1. Introduction

The problem of satellite orbit determination using Earth’s magnetic field measurements can be approached from various directions. Shorshi and Bar-Izthack proposed an autonomous navigation method using the extended Kalman filter (EKF).1) Psiaki suggested a batch filter scheme for autonomous orbit and magnetic field determination,2) where he used the Gauss-Newton technique for the minimization of a certain least-squares cost function. The EKF provides a suboptimal solution of the nonlinear estimation problem. However, an initial state far from the true one may cause a divergence problem due to the linearization of a highly nonlinear system equation. With the optimization techniques using (local) gradient information, convergence to the global minimum is dependent upon an educated guess of the initial state. With an inappropriate initial state, the iteration may converge to a local minimum.3) On the other hand, (global) optimization methods such as the simulated annealing technique have the ability to escape from local minima and to converge to the global minimum.4,5)

This paper presents the generalized simulated annealing technique to find the initial state for the autonomous orbit determination using an onboard magnetometer. It uses the magnetometer measurement only, and does not require additional information, such as the attitude of the satellite, ground station intervention, or the Global Positioning System (GPS) signal.

2. Algorithm Description

Our choice of state-vector elements is the classical orbital elements:

\[
x = [a \ e \ i \ \Omega \ \omega \ M_0]^T,
\]

where \(a\) is the semi-major axis, \(e\) is the eccentricity, \(i\) is the inclination, \(\Omega\) is the longitude of the ascending node, \(\omega\) is the argument of periapsis, and \(M_0\) is the mean anomaly at epoch. Because the mean anomaly varies uniformly in time, the mean anomaly at epoch is chosen as the sixth element instead of the true anomaly \(v\), the eccentric anomaly, or the time of periapsis passage. This simple linear property reduces the complexity of our equation for orbital dynamics.6)

In this paper, the \(J_2\) effect and drag influence are ignored for the sake of simple formulation. Therefore, the backward state-vector propagation from the current time \(t\) to the epoch time \(t_0\) depends only on the mean anomaly \(M\), and is formulated by the continuous time equation:

\[
M = M_0 + n(t - t_0),
\]

where \(n\) is the mean motion. The true anomaly \(v\) can be obtained by a function of the eccentric anomaly which is determined by solving Kepler’s equation numerically.7)

The Cartesian position vector \(p\) of a satellite in the Earth-centered Earth-fixed (ECEF) frame is obtained by

\[
p = T_2(t - t_0)T_1(i, \Omega, \omega)p_{\text{w}},
\]

where \(p_{\text{w}}\) is the Cartesian position vector in the perifocal frame, \(T_1\) is the transformation matrix from the perifocal frame to the Earth-centered inertial (ECI) frame, and \(T_2\) is the transformation matrix from ECI frame to ECEF frame. The spherical position \(p_s = [r \ \theta \ \phi]^T\) of the vector \(p\) is the nonlinear function of state elements. Here, \(r\) is the geocentric distance, \(\theta\) is the colatitude, and \(\phi\) is the east longitude from the Greenwich meridian. If we consider the discrete measurements, then the dynamics of the spherical position become

\[
[r_k \ \theta_k \ \phi_k]^T = f(x, t_k),
\]

where \(f\) is a nonlinear vector function and \(t_k\) is the measurement time.

The elements of the spherical position vector are used to determine the geomagnetic field vector \(b_{\text{igrf}}(r, \theta, \phi)\), obtained from the International Geomagnetic Reference Field (IGRF) model. The system measurement is the magnitude of the geomagnetic field vector \(b_{\text{m,k}}\) obtained from the onboard magnetometer at time \(t_k\). We choose the magnitude instead of the individual components since the magnitude of the geomagnetic field is independent of the attitude of the satellite.

