Transient Lift Based on Extended Theodorsen Function for Airfoil with Wake*

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Transient lift on a flat-plate airfoil in an incompressible uniform flow for a step increase in angle of attack is solved by applying Fourier transformation to a previously reported extended lift on an oscillating flat-plate airfoil with wake vortices assumed to move downstream at an arbitrary and constant velocity. The transient lift, which depends both on the position of the airfoil rotation axis and on the convection velocity of the wake vortices, includes a new term that is not reported in the classical theories. This term yields initial impulsive variations in lift. The relationship between the present solution and Wagner's classical function is also clarified.

Key Words: Airfoil, Theodorsen, Transient, Two-Dimensional, Wagner

Nomenclature

- A : arbitrary constant to give AU ($A > 0$) where AU represents arbitrary convection velocity of wake vortices (Fig. 1)
 C : Theodorsen's function (Eqs. (3)–(5))
 C_A : complementary function coupled with Theodorsen's function C (Eqs. (6)–(8))
 F : real part of Theodorsen's function (Eqs. (3) and (4))
 F_A : real part of C_A (Eqs. (6) and (7))
 F_L : real part of right side of Eq. (2)
 G : imaginary part of Theodorsen's function (Eqs. (3) and (5))
 G_A : imaginary part of C_A (Eqs. (6) and (8))
 G_L : imaginary part of right side of Eq. (2)
 J_0, J_1 : Bessel functions of first and second kind (Eqs. (4), (5), (7) and (8))
 L_A : oscillating lift (per unit span) on oscillating airfoil (Eq. (1))
 $L_{A\theta}$: transient lift (per unit span) on airfoil for step increase in angle of attack (Eq. (18))
 U : velocity of free stream
 Y_0, Y_1 : Neumann functions of first and second kind (Eqs. (4), (5), (7) and (8))
 a : non-dimensional parameter to show streamwise position of airfoil rotation axis (Fig. 1)
 c : chord length of airfoil ($c \equiv 2b$)
 b : semi-chord length of airfoil (Fig. 1)
 h : distance (ordinate) in cross-stream direction to show position of airfoil rotation axis (Fig. 1)
 $\dot{h}, \ddot{h}$: first and second time derivatives of h
 i : $(-1)^{1/2}$
 k : non-dimensional (or reduced) frequency ($k \equiv \omega b/U$)
 $\tilde{k}$: extended non-dimensional (or reduced) frequency ($\tilde{k} \equiv k/A$)
 ℓ : lift defined by Eq. (11)
 s : non-dimensional time ($s \equiv Ut/b$)
 t : time
 x : distance (abscissa) parallel to free stream (Fig. 1)
 Θ : Fourier transform of $\theta(t)$ (Eqs. (13)–(15))
 Φ : non-dimensional transient lift ($\Phi \equiv \Phi(s, A, a)$, Eq. (23))
 α : angle of attack of oscillating airfoil
 $\dot{\alpha}, \ddot{\alpha}$: first and second time derivatives of α
 ε : either amplitude of pure pitching oscillations (Eq. (9)) or amount of step increase in angle of attack (Eq. (12) and Fig. 2)
 θ : angle of attack of airfoil in transient motion (Eq. (12) and Fig. 2)
 ρ : density of fluid
 ω : angular frequency
 Superscript
 — : mean value

1. Introduction

More than 50 years have passed since the classical theories^{1–3)} for an oscillating flat-plate airfoil were reported. Two^{1,3)} of the three theories have been summarized elsewhere.⁴⁾

Recently, the author extended the essential part of these classical theories by assuming an arbitrary and constant convection velocity, AU , for the wake vortices,^{5,6)} showing that this assumption yields extended forms of the well-known Theodorsen function. Here, the extended results for an oscillating airfoil are applied to transient motion of an airfoil, or step increase in the angle of attack. The classical treatment⁴⁾ of the same subject was confined to an unrealistic case ($A = 1$) but the present study aims to remove this restriction.

