Low-Thrust, Multi-Revolution Orbit Transfer under the Constraint of a Switch Function without Prior Information

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In this paper, a low-thrust, multi-revolution orbit transfer under the constraint of a complex switch function is investigated. First, the effect of the switch function, especially that of a switch function with no prior information, is analyzed. Then, by utilizing the concept of the Lyapunov feedback control law, the semi-analytical expressions of suboptimal thrust angles are derived, and a near-optimal solution could approach the optimal solution by adjusting the five weights in the Lyapunov function using sequential quadratic programming (SQP). In the novel method, except for optimization of the weight factors in the Lyapunov function, no iteration is contained in the process of design and optimization, so the method is rapid and has good convergence. This method also overcomes the drawbacks about convergence of traditional calculus of variation (COV). It is an effective method for the design of low-thrust, multi-revolution orbit transfer with no prior information switch functions.

Key Words: Low-Thrust, Switch Function, Solar Electric Propulsion, Orbit Transfer

1. Introduction

Low-thrust propulsion systems have been widely used for both near-Earth and deep-space spacecraft. Their high specific impulse characteristics greatly increase the mass ratio of payload. But it is now well known that the work of low-thrust propulsion systems is restricted by many constraints. Some examples include: when a highly precise orbit measurement is required, the thrusters need to be closed to eliminate perturbation; as a type of widely used low-thrust propulsion system, solar electric propulsion (SEP) requires solar energy to power it, just as its name implies, can not work in the shadow of a planet; and in order to ensure the supply of electrical energy, a SEP system may not work when scientific instrument is working. All of the constraints enumerated above could be described as switch functions. In general, a switch function is a function of time; however, in some cases, it can not be described as a function of time and must be calculated during the process of propagation of the dynamic equations. Whether and when the thruster is on greatly depends on the initial states and the history of control. So the trajectory is very sensitive to initial states and costates. Nonlinearity and sensitivity increase with the constraints of this type of switch function.1

Up to now, calculus of variation (COV) and direct methods have always been regarded as two effective approaches to design low-thrust trajectories. But the nonlinearity of the system produces bad convergence when solved utilizing COV. The direct method also meets the same difficulty about convergence, and it is always a time-consuming algorithm. In a word, solving this type of problem using traditional methods is far from an easy task.

Recently, some investigators3–6 presented novel design methods for low-thrust, multi-revolution orbit transfer by combining the concept of Lyapunov feedback control with variational equations of classical orbital elements. Although these methods decrease the computational complexity effectively, and avoid the difficulty of convergence in traditional methods, a limitation still exists. In fact, orbit elements usually change continuously during low-thrust orbit transfer; hence, the singularities in variational equations of classical orbital elements can scarcely be avoided.

In this paper, in order to avoid the singularities of the variational equations, the Lyapunov feedback control law based on a set of equinoctial elements is derived and used to design a low-thrust multi-revolution transfer orbit with a switch function without prior information. Except for optimization of the weight factors in the Lyapunov function, this method does not contain any iteration in the calculation process. At an arbitrary position in the transfer orbit, a local optimal thrust direction can be obtained using the Lyapunov feedback control law; thus it overcomes the drawbacks of convergence and computing time in traditional methods, and avoids the singularities of variational equations in some Lyapunov-based methods. At the end of this paper, a low-thrust orbit transfer from low Earth orbit (LEO) to middle Earth orbit (MEO) with the influence of Earth shadow is investigated as an example to validate the effect of this method.

2. Design of Transfer Trajectory

As noted in the Introduction, the existence of switch functions without prior information greatly increases the difficulty of trajectory design and optimization. Thus, traditional
methods always face difficulty in convergence. In this section, an analytical design method for this type of trajectory is given based on analyzing the change in each orbit element as an effect of thrust using the concept of Lyapunov feedback control. This method can effectively avoid the difficulty of convergence.

