A Sliding Mode Control with Optimized Sliding Surface for Aircraft Pitch Axis Control System

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A sliding mode controller with an optimized sliding surface is proposed for an aircraft control system. The quadratic type of performance index for minimizing the angle of attack and the angular rate of the aircraft in the longitudinal motion is used to design the sliding surface. For optimization of the sliding surface, a Hamilton-Jacobi-Bellman (HJB) equation is formulated and it is solved through a numerical algorithm using a Generalized HJB (GHJB) equation and the Galerkin spectral method. The solution of this equation denotes a nonlinear sliding surface, on which the trajectory of the system approximately satisfies the optimality condition. Numerical simulation is performed for a nonlinear aircraft model with an optimized sliding surface and a simple linear sliding surface. The simulation result demonstrates that the proposed controller can be effectively applied to the longitudinal maneuver of an aircraft.

Key Words: Sliding Mode Control, Optimal Sliding Surface, Aircraft Pitch Control, Hamilton-Jacobi-Bellman Equation

Nomenclature

α: angle of attack [deg]
q: angular velocity [deg/s]
δε: elevator deflection angle [deg]
u: control input
J: performance index
S: sliding surface
ϕ: basis function
Subscripts
α: function of angle of attack
q: function of angular velocity
δε: function of elevator deflection angle

1. Introduction

The sliding mode control (SMC) theory has been applied successfully to the aircraft vehicle maneuver problem.1–5) Xu et al. designed a multi-input/multi-output adaptive sliding mode controller for the longitudinal model of a generic hypersonic air vehicle.1) This vehicle model is nonlinear, multivariable and unstable, and includes uncertain parameters. This paper shows that the combination of the observer with the adaptive sliding mode controller leads to the final adaptive SMC, and the combined adaptive sliding controller-observer has good tracking performance and robustness. Ashari and Khaloozadeh proposed a new control law by redesigning the sliding-surface based on the eigenstructure assignment technique.2) This method is implemented to design a flight controller.

Usually, system uncertainties as well as actuator faults severely affect the performance of the control system. Ure and Inalhan designed a higher order sliding mode controller for the nonlinear dynamics of an unmanned combat air vehicle.3) In this paper, the overall system was numerically tested on an F-16 model, and the simulations demonstrated that final design was capable of autonomously tracking the reference trajectory in the presence of unmodeled dynamics, disturbances and nonminimum phase outputs. Hao et al. designed a pitch rate track system for an aircraft by applying the sliding mode controller.4) The simulation result demonstrated that the proposed controller can be effectively applied to an aircraft pitch rate track system. Promtun and Seshagiri designed a new sliding mode controller for the pitch-rate of an F-16 aircraft, based on the conditional integrator design.5) The simulation results showed that the method outperforms, without any scheduling requirement, the transient and steady-state performance of a conventional gain-scheduled model-following controller. Bahrami et al. designed a controller for nonlinear non-minimum phase supersonic flight vehicle longitudinal dynamics using a dynamic sliding manifold, which is a modified switching manifold in SMC theory. The simulation results showed the excellent performance of the dynamic sliding manifold.6)

There are two main advantages of SMC. Firstly, the dynamic behavior of the system may be tailored by the particular choice of sliding surface. Secondly, the closed-loop response becomes totally insensitive to a particular class of uncertainties and disturbances.

The sliding mode controller is composed of two control phases: the reaching phase to the sliding surface, and the maintaining phase on the sliding surface. The design process of the sliding mode controller is straightforward once the sliding surface has been chosen. Usually, the sliding surface is defined by the desired dynamics. In other words, the state
errors about the desired states are used for the selection of the appropriate sliding surface.\textsuperscript{1,3–5} In Ref. 2), a linear sliding surface is defined for the sake of simplicity. However, the SMC theory itself does not give any guideline to the design of the sliding surface, which is essential in the performance of the controller. Therefore, the sliding surface in the SMC theory should be carefully selected so that it can guarantee good performance as well as stability of the closed-loop system. In this paper, an optimized sliding surface is designed for an aircraft longitudinal maneuver problem. Unlike the existing approaches, the sliding surface is modeled as a general nonlinear function. The introduction of a general nonlinear function gives flexibility and depth to the design of the sliding surface. Therefore, a more effective sliding surface can be designed. To solve a Hamilton-Jacobi-Bellman (HJB) equation for optimization, the general nonlinear sliding surface is expanded by a series of polynomial functions, and then the Galerkin approximation is applied.\textsuperscript{7} To verify the effectiveness of the proposed method, numerical simulation is performed.