Section 2 presents the main part of the extension for a step

increase in the angle of attack using Fourier transformation applied after the classical method.⁴⁾ An extended expression for transient lift is obtained and visualized in Section 3 where Wagner's classical function is included. Section 4 presents some examples of unusual pulses in lift found in the initial stage of the airfoil motion. The relationship between Wagner's classical function and the present results is also clarified. Mathematical tables, particularly a Russian table⁷⁾ for integral formulas, contributed to the derivations.

2. Extension of Transient Lift on Flat-Plate Airfoil for Step Increase in Angle of Attack by Assuming Arbitrary and Constant Convection Velocity of Wake Vortices

The author described⁵⁾ the following extended form of lift acting on a flat-plate airfoil oscillating in an incompressible flow (Fig. 1) as

$$L_A = \pi \rho b^2 (\ddot{h} + U\dot{\alpha} - \ddot{\alpha}ba) + 2\pi \rho U b (F_L + iG_L) \left[\dot{h} + U\alpha + b \left(\frac{1}{2} - a \right) \dot{\alpha} \right] \quad (1)$$

where

$$F_L + iG_L = C(\tilde{k})A + C_A(\tilde{k})(1 - A) \quad (2)$$

$$C(\tilde{k}) = F + iG \quad (3)$$

$$F = \frac{J_1(J_1 + Y_0) + Y_1(Y_1 - J_0)}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \quad (4)$$

$$G = -\frac{Y_1 Y_0 + J_1 J_0}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \quad (5)$$

$$C_A(\tilde{k}) = F_A + iG_A \quad (6)$$

$$F_A = \frac{2}{\pi \tilde{k}} \frac{\sin(\tilde{k})(J_1 + Y_0) - \cos(\tilde{k})(Y_1 - J_0)}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \quad (7)$$

$$G_A = \frac{2}{\pi \tilde{k}} \frac{\cos(\tilde{k})(J_1 + Y_0) + \sin(\tilde{k})(Y_1 - J_0)}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \quad (8)$$

Note that $C(\tilde{k})$ is Theodorsen's function and $C_A(\tilde{k})$ is a new function, both of the variable $\tilde{k}$. Bessel functions J_0 , J_1 , Y_0 and Y_1 are also functions of $\tilde{k}$.

For pure pitching oscillations, we can define

$$h = 0, \quad \alpha = \varepsilon e^{i\omega t}; \quad \text{for } 0 \leq \varepsilon \ll 1 \quad (9)$$

Substitution of Eq. (9) into Eq. (1) yields

$$L_A = \ell(\omega) \varepsilon e^{i\omega t} \quad (10)$$

where

$$\ell(\omega) = \pi \rho b^2 (U i \omega + \omega^2 b a) + 2\pi \rho U b (F_L + iG_L) \left[U + b \left(\frac{1}{2} - a \right) i \omega \right] \quad (11)$$

Apart from oscillations, we can consider the step function $\theta(t)$ in the angle of attack as shown in Fig. 2, by defining

$$\begin{aligned} \theta(t) &= 0; \quad \text{for } t < 0, \\ \varepsilon/2; &\quad \text{for } t = 0, \quad 0 \leq \varepsilon \ll 1, \\ \varepsilon; &\quad \text{for } t > 0, \quad 0 \leq \varepsilon \ll 1 \end{aligned} \quad (12)$$

Fig. 1. Coordinates for oscillating flat-plate airfoil.

The convection velocity of the wake vortices is expressed by AU . The ordinate ba shows the position of the airfoil rotation axis. The cross-stream displacement of this axis is expressed by the ordinate h .

Fig. 2. Step function in angle of attack.