2.1 Dynamic model

From the dynamic equations of classical orbit elements, it is facile to find that, when \( e = 0 \) or \( i = 0 \), singularities will occur. This leads us to adopt a set of equinoctial elements (i.e., modified equinoctial elements) that can avoid these singularities and accommodate all possible conic orbits except for the case where \( f = 180^\circ \). The set of equinoctial elements can be described as follows:

\[
\begin{align*}
p &= a(1 - e^2) \\
f &= e \cos(\omega + \Omega) \\
g &= e \sin(\omega + \Omega) \\
h &= \tan(i/2) \cos \Omega \\
k &= \tan(i/2) \sin \Omega \\
L &= \Omega + \omega + \nu
\end{align*}
\]

where \( a, e, i, \Omega, \omega \) and \( \nu \) correspond to semi-major axis, eccentricity, inclination, longitude of ascending node, argument of perigee and true anomaly, respectively.

Variational equations for the modified equinoctial elements can be expressed as follows:

\[
\begin{align*}
\dot{p} &= \sqrt{\frac{p}{\mu}} \frac{2p}{w} f_i \\
\dot{f} &= \sqrt{\frac{p}{\mu}} \left\{ f_i \sin L + [(1 + w) \cos L + f] \frac{f_i}{w} \\
&- (h \sin L - k \cos L) \frac{g \cdot f_n}{w} \right\} \\
\dot{g} &= \sqrt{\frac{p}{\mu}} \left\{ -f_r \cos L + [(1 + w) \sin L + g] \frac{f_i}{w} \\
&+ (h \sin L - k \cos L) \frac{f \cdot f_n}{w} \right\} \\
\dot{h} &= \sqrt{\frac{p}{\mu}} \frac{s^2 f_n}{w} \cos L \\
\dot{k} &= \sqrt{\frac{p}{\mu}} \frac{s^2 f_n}{w} \sin L \\
\dot{L} &= \sqrt{\mu} \left\{ \frac{f_n}{w} \left\{ \frac{1}{w} \frac{p}{\sqrt{\mu}} (\sin L - k \cos L)f_n \right\} \right\}
\end{align*}
\]

where \( w = 1 + f \cos L + g \sin L \), \( s^2 = 1 + h^2 + k^2 \), and \( \mu \) denotes gravity constant. The components of the thrust acceleration, \( f_r, f_i, f_n \), are given in terms of the thrust acceleration and local pitch and yaw steering angles as:

\[
\begin{align*}
f_r &= F \cos \beta \sin \alpha \\
f_i &= F \cos \beta \cos \alpha \\
f_n &= F \sin \beta
\end{align*}
\]

In Eq. (3), \( F \) denotes the magnitude of thrust acceleration and \( \alpha, \beta \) correspond to local pitch and yaw steering angles, respectively.

In multi-revolution orbit transfer, we are interested in attaining prescribed changes in the elements except for true anomaly.3,4) This means that the position of departure from the initial orbit and the position of arrival onto the final orbit are often immaterial. In modified equinoctial elements, the true longitude, \( L \), is the only element which contains the information about true anomaly. So true longitude \( L \) can be free, and we concern ourselves only with the first five variational equations.

We pay attention to the thrust angle which maximizes the rate of change of that orbit element at any given position in the orbit. For a particular orbit element, the thrust angles maximizing its rate of increase are found by setting the partial derivative on the right-hand side of the variational equations for that element with respect to the thrust angles to zero and solving these nonlinear equations to obtain optimal thrust angles. Then, substituting these thrust angles into the variational equation, the maximum rate of increase of the element is given.

Five elements except for true longitude \( L \) could determine the shape of an orbit. At different positions (different \( L \)) in a particular orbit, the maximum rate of changing a particular orbit element is different. For example, the maximum rate of semi-major axis change could be achieved at perigee. For efficient analysis, we must find the position on the orbit where the maximum rate is achieved. This position could be found by setting the partial derivative of the maximum of a function with respect to the true longitude to zero, and the thrust angles are independent variables in this function. Then, by substituting this true longitude into the expression of the maximum over thrust angle, the maximum values for both true longitude and thrust angle can be obtained. In the following text, five modified equinoctial elements we are concerned with are investigated thoroughly.