The cost function \(J(x)\) we have chosen is the sum of the squares of the difference between the 2-norm of the measured geomagnetic field vector and the IGRF model:

\[
J(x) = \frac{1}{N\max_k \|b_{m,k}\|^2} \sum_{k=1}^{N} (\|b_{m,k}\|_2 - \|b_{\text{igrf}}(x, t_k)\|_2)^2,
\]

where \(N\) is the number of measurements.
where $N$ is the number of the measurements and the denominator $N(\max_k \|b_{mk}\|^2)^2$ is the normalizing factor that reduces the influence of the altitude on the cost function magnitude. To minimize the cost function, we use the generalized simulated annealing technique, which finds the global minimum along iterations with a fixed control parameter $\beta$ and a random walk. After each iteration, the control parameter $\beta$ increases and the size of the random walk decreases. The new state caused by the random walk can be accepted or rejected by the Metropolis criteria. If the value of cost function decreases at the new state, the state is accepted. Otherwise, it is accepted with a probability $p = \exp(-\beta J_0 \Delta J)$, which is able to escape from local minima. The initial state $x_0$ can be chosen arbitrarily anywhere in the bounded search space $\Gamma \subset \mathbb{R}^6$. Since we consider LEO satellites with direct and polar orbital motions, the search space $\Gamma$ is $a_i < a < a_u$, $0 < e < 1$, $0 \leq i \leq \pi/2$, and $0 < \Omega$, $\omega$, $M_0 < 2\pi$. Here, the lower and upper bounds of the semi-major axis are $a_l = 6900\, \text{km}$ and $a_u = 7900\, \text{km}$, respectively, because the altitude of the LEO satellite is between $500\, \text{km}$ and $1500\, \text{km}$. The simulated annealing technique is summarized below:

### Algorithm (Simulated Annealing)

Choose $x_0$
Set $b_0$
Compute $J_0 = J(x_0)$ using eq. (4)
If $|J_0| < \epsilon$, then break

**End If**
For $k_1 = 1, 2, \ldots, \kappa_1$
$k_2 = 1$
While $(|J_0| > \epsilon)$ and $(k_2 < \kappa_2)$
1. Generate $\Delta \gamma \circ u$
Set $x_1 = x_0 + \Delta \gamma \circ u$
If $x_1 \in \Gamma$, then set $J_1 = J(x_1)$ and $\Delta J = J_1 - J_0$
Else, goto step I

**End If**
$k_2 = k_2 + 1$
If $J_1 \leq J_0$, then set $x_0 = x_1$ and $J_0 = J_1$
If $|J_0| < \epsilon$, then break
Else, goto step I

**End If**
Else, set $p = \exp(-\beta J_0^{-1} \Delta J)$
If $V \geq P$, then goto step I
Else, set $x_0 = x_1$ and $J_0 = J_1$, then goto step I

**End If**

**End While**

$\Delta \gamma = \gamma \Delta \gamma$
$\beta = \beta / \gamma$

**End For**

### 3. Simulation Results

To verify that the algorithm can be applied for any low Earth orbits, we consider nine low Earth orbits whose state elements are listed in Table 1. We classify the nine orbits into three cases according to element variation. At case A, we vary the semi-major axis for three subcases. Cases I and O are considered by varying the inclination and the longitude of the ascending node, respectively. For each subcase, the

<table>
<thead>
<tr>
<th>Case</th>
<th>true state</th>
<th>estimates</th>
<th>$a$ (km)</th>
<th>$e$</th>
<th>$i$ (°)</th>
<th>$\Omega$ (°)</th>
<th>$\omega$ (°)</th>
<th>$M_0$ (°)</th>
<th>$J \times 10^{-5}$</th>
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<tr>
<td>A1</td>
<td>7000.000</td>
<td>6999.379</td>
<td>0.05000</td>
<td>60.000</td>
<td>120.000</td>
<td>70.000</td>
<td>50.000</td>
<td>1.515</td>
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<tr>
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<td>7400.000</td>
<td>7400.000</td>
<td>0.05000</td>
<td>60.000</td>
<td>120.000</td>
<td>70.000</td>
<td>50.000</td>
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<tr>
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<td>0.05000</td>
<td>60.000</td>
<td>120.000</td>
<td>70.000</td>
<td>50.000</td>
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<tr>
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<td>7400.000</td>
<td>0.04974</td>
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<td>70.186</td>
<td>49.713</td>
<td>0.943</td>
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<tr>
<td>I2</td>
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<td>45.419</td>
<td>118.563</td>
<td>70.762</td>
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<tr>
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<td>0.04995</td>
<td>85.585</td>
<td>120.555</td>
<td>70.353</td>
<td>49.417</td>
<td>0.595</td>
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<tr>
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<td>7000.000</td>
<td>7001.061</td>
<td>0.05026</td>
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</table>