The function $\theta(t)$ is connected with its Fourier transform $\Theta(\omega)$ by

$$\theta(t) - \overline{\theta(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta(\omega) e^{i\omega t} d\omega \quad (13)$$

and

$$\Theta(\omega) = \int_{-\infty}^{\infty} [\theta(t) - \overline{\theta(t)}] e^{-i\omega t} dt \quad (14)$$

where $\overline{\theta(t)} (= \varepsilon/2)$ expresses the mean value of $\theta(t)$. Two cases for Eq. (14) are considered separately as follows:

$$\begin{aligned} \text{(i) } \omega \neq 0 \quad \Theta(\omega) &= \int_{-\infty}^{\infty} \theta(t) e^{-i\omega t} dt - \frac{\varepsilon}{2} \int_{-\infty}^{\infty} e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \theta(t) e^{-i\omega t} dt \end{aligned} \quad (14a)^{\dagger}$$

$$\text{(ii) } \omega = 0 \quad \Theta(\omega) = \int_{-\infty}^{\infty} [\theta(t) - \overline{\theta(t)}] dt = 0 \quad (14b)^{\ddagger}$$

Substituting Eq. (12) into Eq. (14a) and using Eq. (14b), we have

$$\begin{aligned} \Theta(\omega) &= \frac{\varepsilon}{i\omega}; \quad \text{for } \omega \neq 0, \\ &= 0; \quad \text{for } \omega = 0 \end{aligned} \quad (15)$$

For details, refer to App. 1. Substitution of Eq. (15) into Eq. (13) yields

$$\theta(t) = \frac{\varepsilon}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\omega} \varepsilon e^{i\omega t} d\omega \quad (16)$$

Note that the integrand = 0 at $\omega = 0$. For $\omega \neq 0$, the integrand $(i\omega)^{-1} \varepsilon e^{i\omega t}$, representing a component of the oscillation with amplitude equal to $(i\omega)^{-1} \varepsilon$, should cause the fol-

[†]The integration of $e^{-i\omega t}$ in Eq. (14a) vanishes. Refer to App. 4 by applying a temporal variable-exchange $\omega \rightleftharpoons t$.

[‡]The integrand in Eq. (14b) is odd.

lowing oscillating lift, based on Eq. (10).

$$\ell(\omega) \frac{1}{i\omega} \varepsilon e^{i\omega t} \quad (17)$$

Therefore, total lift is given by

$$L_{A\theta}(t) = \ell(0) \frac{\varepsilon}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \ell(\omega) \frac{1}{i\omega} \varepsilon e^{i\omega t} d\omega \quad (18)$$

Note again that the integrand = 0 at $\omega = 0$ because we have used Eq. (16), but this should have no substantial effect on the integration because it occurs only at $\omega = 0$ without width. The first term $\ell(0)\varepsilon/2$ shows the steady lift for the angle of attack $\varepsilon/2$, which is equal to $2\pi\rho U^2 b \cdot \varepsilon/2$ from Eqs. (10) and (11). Note that $F_L + iG_L = 1$ at $\omega = k = 0$.

Equation (18) can be simplified as follows by substituting Eq. (11) into Eq. (18) to obtain

$$\begin{aligned} L_{A\theta}(t) = & 2\pi\rho U^2 b \frac{\varepsilon}{2} + \frac{1}{2} \rho b^2 \varepsilon \int_{-\infty}^{\infty} (U - i\omega b a) e^{i\omega t} d\omega \\ & \text{-----①-----} \\ & + \rho U b \varepsilon \int_{-\infty}^{\infty} (F_L + iG_L) \left[\frac{U}{i\omega} + b \left(\frac{1}{2} - a \right) \right] e^{i\omega t} d\omega \\ & \text{-----②-----} \end{aligned} \quad (19)$$

Symmetry/singularity considerations alter the above integrations ① and ② as follows. For details, refer to App. 3.

