Electric Current Analysis of CFRP using Perfect Fluid Potential Flow*

By Akira Todoroki
Tokyo Institute of Technology, Tokyo, Japan
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A new analytical method to calculate the electric current density between two probes in carbon-fiber-reinforced plastic (CFRP) is presented. Unidirectional CFRP has strongly orthotropic electric conductance. Even when electric current is applied to a CFRP plate using two probes on a single surface, the electric current density is not uniform along the cross-section. The electric current is concentrated near the surface where an electric current is applied. Although it is important to know the electric current density in the CFRP plate for the analysis of lightning effects, the density is difficult to calculate using a three-dimensional finite element model. In the present study, the orthotropic coordinate is transformed into a uniform coordinate. Laplace’s equation is solved using the potential theory for a perfect fluid. Equations solved employing an infinite-body approximation are verified with a finite element model. As a result, the new analysis method is demonstrated to be efficient for unidirectional CFRP. The limitations of the method are also discussed.

Key Words: Composite Materials, Potential Flow, Analysis, Electric Current, Orthotropic Electric Conductivity

1. Introduction

Carbon-fiber-reinforced plastic (CFRP) composite structures have been used for many aerospace components. The Boeing 787 adopts CFRP components for primary structures such as the main wings and fuselage. For these CFRP components, electric current flow resulting from a lightning strike is an important issue.

Laminated CFRP structures comprise highly conductive carbon fibers and insulator resin matrix. The combination provides strongly orthotropic electrical conductance of the laminated CFRP structures.1)

Electric conductance in the fiber direction of the CFRP is linearly proportional to the fiber volume fraction. On the other hand, the electric conductance in the transverse direction and the conductance in the thickness direction depend on the contact between fibers and plies. This results in small electric conductance in the transverse and thickness directions. For example, laminated CFRP composites of 60% fiber volume fraction, which are widely adopted for aerospace components, have only 1% electric conductance in the thickness direction compared with that in the fiber direction.

High-toughness CFRP, which has typically been adopted for recent aerospace components to prevent delamination cracking, has resin-rich layers. A resin-rich layer provides quite small electric conductance in the thickness direction.2)

When a laminated CFRP structure is struck by lightning, it is difficult to obtain the exact electric current flow path without considering the effect of the strongly orthotropic electric conductance. Usually, copper mesh or copper foil is attached to the CFRP structural surface to prevent lightning damage. Most of the current flows in the anti-lightning copper system, but some electric current leaks from the copper system. The leaked electric current flows within the CFRP structure. The leaked electric current is similar to steady electric current flow. To analyze the effect of the leaking electric current, a calculation method that considers the effect of the strongly orthotropic electric conductance is indispensable.

The finite element method (FEM) is one analysis tool that considers the effect of anisotropy. It is, however, difficult to analyze the electric current using a three-dimensional (3D) cubic mesh for a huge laminated CFRP structure: a large number of mesh divisions are required in the thickness direction. An adequate approximation analysis has not been proposed.

An analytical method for electric current flow in strongly orthotropic material is proposed here as a first step to analyze electric current flow in laminated CFRPs. In the present study, potential flow analyses of two-dimensional (2D) and 3D perfect fluids of infinite plates are adopted as an approximation method. Here, the infinite-body approximation method employing potential flow analysis of a perfect fluid is applied to a thick unidirectional CFRP plate; the plate has two sources (electric current sources) and a sink (electrical ground) on the surface. The results are compared with the FEM results. The limit of the effectiveness of the infinite-body approximation analysis is discussed here. The infinite-body approximation method is also applied to a cross-ply CFRP laminate, and the results are compared with FEM results.

2. Analytical Method

2.1. Theory of electric current for orthotropic material

Unidirectional CFRP has orthotropic electric conductance. The theory of orthotropic electric conductance is sim-
ilar to the theory of heat conductance. Let us consider that the $x$-direction is the fiber direction in a unidirectional CFRP ply, the $y$-direction is the transverse direction and the $z$-direction is the thickness direction. The electric densities in the $x$, $y$ and $z$ directions ($i_x$, $i_y$, $i_z$) can be obtained using the electric potential $\phi$:

$$
    i_x = -\sigma_x \frac{\partial \phi}{\partial x}, \quad i_y = -\sigma_y \frac{\partial \phi}{\partial y}, \quad i_z = -\sigma_z \frac{\partial \phi}{\partial z},
$$

(1)

where $\sigma_x$, $\sigma_y$, and $\sigma_z$ are the electric conductance of the unidirectional CFRP in the fiber, transverse and thickness directions, respectively. For the laminated CFRP, the electric conductivity in the thickness direction includes that of the resin-rich layer. To simplify the problem, it is evaluated as a uniform material here. When there is no electric current source, the continuity of electric current is

$$
    \sigma_x \frac{\partial^2 \phi}{\partial x^2} + \sigma_y \frac{\partial^2 \phi}{\partial y^2} + \sigma_z \frac{\partial^2 \phi}{\partial z^2} = 0.
$$