2.2 Analysis of each element

It is easy to know from the variational equation that the rate of increase of element \( p \) is maximized when thrust is along the tangential component, while the rate of decrease is maximized when thrust is against the tangential component. The two rates are equal. The partial derivatives of the right hand side of the variational equation for element \( p \) with respect to the two thrust angles and the maximum rate of change are expressed as follows:

\[
\begin{align*}
\frac{\partial p}{\partial \alpha} &= -\frac{2p}{w} \sqrt{\frac{p}{\mu}} F \cdot \sin \alpha \cdot \cos \beta \\
\frac{\partial p}{\partial \beta} &= -\frac{2p}{w} \sqrt{\frac{p}{\mu}} F \cdot \sin \alpha \cdot \sin \beta \\
\max p &= \max \frac{p}{\alpha, \beta} \sqrt{\frac{p}{\mu}} F
\end{align*}
\]

When the two thrust angles are equal to zero, the rate of increase of element \( p \) is maximized. The rate of increase is not related to true longitude \( L \); thus, it shows that the thrust efficiency of changing element \( p \) is the same at every position in the orbit.

Variational equations of \( f \) and \( g \) are more complex than
those of other elements because they are related to all three thrust components. The partial derivatives of the right-hand side of the variational equations for \( f \) and \( g \) with respect to the two thrust angles and the maximum increase rates of the two elements are given by:

\[
\frac{\partial f}{\partial \alpha} = \sqrt{\frac{p}{\mu}} \left( \cos \alpha \cos \beta \sin L - [w + 1] \cos L + f \right) F
\]

\[
\frac{\partial f}{\partial \beta} = \sqrt{\frac{p}{\mu}} \left( -\sin \alpha \sin \beta \sin L - (w + 1) \cos L + \frac{1}{w} \cos \alpha \sin \beta - \frac{1}{w} \sin L - k \cos L \right) \frac{g}{w} \cos \beta F
\]

\[
\frac{\partial g}{\partial \alpha} = \sqrt{\frac{p}{\mu}} \left( \sin \alpha \sin \beta \cos L - [(w + 1) \sin L - g] \frac{1}{w} \cos \alpha \sin \beta + (h \sin L - k \cos L) \frac{f}{w} \cos \beta \right) F
\]

\[
\frac{\partial g}{\partial \beta} = \sqrt{\frac{p}{\mu}} \left( \sin \alpha \sin \beta \cos L - [(w + 1) \sin L + g] \frac{1}{w} \cos \alpha \sin \beta + (h \sin L - k \cos L) \frac{f}{w} \cos \beta \right) F
\]

\[
\max_{\alpha \beta} \dot{f} = \sqrt{\frac{p}{\mu}} \cdot \frac{F}{w} \sqrt{w^2 \sin^2 L - [(w + 1) \sin L + f]^2 + (h \sin L - k \cos L)^2 g^2}
\]

\[
\max_{\alpha \beta} \dot{g} = \sqrt{\frac{p}{\mu}} \cdot \frac{F}{w} \sqrt{w^2 \cos^2 L - [(w + 1) \sin L + g]^2 + (h \sin L - k \cos L)^2 f^2}
\]

The maximum value in terms of true longitude and thrust angle has no analytic solution for these two elements, but it is easy to find numerical solutions by solving nonlinear equations.

By defining of \( h \) and \( k \), we can find that these elements are only related to inclination and longitude of the ascending node, which reflects the position of the orbit in the space and has no relationship with the shape of the orbit. So only the normal component of the thrust can change \( h \) and \( k \). We could easily obtain the partial derivatives of the right-hand side of variational equations for \( h \) and \( k \) with respect to the two thrust angles:

\[
\frac{\partial h}{\partial \alpha} = 0
\]

\[
\frac{\partial h}{\partial \beta} = \sqrt{\frac{p}{\mu}} \cdot \frac{s^2 \cos L}{2w} \cos \beta \cdot F
\]

\[
\frac{\partial k}{\partial \alpha} = 0
\]

\[
\frac{\partial k}{\partial \beta} = \sqrt{\frac{p}{\mu}} \cdot \frac{s^2 \sin L}{2w} \cos \beta \cdot F
\]