$$\text{①} = \infty; \text{ for } t = 0,$$

$$0; \text{ for } t \neq 0 \quad (20)$$

$$\begin{aligned} \text{②} = & 2U \int_0^{\infty} \frac{F_L \sin \omega t + G_L \cos \omega t}{\omega} d\omega + 2b \left(\frac{1}{2} - a \right) \\ & \times \int_0^{\infty} (F_L \cos \omega t - G_L \sin \omega t) d\omega; \\ & \text{for } -\infty < t < \infty \end{aligned} \quad (21)$$

Using the symbols ① and ② temporarily and dividing Eq. (19) by $2\pi \cdot (1/2)\rho U^2 \cdot 2b \cdot \varepsilon$, we obtain

$$\frac{L_{A\theta}(t)}{2\pi \cdot (1/2)\rho U^2 \cdot 2b \cdot \varepsilon} = \frac{1}{2} + \frac{\text{①}b}{4\pi U^2} + \frac{\text{②}}{2\pi U}; \quad (22)$$

for $-\infty < t < \infty$

Hereafter, we avoid the origin ($t = 0$) in order to escape infinity in ① (see Eq. (20)). This leads to elimination of ① in Eq. (22). By using the notation, Φ , we obtain

$$\Phi = \frac{1}{2} + \frac{\text{②}}{2\pi U}; \quad \text{for } -\infty < t < \infty, \quad t \neq 0 \quad (23)$$

Substituting Eq. (21), we obtain the non-dimensional lift

$$\begin{aligned} \Phi \equiv \Phi(s, A, a) = & \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{F_L \sin ks + G_L \cos ks}{k} dk \\ & + \frac{1}{\pi} \left(\frac{1}{2} - a \right) \int_0^{\infty} (F_L \cos ks - G_L \sin ks) dk; \\ & \text{for } -\infty < s < \infty, \quad s \neq 0 \end{aligned} \quad (24)$$

Note that the final term on the right side of Eq. (24) appears in this paper. Equation (24) can be modified as follows by applying the same method previously described elsewhere.⁴⁾

Let the non-dimensional time s be positive, so for negative values of time $-s$ (< 0), the lift must be zero and Eq. (24) becomes

$$\begin{aligned} 0 = & \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{-F_L \sin ks + G_L \cos ks}{k} dk \\ & + \frac{1}{\pi} \left(\frac{1}{2} - a \right) \int_0^{\infty} (F_L \cos ks + G_L \sin ks) dk; \\ & \text{for } s > 0 \end{aligned} \quad (25)$$

The subtraction (Eq. (24) – Eq. (25)) and addition (Eq. (24) + Eq. (25)) give two different forms of $\Phi(s, A, a)$ as follows:

$$\begin{aligned} \Phi(s, A, a) = & \frac{2}{\pi} \int_0^{\infty} \frac{F_L \sin ks}{k} dk \\ & \text{-----} \\ & - \frac{2}{\pi} \left(\frac{1}{2} - a \right) \int_0^{\infty} G_L \sin ks dk; \\ & \text{for } s > 0 \end{aligned} \quad (26)$$

$$\begin{aligned} \Phi(s, A, a) = & 1 + \frac{2}{\pi} \int_0^{\infty} \frac{G_L \cos ks}{k} dk \\ & \text{-----} \\ & + \frac{2}{\pi} \left(\frac{1}{2} - a \right) \int_0^{\infty} F_L \cos ks dk; \\ & \text{for } s > 0 \end{aligned} \quad (27)$$

where $F_L \equiv F_L(\tilde{k})$ and $G_L \equiv G_L(\tilde{k})$. Although the final terms in Eqs. (26) and (27) are not found elsewhere,⁴⁾ the initial simplification of $a = 1/2$ is.⁴⁾ The underlined parts reduce to Wagner's function⁴⁾ if $A = 1$ (Section 4.3).