(2)

Substituting Eq. (1) into Eq. (2) gives

$$
    \frac{\partial}{\partial x}(i_x) + \frac{\partial}{\partial y}(i_y) + \frac{\partial}{\partial z}(i_z) = 0.
$$

(3)

As shown, Eq. (3) is not a Laplacian equation, but it can be transformed by magnifying the coordinate axes. To simplify the problem, the $x$ and $y$ coordinates (2D problem) are considered here. The $x$–$y$ coordinates are transformed into $\xi$–$\eta$ coordinates according to

$$
    \xi = \frac{x}{\sqrt{\sigma_x}}, \quad \eta = \frac{y}{\sqrt{\sigma_y}}.
$$

(4)

The coordinate conversion transforms Eq. (3) into a Laplacian equation:

$$
    \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0.
$$

(5)

As the direction of the $\xi$–$\eta$ coordinates is the same as that of the $x$–$y$ coordinates, the electric current density of $i_\xi$ and $i_\eta$ are

$$
    i_x = \frac{A}{d\eta dz} = \frac{A}{\sqrt{\sigma_x} \, d\eta \, dz} = \frac{i_\xi}{\sqrt{\sigma_x}},
$$

$$
    i_y = \frac{A}{dx \, dz} = \frac{A}{\sqrt{\sigma_y} \, dx \, dz} = \frac{i_\eta}{\sqrt{\sigma_y}},
$$

(6)

$$
    \therefore i_x = \sqrt{\sigma_x} i_\xi = -\sqrt{\sigma_x} \sigma_y \frac{\partial \phi}{\partial \xi} = -\sqrt{\sigma_x} \sigma_y \frac{\partial \phi}{\partial \xi},
$$

$$
    i_y = \sqrt{\sigma_y} i_\eta = -\sqrt{\sigma_y} \sigma_x \frac{\partial \phi}{\partial \eta} = -\sqrt{\sigma_y} \sigma_x \frac{\partial \phi}{\partial \eta},
$$

(7)

where $A$ is the sum of electric charge that passes through the small target cross-section.

The transformed Eqs. (5) and (7) indicate that the electric current flow is similar to the potential flow of a perfect fluid without vortices.

### 2.2. Source in an infinite plate

Let us consider that the electrode is a dot without dimension and the ground of 0 V is located at infinite distance. The dot electric current source is placed at the origin of the coordinate system of the orthotropic plate; the $x$-coordinate is the fiber direction and the $y$-coordinate is the transverse direction.

The $x$–$y$ coordinate system can be transformed to the $\xi$–$\eta$ coordinate system using the coordinate conversion. In the $\xi$–$\eta$ coordinate system, the isotropic Laplacian equation indicates that the electric current emanates equally in all directions.

This is similar to flow from a source of perfect fluid without vortices; the coefficients of Eq. (7) are one for perfect-fluid flow. Therefore, the velocity potential of a source flow of perfect fluid can be applied to the present problem of the electric current flow with a minor correction. The velocity potential of flow from a source of perfect fluid without vortices is

$$
    \phi = q \log r, \quad r = \sqrt{\xi^2 + \eta^2},
$$

(8)

where $q$ is a coefficient, and a polar coordinate of $r$–$\theta$ is adopted. $\theta$ is the rotation angle from the $x$-axis; counterclockwise rotation is positive. From Eqs. (7) and (8), the electric current densities of $i_\xi$ and $i_\eta$ can be obtained:

$$
    i_\xi = -\sqrt{\sigma_x \sigma_y} \frac{\partial \phi}{\partial \xi} = -q \sqrt{\sigma_x \sigma_y} \frac{\xi}{r^2},
$$

$$
    i_\eta = -\sqrt{\sigma_x \sigma_y} \frac{\partial \phi}{\partial \eta} = -q \sqrt{\sigma_x \sigma_y} \frac{\eta}{r^2}.
$$

(9)

When input electric current per unit thickness from the source is $lA/m$, the sum of electric current in the radial direction ($i_r$) around the source per unit thickness is $I$:

$$
    I = \int_0^{2\pi} i_r d\theta.
$$

(10)

$i_r$ at an angle $\theta$ can be calculated using the current densities $i_\xi$ and $i_\eta$:

$$
    i_r = i_\xi \cos \theta + i_\eta \sin \theta.
$$

(11)