In the algorithm, $\epsilon$ is a threshold, $\Delta \gamma$ is a step size, $u$ is a unit random direction whose element is drawn from uniform distribution on $[-1, 1]$, $\odot$ denotes the elementwise product, $\beta$ is a control parameter, $V$ is a uniform variate on $[0, 1]$, $\kappa_1$ is the number of control steps, $\kappa_2$ is the number of iterations per a fixed control parameter, and $\gamma$ is a reduction factor of less than one.
first and second rows are the true and estimated state values, respectively. In our simulations, it is assumed that the Earth is rotating eastward at a rate of \(7.29211593 \times 10^{-5}\) rad/sec. The time interval between measurements is 10 minutes, and the number of measurements \(N\) is 30. The 10th degree, 10th order IGRF model is used to determine the geomagnetic field. White Gaussian measurement noises with zero-mean and \(\sigma = 100, 150,\) and \(200\) nT standard deviations are added to each component of the measured geomagnetic field vector according to the semi-major axes \((6900, 7400,\) and \(7900\) km) of the orbit, respectively.

Figure 1 shows the selected two-dimensional cost functions for case A3. The \(\odot\) mark denotes the true values. The two-dimensional cost functions depicted in Fig. 1 ascertain that several local minima occur in the search space. In Fig. 1(a), the variation of the cost function along the semi-major axis (x axis) is about twice as much as other state elements because the magnitude of the geomagnetic field rapidly decreases as the altitude increases. The annealing schedule used for all cases is composed of the control parameter \(\beta = 0.45\) with reduction factor \(\gamma = 1/\sqrt{2}\), the step size \(\Delta r = [300\text{ km} 0.3^\circ 108^\circ 108^\circ 108^\circ]^T\), the control step \(\kappa_1 = 28\) and the number of iterations \(\kappa_2 = 500\). The threshold \(\epsilon\) is \(5 \times 10^{-6}\). We start the iteration with the initial state \(x_0 = [7400\text{ km} 0.5^\circ 45^\circ 180^\circ 180^\circ 180^\circ]^T\). It is the center of the search space. The major computational load comes from the calculation of the cost function. For each iteration, 117,840 flops (floating point multiplications and additions) and 238 trigonometric calculations are needed. The estimates of each case are given in Table 1. The last column of Table 1 means the discrepancy between the true and the estimated states. Although function value at the final stage is smaller than the true state, it does not mean that estimate is more accurate because of measurement noise and uncertainty of the IGRF model. Figure 2 depicts the convergence behavior of each state element for case I3. The estimate converges near the global minimum prior to the 5th control step. Note that all elements have the same convergence rate, since the relative element magnitudes for the step size to the search space are equal. For case A3, the convergence behavior of the proposed method is displayed in
Fig. 3. The dots denote the trajectory states of the accepted steps in the iteration stages. Note that the estimate near local minima does not get trapped.

4. Conclusions

We have proposed a low-cost, robust and autonomous initial orbit determination method for an LEO-satellite using the simulated annealing technique. Our method has the following advantageous features: First, it only uses an inexpensive magnetometer. Second, it does not require attitude information, ground intervention, or a GPS signal. Finally, our method does not need a-priori information on the orbital elements, because the simulated annealing technique finds the global minimum.

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References