Maximum increase rates in terms of thrust angle may be written as:

\[
\max_{\alpha \beta} \dot{h} = \sqrt{\frac{p}{\mu}} \cdot \frac{s^2 \cos L}{2w} \frac{F}{2}
\]

\[
\max_{\alpha \beta} \dot{k} = \sqrt{\frac{p}{\mu}} \cdot \frac{s^2 \sin L}{2w} \frac{F}{2}
\]

2.3. Concept of Lyapunov feedback control law

The purpose of orbit design is determining the two thrust angles that can drive the current orbit elements to the targeted elements while consuming the least amount of propellant. These two thrust angles can be described using a set of analytical expressions, which are derived based on the concept of Lyapunov feedback control law.

The Lyapunov function is defined as:

\[
V = \sum_{\zeta} W(x) \left( \frac{\zeta - \zeta^*}{\max_{\zeta} \xi} \right)^2 (\zeta = p, f, g, h, k),
\]

where \( W(x) \) is the weight of element \( \zeta \) and \( \zeta^* \) is the target orbit element.

The weights are always greater than zero. Thus, \( V \) is always greater or equal to zero. When the spacecraft transfers to the target orbit, the Lyapunov function \( V \) will be equal to zero. \( V \) is a function of the first five elements, and has no relation to true longitude and thrust angles. Thus, the derivative of \( V \) with respect to time is described as follow:

\[
\frac{dV}{dt} = \sum_{\zeta} \frac{\partial V}{\partial \zeta} \dot{\zeta}
\]

where \( \dot{\zeta} \) is the right-hand side of the variational equations for the modified equinoctial elements. By substituting Eqs. (2) and (21) into Eq. (22), we could find that \( \dot{V} \) is a function of the two thrust angles. If the thrust angles that can make \( \dot{V} \) most negative at every position are given, the orbit trans-
fer can be finished in the shortest time under the control of
the thrust angles. For the convenience of derivation, we de-

scribe the partial derivatives of Lyapunov function \( V \) with respect to
the first five modified equinoctial elements.

\[
K_p = \frac{\partial V}{\partial p}, \quad K_f = \frac{\partial V}{\partial f}, \quad K_g = \frac{\partial V}{\partial g}, \quad K_h = \frac{\partial V}{\partial h}, \quad K_k = \frac{\partial V}{\partial k}
\]

K\(_p\), K\(_f\), and K\(_k\) can be derived analytically. K\(_f\) and K\(_g\) must
be determined numerically, because for elements f and g,
the maximum values in terms of true longitude and thrust
angle have no analytic solution. The five weights \( W_i \), (\( i = p, f, g, h, k \)) are contained in K\(_p\), K\(_f\), K\(_g\), K\(_h\), and K\(_k\), respectively.

The thrust angles which make \( \dot{V} \) most negative have an-
lytic solutions, and these analytic solutions are described
as follows:

\[
\alpha = \arctan\left\{ \frac{(K_f \sin L - K_g \cos L)w}{2pK_p + (K_f + K_g)(w + 1) \cos L + f} \right\}
\]

(23)

\[
\beta = \arctan\left\{ \frac{2pK_p \cos \alpha - (h \sin L - k \cos L)(-K_f g + K_g f) - \frac{\gamma^2}{2}(K_h \cos L + K_k \sin L)}{(K_g \cos L - K_f \sin L)w \sin \alpha - [(w + 1) \cos L + f]K_f \cos \alpha - [(w + 1) \sin L + g]K_g \cos \alpha} \right\}
\]

(24)

These expressions of thrust angles are obtained by setting
the first derivative of \( \dot{V} \) with respect to thrust angles equal to
zero, and the only necessary condition is being the minimum
value. In order to judge whether it is the minimum value or
not, the second derivative must be calculated.