Since Eqs. (26) and (27) are mutually equivalent and the parameter $(1/2 - a)$ is independent of the others, we get the following two relationships.

$$\frac{2}{\pi} \int_0^{\infty} \frac{F_L \sin ks}{k} dk = 1 + \frac{2}{\pi} \int_0^{\infty} \frac{G_L \cos ks}{k} dk; \quad \text{for } s > 0 \quad (28)$$

$$- \int_0^{\infty} G_L \sin ks dk = \int_0^{\infty} F_L \cos ks dk; \quad \text{for } s > 0 \quad (29)$$

These can be used to modify Eqs. (26) or (27). We chose one such modification by substituting Eq. (29) into Eq. (26), yielding

$$\begin{aligned} \Phi(s, A, a) = & \frac{2}{\pi} \int_0^{\infty} \frac{F_L \sin ks}{k} dk \\ & + \frac{2}{\pi} \left(\frac{1}{2} - a \right) \int_0^{\infty} F_L \cos ks dk; \\ & \text{for } s > 0 \end{aligned} \quad (30)$$

where the function F_L is given as follows using Eqs. (2), (3) and (6).

$$F_L = F \cdot A + F_A \cdot (1 - A) \quad (31)$$

(a) $a = -1/2$ (b) $a = 0$ (c) $a = +1/2$

Fig. 3. Normalized transient lift $\Phi(s, A, a)$ ($s \neq 0$) for step increase in angle of attack.

The calculated values of $\Phi(s, A, a)$ are plotted against non-dimensional time s for different values of a and A .

3. Numerical Results

In order to calculate the two integrations in Eq. (30), the functions $F \equiv F(\tilde{k})$ and $F_A \equiv F_A(\tilde{k})$ in Eq. (31) were replaced by approximated functions composed of elementary functions such as $\tilde{k}^{1/2}$ and $\exp(-\tilde{k})$.⁸⁾ The approximated F_L ($= F \cdot A + F_A \cdot (1 - A)$) was substituted into Eq. (30) and integrated analytically to give $\Phi(s, A, a)$ shown in Fig. 3 against s for different values of a and A . It is clear that the magnitude of A affects the overall variation of $\Phi(s, A, a)$ while the influence of a is predominant at small values of s .

4. Considerations

4.1. Abnormal effect of position of airfoil rotation axis

Some of the present results are shown again in Fig. 4

(a) With normal scales for abscissa and ordinate

(b) With enlarged scale for abscissa

Fig. 4. Examples showing abnormal effect of a .
 $A = 0.8$.

which demonstrates an effect of a especially at $s \rightarrow 0$. The data for $a = -1/2$ show an impulsive variation at small values of s , while those for $a = +1/2$ do not. The impulsive variation originates from function $F_A(\tilde{k})$ in F_L in the second term of Eq. (30). This can be confirmed directly by examining Eq. (30) at $s \rightarrow 0$. Therefore, the function approximations adopted to Eqs. (30) and (31) (Section 3) are excluded as possible origins for the impulsive variation. The impulsive variation cannot be attributed to any virtual-mass effect because this effect has been avoided after Eq. (22) by ignoring the origin of time t . From Eq. (30) it can be seen that $\Phi(s, A, a)$ approaches infinity by the factor $s^{-1/2}$, meaning that the impulsive variation is not of the delta function. This suggests that the impulsive variation could be caused by a viscous effect, such as initial wake vortices, although the true origin is unknown at present.

4.2. Similar singular behaviour at $A \cong 1$

Figure 5 shows a switching of the direction in the impulsive variation against an increase in A from 0.8 to 1.2 across 1.0. It is seen that the impulsive variation disappears at $A = 1.0$. The switching comes from the sign of the term $(1 - A)$ in Eq. (31) that operates in the second term of the right side of Eq. (30). Based on Figs. 4 and 5, we can say that the classical condition of $a = +1/2$ and $A = 1.0$ is a very unique case.