2. SMC with Optimized Sliding Surface

SMC is a nonlinear control method that alters the dynamics of a nonlinear system by application of high-frequency switching control. It drives the state trajectory onto the sliding or switching function, a specified and user-chosen surface in the state space, and maintains the system’s state trajectory on this surface for all subsequent time.

Since the system dynamics are theoretically restricted to this surface, once the state trajectory reaches it, the controlled system’s behavior is determined by the sliding surface. The behavior of the system, if the state is kept on the sliding surfaces, has strong robustness against model uncertainties that satisfy a matching condition.\textsuperscript{8} Therefore, a proper design of the sliding surface is crucial to the performance of the control system. However, in most previous literature on SMC laws, the sliding surfaces have been chosen arbitrarily or as simple linear functions. If a systematic method for selecting the sliding surfaces can be found, a better closed-loop performance can be obtained.

2.1. Aircraft equations of motion

The aircraft longitudinal equation of motion is considered. If neglecting the effect of gravity, the following two degree-of-freedom equations of motion are obtained.\textsuperscript{9,10}

\begin{equation}
\dot{\alpha} = Z_{\alpha\alpha} + Z_{\alpha\delta} + (Z_{q\alpha} + 1)q + Z_{\delta\delta}\delta_e
\end{equation}

\begin{equation}
\dot{q} = M_{\alpha\alpha} + M_{\alpha\delta} + M_{q\delta} + M_{\delta\delta}\delta_e
\end{equation}

$Z_{\alpha\alpha}$ and $M_{\alpha\alpha}$ are the aerodynamic, moment and force coefficients at zero angle of attack. The coefficients $Z_{\alpha\delta}, Z_{q\alpha}, Z_{\delta\delta}, M_{\alpha\delta}, M_{q\delta}$ and $M_{\delta\delta}$ are the nonlinear functions of the angle of attack. The stability derivative $M_{\alpha}$ is the primary quantitative measure. The sign of $M_{\alpha}$ is positive when the aircraft becomes statically unstable. Figures 1 and 2 give the aerodynamic coefficients $M_{\alpha}$ and $M_{\delta}$ with respect to the angle of attack. Figure 1 shows that with the increase in angle of attack the aircraft becomes statically-unstable in the region of $-1.0^\circ$ to $12.7^\circ$, and regains stability with further increase in the angle of attack.

The direct effect of the elevator deflection on the angle of attack is very small, and therefore $Z_{\delta}$ can be neglected. Then, the following simplified equations of motion can be obtained.

\begin{equation}
\dot{\alpha} = Z_{\alpha\alpha} + Z_{\alpha\delta} + (Z_{q\alpha} + 1)q
\end{equation}

\begin{equation}
\dot{q} = M_{\alpha\alpha} + M_{\alpha\delta} + M_{q\delta} + M_{\delta\delta}\delta_e
\end{equation}

2.2. Controller design procedure

Consider the aircraft longitudinal equation of motion given in Eqs. (3) and (4), and let us define the sliding surface as:

\begin{equation}
S = q + k(\alpha)
\end{equation}

If the initial states are on the sliding surface, the trajectory of the longitudinal motion follows the sliding surface exactly; i.e. the sliding condition $S = 0$ is always satisfied. This gives the relation as $q = -k(\alpha)$.

If we note that the aircraft model is a cascade system, the pitch rate $q$ can be considered as the pseudo-control input for the angle of attack dynamics. Therefore, an optimal sliding surface can be designed for the whole system in Eqs. (3)
and (4) considering the pitch rate \( q \) as an input variable.