If \( \frac{\partial^2 \dot{V}}{\partial \alpha^2} < 0 \), the \( \alpha \) makes \( \dot{V} \) most positive, and \( \alpha + \pi \) will
make \( \dot{V} \) negative, accordingly. So we must set \( \alpha = \alpha + \pi \). Similarly, if \( \frac{\partial^2 \dot{V}}{\partial \beta^2} < 0 \), we set \( \beta = -\beta \).

With these thrust angles, the Lyapunov function is being
sent towards zero as quickly as possible at each instant, and
as the Lyapunov function decreases, the spacecraft transfers
from its initial orbit to the target orbit.

3. Trajectory Optimization

An analytic feedback control law is derived in the above
paragraphs. Strictly speaking, the maximum of \( f, g \) in terms
of the true longitude and two thrust angles must be obtained
by solving two nonlinear equations, so the control law is
semi-analytic. With this control law, spacecraft can move
towards the target orbit, but the transfer orbit is very sensi-
tive to these weights in the Lyapunov function. In order to
obtain the optimal transfer orbit, we utilize a sequential
quadratic programming (SQP) method to optimize these
weights. The five weights and the time of flight are the free
variables. At the end of the flight, the first five elements must
be equal to the target elements, and these are the equation
constraints. For propellant-optimal missions, the total time
length when the thruster is on is the performance index.
Thus, the optimization with nonlinear constraints can be
described as follows:

\[
Z = [W_p, W_f, W_g, W_h, W_k, f_t]
\]

Constraints:

\[
\psi_j[x_j(Z), f_j(Z)] = \xi_j(Z) - \xi_j^*(Z) = 0
\]

(26)

\[
\xi_j^* = p^*, f^*, g^*, h^*, k^*, \quad j = 1, \ldots, 5
\]

(27)

Index performance:

\[
J(Z) \to \min
\]

Here \( J \) denotes the total time length when the thruster is
on.

4. Numerical Examples

In order to validate the effect of the design method pre-
sented, we take an orbit transfer from LEO to MEO as an
example, and the Earth’s shadow, a typical switch function
without prior information, is considered as the constraint
of the problem. Thrusters are closed while the spacecraft
in the Earth’s shadow. The shadow of the Earth is a common
stellar phenomenon, which is called an eclipse, so the
gometry and the influence of the eclipse are first analyzed
thoroughly.

4.1. Eclipse geometry

Satellites routinely enter eclipse (Earth shadow) periods
during their orbits, and this can cause a great fluctuation
in the available solar radiation for the solar array panels.
As discussed in the above paragraph, the SEP system’s work
is constrained by this stellar phenomenon. Figure 1 shows a
basic eclipse geometry.\(^7\)

The general geometry of eclipses involves a primary body
(Earth) and a secondary body (Sun). Eclipses occur within
the umbra and penumbra regions. The umbra is totally
eclipsed by the primary body (Earth); the penumbra is only
partially obscured by the primary body. Notice the scales are
greatly exaggerated; the actual angular departure of the pe-
numbra from the umbra region is rather small. For the Sun-
Earth system, the actual angles are found using right trian-
gles as follows:
\[
\tan \alpha_{\text{umb}} = \frac{r_s - r_p}{R_p} = 0.26411888 \\
\tan \alpha_{\text{pen}} = \frac{r_s + r_p}{R_p} = 0.26900424
\]

\[r_s = 6.960 \times 10^5 \text{ km}, \quad r_p = 6378.145 \text{ km}, \]
\[R_p = 1.4959987 \times 10^8 \text{ km}\]

where \(r_s\), \(r_p\) and \(R_p\) are the radius of the Sun, radius of the Earth and the distance between the Sun and Earth.

Combining the two angles and the position vectors of the Sun and spacecraft with triangles, we can judge whether the spacecraft is involved in the shadow, and we can also distinguish if the spacecraft is in the umbra or penumbra.\(^8\)

### 4.2. Orbit transfer with the constraint of Earth shadow

A satellite transfer from a 200 km quasi-circular parking orbit to a 1,000 km circle orbit in the equatorial plane is assumed. During the propagation of dynamic equations, the direction of thrust is determined by the semi-analytical expressions of the control angles, and at any point, whether the thrusters work is determined by utilizing the model of eclipses.