4.3. Extended Wagner's function

For $A = 1$ and $a = +1/2$, Eq. (30) reduces to

$$\Phi(s, 1, 1/2) = \Phi(s) = \frac{2}{\pi} \int_0^\infty \frac{F(k) \sin ks}{k} dk \quad (32)$$

Fig. 5. Singular behavior of $\Phi(s, A, a)$ at $A \equiv 1.0$ with enlarged abscissa and reduced ordinate.
 $a = -1/2$.

or Wagner's function.⁴⁾ The curve for Eq. (32) is included in Fig. 3(c) ($A = 1$). Therefore, we may say that Fig. 3(c) expresses an extension of Wagner's function.

The reason why Wagner's function, which should be defined in a starting-flow problem, appears in the present uniform-flow problem is attributable to the multiplied form between $(1/2 - a)$ and $\dot{\alpha}$ in the second term of the right side of Eq. (1). In other words, the selection of $a = +1/2$ is effectively equivalent to the selection of $\dot{\alpha} = 0$, reducing a uniform-flow problem to a starting-flow problem where $\dot{\alpha} = 0$. In other words, the classical result⁴⁾ is valid only at $a = +1/2$. If $a = +1/2$, the second term of the right side of Eq. (30) must be taken into account in uniform-flow problems. This important term is not found elsewhere.⁴⁾

5. Conclusion

The previously reported extended lift⁵⁾ on an oscillating airfoil obtained by assuming an arbitrary and constant convection velocity of the wake vortices was Fourier-transformed into transient lift for a step increase in angle of attack. This gave an extended form of the conventional transient lift.

The new form of transient lift includes a new term not found in classical expressions. This term gives characteristic initial ($s \equiv 0$) impulsive variations in the transient lift. Both a and A affect the lift at $s \equiv 0$, while only A predominantly affects the lift as s increases. In general, the response of the transient lift is delayed by the decrease in A . However, at $a = +1/2$ (rear $(1/4)$ c -axis), the above-mentioned new lift term vanishes and the lift reduces to a single-term expression including Wagner's classical ($A = 1$) function.

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Appendix

Appendix 1. Derivation of Eq. (15)

Equation (12) is substituted into Eq. (14a) to give

$$\begin{aligned}\Theta(\omega) &= \varepsilon \int_0^\infty e^{-i\omega t} dt \\ &= \frac{\varepsilon}{-i\omega} [e^{-i\omega t}]_0^\infty \\ &= \frac{\varepsilon}{-i\omega} (e^{-i\omega\infty} - 1) \\ &\downarrow \text{App. 2} \\ &\rightarrow \frac{\varepsilon}{i\omega} \\ &\quad (\omega \neq 0)\end{aligned}$$

Appendix 2. $e^{\pm i\omega\infty} (\omega \neq 0) (\rightarrow \text{App. 1})$

It is known that the term $e^{\pm i\omega\infty} (\omega \neq 0)$ converges to zero. A suggested⁴⁾ simplified proof for this is shown below where ν is a fixed and infinitesimally small constant.

$$e^{\pm i\omega\infty} \doteq \lim_{0 < \nu \rightarrow 0} e^{\pm i(\omega \pm i\nu)\infty} \doteq \lim_{0 < \nu \rightarrow 0} e^{-\nu\infty} e^{\pm i\omega\infty} \rightarrow 0$$

Appendix 3. Derivation of Eqs. (20) and (21)

$$\textcircled{1} = \int_{-\infty}^{\infty} (U - i\omega ba) d\omega = U \int_{-\infty}^{\infty} d\omega \rightarrow \infty$$

↑
The second integrand $-i\omega ba$ is odd.
($t = 0$)

$$U \int_{-\infty}^{\infty} e^{i\omega t} d\omega - iba \int_{-\infty}^{\infty} \omega e^{i\omega t} d\omega \rightarrow 0,$$