Now, let us design a control law \( q = -k(\alpha) \), which minimizes the following performance index.

\[
J = \int_0^\infty (Q\alpha^2 + Rq^2)dt
\]

subject to Eq. (3) where \( Q \) and \( R \) are the weightings.

By applying the optimality conditions, the following relation can be obtained.

\[
q^*(\alpha) = -\frac{1}{2}R^{-1}(Z_q + 1)\frac{\partial V^*}{\partial \alpha}
\]

where \( V^* \) is the solution of the following HJB equation.

\[
HJB(V^*) = \frac{\partial V^*}{\partial \alpha}Z_q \alpha + Q\alpha^2 - \frac{1}{4R}(Z_q + 1)^2\left(\frac{\partial V^*}{\partial \alpha}\right)^2 = 0
\]

Note that the HJB equation is very difficult to solve, especially for nonlinear systems. A numerical algorithm to solve the HJB equation will be presented in the subsequent section.

Once the sliding surface is designed, it is required that the state trajectory remains on the sliding surface when the state reaches the surface. The control that ensures this condition is called the equivalent control. By differentiating Eq. (5) with respect to time, and substituting Eqs. (3) and (4) in the resulting equation, we have

\[
\dot{S} = q + \left(\frac{\partial k}{\partial \alpha}\right)\alpha
\]

\[
= M_\alpha + M_\alpha q + M_\delta \delta
\]

\[
+ \left(\frac{\partial k}{\partial \alpha}\right)\left[Z_\alpha + Z_o \alpha + (Z_q + 1)q\right]
\]

To satisfy \( S = 0 \) on the sliding surface, the equivalent control using the optimized sliding surface can be given by

\[
\delta = -\frac{1}{M_\delta} \left[M_\alpha + M_\alpha q + M_\delta \delta\right]
\]

\[
+ \left(\frac{\partial k}{\partial \alpha}\right)\left[Z_\alpha + Z_o \alpha + (Z_q + 1)q\right]
\]

Once the state trajectory reaches the sliding surface \( S(q, \alpha) = 0 \), the equivalent control guarantees that the trajectory remains in the sliding surface under the ideal condition. Note that \( q = -k(\alpha) \) is a stabilizing control because it has been found by solving the optimal control problem and the state trajectory eventually goes to the trim condition on the sliding surface. Finally, the SMC input is obtained as

\[
u = \delta_c + \delta_s
\]

where \( \delta_c \) is an equilibrium control, and \( \delta_s \) is a switching control expressed as follows:

\[
\delta_s = l \cdot \text{sign}(S), \quad l < 0
\]

where the sign function of Eq. (13) is defined as

\[
\text{sign}(S) = \begin{cases} 
1 & \text{if } S > 0 \\
0 & \text{if } S = 0 \\
-1 & \text{if } S < 0
\end{cases}
\]

In Eq. (12), the use of the sign function generates chattering. The widely used method to reduce the chattering is to replace the sign function by the following saturation function.

\[
\delta_s = l \cdot \text{sat}(S)
\]

where the saturation function of Eq. (13) is defined as

\[
\text{sat}(S) = \begin{cases} 
1 & \text{if } S > \varepsilon \\
-1 & \text{if } S < -\varepsilon \\
\frac{S}{\varepsilon} & \text{if } -\varepsilon < S < \varepsilon
\end{cases}
\]

3. Numerical Algorithm for Solving HJB Equation

In this section, a numerical algorithm is introduced for solving the HJB equation to obtain the approximate closed-loop solutions.

The HJB equation is approximated via two steps. The first step is the Generalized HJB (GHJB) equation. The HJB equation, which is a nonlinear partial differential equation, becomes a sequence of linear partial differential equations by the GHJB. The second step is to use the Galerkin spectral method to approximate the GHJB equation. This algorithm has been developed by Beard et al. and the details can be found in Ref. 6.