The initial and final orbit elements are listed in Table 1.

Because the target orbit is a circle orbit, \(\Omega\) and \(\omega\) can be free, they will not change the shape and attitude of the orbit in space. But for other kinds of target orbits, these two parameters can also be constrained by this method.

The values of the five weights in the Lyapunov function are shown in Table 2, before and after optimization.

Table 1. Initial and target orbit elements.

<table>
<thead>
<tr>
<th>(a)</th>
<th>Initial orbit</th>
<th>Target orbit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e)</td>
<td>0.0005</td>
<td>0</td>
</tr>
<tr>
<td>(i)</td>
<td>28.5°</td>
<td>0°</td>
</tr>
<tr>
<td>(\Omega)</td>
<td>free</td>
<td>free</td>
</tr>
<tr>
<td>(\omega)</td>
<td>free</td>
<td>free</td>
</tr>
<tr>
<td>(\nu)</td>
<td>free</td>
<td>free</td>
</tr>
</tbody>
</table>

Table 2. Values of weights before and after optimization.

<table>
<thead>
<tr>
<th>(W_p)</th>
<th>(W_f)</th>
<th>(W_g)</th>
<th>(W_h)</th>
<th>(W_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial value</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Final value</td>
<td>0.4680</td>
<td>0.9870</td>
<td>0.9639</td>
<td>0.9981</td>
</tr>
</tbody>
</table>

With the final values of these weights, an optimal transfer trajectory under the effect of the Earth’s shadow can be generated, and the trajectory is shown in Fig. 2. For convenience of comparison, the same trajectory transfer without the effect of the Earth’s shadow is generated by the same method, and shown in Fig. 3. The marks “\(\triangle\)” and “\(\bigcirc\)” represent the initial and final positions of the transfer trajectory, respectively.

The tendencies of change about the two orbits are very different. Without the effect of the Earth’s shadow, the semi-major axis changes smoothly and overshooting does not appear. The orbit is sparser than that with the influence of the Earth’s shadow. This phenomenon implies that the same orbit transfer requires a longer time when the Earth’s shadow is present. The key characters of the two trajectories are different too, and these characters are summarized in Table 3.

Table 3 reconfirms that the presence of the Earth’s shadow will greatly extend the flight time. The ratio between the time of thruster working and the whole flight time is 55.44%, and for other problems, which are transferring to
a higher target orbit, the ratio will increase. The constraint of shadowing prevents the thrusters from working at some positions, which can change the orbit elements towards the target values with high efficiency, so the same orbit transfer requires more propellant to finish.

Time histories of semi-major axis, eccentricity and inclination are illustrated in Fig. 4. The curves of the three elements are scalariform, because orbit elements do not change in the Earth’s shadow, and the no prior information switch function is defined connotatively by these step-wise curves.

5. Conclusion

In order to design a low-thrust transfer trajectory under the influence of a switch function without prior information, the efficiency of changing each of the elements except for true longitude and the semi-analytical expressions of thrust angles are derived based on the concept of Lyapunov feedback control and the variational equations of equinoctial elements. Utilizing these semi-analytic expressions, an effective method for designing a low-thrust, multi-revolution orbit transfer with the constraints of a complex switch function is presented. A suboptimal transfer trajectory can be generated by the method, which enables approaching optimal solutions by adjusting five weights in the Lyapunov function. Except for optimization of the weight factors in the Lyapunov function, the method does not have any iteration in the calculation process, so it is rapid and has good convergence. Singularities of dynamic equations are eliminated by utilizing a set of modified equinoctial elements.

This method does not depend on the concept of traditional COV, and does not import any costates or conjugate equations. The nonlinearity, which is caused by complex switch functions, is minimized. It is an effective approach to designing and optimizing sophisticated scenarios, which are constrained by very complex switch functions without any prior information, such as the Earth’s shadow effect.

References