↑
App. 4, 5
($t \neq 0$)

$$\begin{aligned} \textcircled{2} = U \int_{-\infty}^{\infty} \frac{(F_L + iG_L)e^{i\omega t}}{i\omega} d\omega \\ \text{-----}\textcircled{2a}\text{-----} \\ + b\left(\frac{1}{2} - a\right) \int_{-\infty}^{\infty} (F_L + iG_L)e^{i\omega t} d\omega; \\ \text{-----}\textcircled{2b}\text{-----} \\ \text{for } -\infty < t < \infty \quad (\text{A-1}) \end{aligned}$$

We can adopt symmetry considerations to $\textcircled{2a}$ and $\textcircled{2b}$ to obtain

$$\begin{aligned} \textcircled{2a} = 2 \int_0^{\infty} \frac{F_L \sin \omega t + G_L \cos \omega t}{\omega} d\omega; \\ \text{for } -\infty < t < \infty \quad (\text{A-2}) \end{aligned}$$

$$\begin{aligned} \textcircled{2b} = 2 \int_0^{\infty} (F_L \cos \omega t - G_L \sin \omega t) d\omega; \\ \text{for } -\infty < t < \infty \quad (\text{A-3}) \end{aligned}$$

Note that we have even- F_L and odd- G_L with respect to ω which bring about real $\textcircled{2a}$ and $\textcircled{2b}$, based on component considerations for Eqs. (2)–(8), where J_0 and Y_1 are even and J_1 and Y_0 are odd with respect to $\tilde{k}$ (or ω) from their original integral forms during the derivations. Equations (A-2) and (A-3) are finally substituted into Eq. (A-1) to give Eq. (21).

Appendix 4. $\int_{-\infty}^{\infty} e^{i\omega t} d\omega (t \neq 0) (\rightarrow \text{App. 5, } \textcircled{1} \text{ in App. 3 and footnote for Eq. (14a)})$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\pm i\omega t} d\omega = \frac{1}{\pm it} (e^{\pm it\infty} - e^{\mp it\infty}) \rightarrow 0 \\ \uparrow \\ \text{App. 2} \\ (t \neq 0) \end{aligned}$$

Appendix 5. $\int_{-\infty}^{\infty} \omega e^{i\omega t} d\omega (t \neq 0) (\rightarrow \text{App. 3})$

$$\begin{aligned} \int_{-\infty}^{\infty} \omega e^{i\omega t} d\omega = \frac{1}{it} [e^{i\omega t} \omega]_{-\infty}^{\infty} - \frac{1}{it} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \rightarrow 0 \\ \uparrow \qquad \qquad \qquad \uparrow \\ \text{Partial integration} \qquad \qquad \text{App. 4, 6} \\ (t \neq 0) \end{aligned}$$

Appendix 6. $e^{i\omega t} \omega (t \neq 0, \omega \rightarrow \pm\infty) (\rightarrow \text{App. 5})$

Similar to App. 2, a simplified proof is shown below.

$$\begin{aligned} \lim_{\omega \rightarrow \pm\infty} e^{i\omega t} \omega &\doteq \lim_{\substack{\omega \rightarrow \pm\infty \\ \nu \rightarrow \pm 0}} e^{i\omega(t+i\nu)} \omega \\ &= \lim_{\substack{\omega \rightarrow \pm\infty \\ \nu \rightarrow \pm 0}} e^{i\omega t} \frac{\omega}{e^{\omega\nu}} \\ &= \lim_{\substack{\omega \rightarrow \pm\infty \\ \nu \rightarrow \pm 0}} e^{i\omega t} \frac{1}{\nu e^{\omega\nu}} \rightarrow 0 \\ &\quad \uparrow \text{ l'Hôpital's rule for } \omega/e^{\omega\nu} \text{ on } \omega \\ &\qquad \qquad \qquad (t \neq 0) \end{aligned}$$

Note: We choose the case $\omega\nu > 0$ where ν is a fixed and infinitesimally small constant independent of ω .