Lu}) - C_{mu} C_{L\alpha}}{C_{L\alpha}(2C_D + C_{Du}) - C_{D\alpha}(2C_L + C_{Lu}) + C_L(2C_L + C_{Lu})} > 0 \quad (49)$$

8. Summary of This Report

This paper has proposed the definition system on the static stability of airplanes in the first report,¹¹⁾ as shown in Table 1. In this second report, “static pitch stability” is discussed in detail, and its criteria in terms of the aerodynamic characteristics of an airplane are obtained as shown in Table 2.

The fundamental definition of static pitch stability is “ $dM/d\Theta < 0$,” and the derived form is “ $d\delta_e/d\Theta < 0$.” On the basis of the general criterion obtained from the fundamental definition, “ $C_{m\alpha} < 0$ ” is deduced as the criterion on static pitch stability in normal and low subsonic regions. Next, it is shown that “ $d\delta_e/dU > 0$ ” is also usually treated as the derived form of “ $dM/d\Theta < 0$,” though “ $d\delta_e/dU > 0$ ” depends not only on the static pitch stability, but also on the sign of $d\Theta/dU$. Moreover, it is shown that the phenomena of “flight path divergence” and “tuck under” can be explained on the basis of the general criterion.

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