Let us consider the HJB equation, Eq. (8), for solving \( V^* \). The GHJB equation is introduced to develop the proposed algorithm.

\[
\text{GHJB}(V^*; q) = \frac{\partial V^*}{\partial \alpha} \left[Z_\alpha \alpha + (Z_q + 1)q\right] + Q\alpha^2 + Rq^2 = 0,
\]

\[
V(0) = 0
\]

By substituting the optimal solution \( V^* \) and \( q^* \) into the GHJB equation, the following HJB equation can be obtained

\[
\text{HJB}(V^*) = \text{GHJB} \left[V^*; -\frac{1}{2}R^{-1}(Z_q + 1)\frac{\partial V^*}{\partial \alpha}\right] = 0
\]

The proposed numerical algorithm requires the initialization and iterative process. The initialization process selects an initial function \( q^{(0)}(\alpha) \) that satisfies the following conditions.

(a) \( q^{(0)}(\alpha) \) is continuous.

(b) \( q^{(0)}(0) = 0 \).

(c) The system \( \dot{q} = M_\alpha \alpha + M_q q + M_\delta \delta \) is Lyapunov stable.

(d) The cost function \( J(\alpha, q^{(0)}) \) is finite for all \( \alpha \).

The iterative process can be summarized as follows. For \( i \geq 0 \), find the performance index \( V^{(i)} \) for \( q^{(i)} \).

\[
V^{(i)}(\alpha; q^{(i)}) = \int_0^\infty \{Q\alpha^2 + R[q^{(i)}]^2\}dt
\]

Update the \( q^{(i+1)}(\alpha) \) function by solving the equation,

\[
\text{GHJB}(V^{(i)}; q^{(i)}) = 0,
\]

as

\[
q^{(i+1)}(\alpha) = -\frac{1}{2}R^{-1}(Z_q + 1)\frac{\partial V^{(i)}}{\partial \alpha}
\]

The condition \( V^*(\alpha) \leq V^{(i+1)}(\alpha) \leq V^{(i)}(\alpha) \) holds for each \( i \geq 0 \) and the solution does not get stuck in local minima.
Therefore, if the unique optimal control \( q^*(\alpha) \) exists, \( q^{(i)}(\alpha) \) converges to the optimal control uniformly.

To apply this method numerically, the Galerkin method is used to approximate \( V^{(i)} \) as

\[
V^{(i)}_N(\alpha) = \sum_{j=1}^{N} c^{(i)}_j \phi_j(\alpha) \tag{19}
\]

where \( \{ \phi_j : \Omega \rightarrow \mathbb{R} \} \) spans the set \( \{ f \in C^4 : f(0) = 0 \} \). By solving the GHJB equation, we have

\[
q^{(i+1)}_N(\alpha) = -\frac{1}{2} R^{-1}(Z_q + 1) \frac{\partial V^{(i)}_N}{\partial \alpha}(\alpha) \tag{20}
\]

It can be proven that the approximate control is guaranteed to stabilize the system if the algorithm is truncated as a finite, but large enough \( i \) and \( N \).

4. Numerical Simulation

4.1. Trim point

The aircraft trim point can be found from Eqs. (3) and (4) by making \( \alpha \) and \( \dot{\alpha} \) equal to 0 as

\[
Z_{\alpha_0} + Z_q \alpha + (Z_q + 1)q = 0 \tag{21}
\]

\[
M_{\alpha_0} + M_{\dot{\alpha} \alpha} + M_{\dot{q} \dot{q}} + M_{\delta e} \delta_e = 0 \tag{22}
\]

where \( Z_{\alpha_0} \) and \( M_{\alpha_0} \) are 0.0153 and 0.0316, respectively. The trim point \( \alpha_{trim} = 0.5872 \) deg is obtained with the initial values \( [\alpha \dot{q}]_{trim}^T = [10 \text{ deg} \ 5 \text{ deg/s}]^T \).

4.2. Optimized sliding surface

To design the optimal sliding surfaces while minimizing the pitch rate and angle of attack at the same rate, weightings \( Q = R = 0.5 \) are used in Eq. (6). To apply the numerical algorithm for solving the HJB equations, even symmetric polynomials up to the 10th degree are taken as a basis function, as follows:

\[
\{ \phi_j \} = \{ \alpha^2, \alpha^4, \alpha^6, \alpha^8, \alpha^{10} \} \tag{23}
\]

Using the initial input \( u_0 = 0 \) and the interval \( \Omega = [-5 \ 10] \) the optimal solution is obtained as follows:

\[
q^* = -1.2479\alpha + 0.0083\alpha^3 + \text{ignorable terms} \tag{24}
\]

Using Eqs. (5) and (10), the following optimized sliding surface and the optimal sliding control input are finally obtained:

\[
S = q + 1.2479\alpha - 0.0083\alpha^3 \tag{25}
\]

\[
u = -\frac{1}{M_{\delta e}} \left[ M_{\alpha_0} + M_{\alpha \alpha} + M_{\dot{q} \dot{q}} \right] + \frac{1}{c M_{\delta e}} \left[ Z_{\alpha_0} + Z_q \alpha + (Z_q + 1)q \right] + \delta_e \tag{26}
\]

4.3. Linear sliding surface

For comparison to the optimized sliding surface, a linear sliding surface is considered. Simply, it can be designed as follows:

\[
S = cq + \alpha \tag{27}
\]

where \( c \) is the coefficient of linear sliding surface.

In the same way as the optimized sliding surface, by differentiating Eq. (27) with respect to time, and substituting Eqs. (3) and (4) in the resulting equation, we have

\[
S = c \dot{q} + \alpha
\]

\[
= c \left[ M_{\alpha_0} + M_{\alpha \alpha} + M_{\dot{q} \dot{q}} + M_{\delta e} \delta_e \right]
\]

\[
+ \left[ Z_{\alpha_0} + Z_q \alpha + (Z_q + 1)q \right]
\]

To satisfy \( S = 0 \) on the sliding surface, the equivalent control using the linear sliding surface can be given by

\[
\dot{u} = -\frac{1}{M_{\delta e}} \left[ M_{\alpha_0} + M_{\alpha \alpha} + M_{\dot{q} \dot{q}} \right] + \frac{1}{c M_{\delta e}} \left[ Z_{\alpha_0} + Z_q \alpha + (Z_q + 1)q \right] + \delta_e \tag{29}
\]

4.4. Simulation result and comparison

Figures 3 to 7 show the comparison result between the linear and the optimal sliding surface. In this simulation, the gain \( l \) in Eq. (13), threshold value \( \varepsilon \) in Eq. (14) and coefficient of linear sliding surface \( c \) in Eq. (27) are selected as 1, 0.1 and 1, respectively.

The linear and the optimal sliding mode controllers lead the aircraft longitudinal motion to the trim condition. That is, the angle of attack is converged to the trim condition.
The control input using the optimized sliding surface is greater than using the linear sliding surface in Fig. 5, because the control input is not included in the performance index. However, taking the elevator deflection range \( \pm 20^\circ \) into consideration, the difference between the linear and the optimized sliding surface would not influence the energy much.

Figure 6 shows the performance index for the two cases. It is obvious that the performance of the optimal sliding surface controller is better than that of a linear sliding surface controller.

Figure 7 shows the state trajectories on the \( \alpha-q \) phase plane. It clearly shows the location of the switching.

5. Conclusion

An optimal sliding mode controller for aircraft longitudinal motion was proposed. To obtain an effective sliding surface, the optimal problem was formulated and the HJB equation was investigated. The HJB equation was solved through a numerical method using the Galerkin spectral approximation, which gave a closed-loop solution to the HJB equation. The equilibrium control and the switching logic were constructed using the optimal switching surface. The simulation was performed to demonstrate the effectiveness of the proposed controller. The result showed that the performance of the proposed controller with the optimized sliding surface was better than that of the controller with an arbitrarily chosen sliding surface.

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References