Substituting Eqs. (10) and (12) into Eq. (11) gives

$$
    I = -q \sqrt{\sigma_x \sigma_y} \int_0^{2\pi} \left( \frac{\xi}{r^2} \cos \theta + \frac{\eta}{r^2} \sin \theta \right) r d\theta
$$

$$
    = -q \sqrt{\sigma_x \sigma_y} \int_0^{2\pi} \left( \cos^2 \theta + \sin^2 \theta \right) d\theta
$$

$$
    = -2\pi q \sqrt{\sigma_x \sigma_y},
$$

(13)

$$
    \therefore q = \frac{I}{2\pi \sqrt{\sigma_x \sigma_y}}.
$$

(14)

As a result, $i_\xi$ and $i_\eta$ are

$$
    i_\xi = \frac{I}{2\pi \sqrt{\sigma_x \sigma_y}} \frac{x}{\sigma_x + \frac{y^2}{\sigma_y}},
$$

$$
    i_\eta = \frac{I}{2\pi \sqrt{\sigma_x \sigma_y}} \frac{y}{\sigma_x + \frac{x^2}{\sigma_y}}.
$$

(15)
When there is a pair of a source (electric current source) and sink (zero voltage; the ground electrode) in an infinite plate, the superposition of Eq. (15) provides the exact electric current flow. Let us consider a source located at \((-a, 0)\) and sink located at \((a, 0)\), where \(a > 0\). The flow around the sink can be obtained as the negative value of Eq. (15):

\[
i_s = \frac{I}{2\pi \sqrt{\sigma_x \sigma_y}} \left\{ \frac{x + a}{(x + a)^2 + y^2} - \frac{x - a}{(x - a)^2 + y^2} \right\}
\]

\[
i_z = \frac{I}{2\pi \sqrt{\sigma_x \sigma_y}} \left\{ \frac{y}{(x + a)^2 + y^2} - \frac{y}{(x - a)^2 + y^2} \right\}
\]

When there is a pair of a source (electric current \(I\)) and sink (zero voltage; electrical ground) on the edge of a semi-infinite plate, half the electric current in Eq. (16) flows in the semi-infinite plate. This means that the solution for the semi-infinite plate with input current \(I\) can be obtained substituting \(2I\) for \(I\) in Eq. (16).

### 2.3. Source in an infinite body

As in the case of the source and sink in an infinite plate (section 2.2), the case of a pair of a source and sink in an infinite body (3D analysis) can be solved using a similar conversion of the \(x-y-z\) coordinates. The orthotropic \(x-y-z\) coordinates are transformed to isotropic \(\xi-\eta-\zeta\) coordinates. In the analysis, the velocity potential of a perfect fluid without vortices can be adopted. When there is a source of strength \(q\) in an infinite uniform body, the velocity potential \(\phi\) can be obtained as:

\[
\phi = \frac{q}{r},
\]

\[
r = \sqrt{\xi^2 + \eta^2 + \zeta^2}.
\]

Similar to the 2D case in section 2.2, the coordinate conversion is applied to the \(x-y-z\) coordinates:

\[
\xi = \frac{x}{\sqrt{\sigma_x}}, \quad \eta = \frac{y}{\sqrt{\sigma_y}}, \quad \zeta = \frac{z}{\sqrt{\sigma_z}}.
\]

Similar to Eq. (10), electric current densities \(i_\xi, i_\eta,\) and \(i_\zeta\) are

\[
i_\xi = -q \sqrt{\sigma_x \sigma_y \sigma_z} \frac{\xi}{r^3},
\]

\[
i_\eta = -q \sqrt{\sigma_x \sigma_y \sigma_z} \frac{\eta}{r^3},
\]

\[
i_\zeta = -q \sqrt{\sigma_x \sigma_y \sigma_z} \frac{\zeta}{r^3}.
\]

Let us consider the spherical coordinates \((r-\theta-\varphi)\) with the source of the electric current placed at the origin. In spherical coordinates, \(\theta\) is the angle in the \(\xi-\eta\) plane, and \(\varphi\) is the angle from the \(\xi-\eta\) plane. When the direction cosine of the plane in the \(r\)-direction is given by \((n_1, n_2, n_3)\), the electric current density \(i_r\) in the \(r\)-direction can be obtained as

\[
i_r = i_\xi n_1 + i_\eta n_2 + i_\zeta n_3.
\]

Similar to Eq. (13) for the 2D case, the integral of the electric current density in all directions is \(I\):

\[
I = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} i_r \cos \theta \, d\theta \, d\varphi
\]

\[
= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left( i_\xi n_1 + i_\eta n_2 + i_\zeta n_3 \right) \cos \theta \, d\theta \, d\varphi
\]

\[
= -q \sqrt{\sigma_x \sigma_y \sigma_z} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left( n_1^2 + n_2^2 + n_3^2 \right) \cos \theta \, d\theta \, d\varphi
\]

\[
= -4\pi q \sqrt{\sigma_x \sigma_y \sigma_z}.
\]

Using Eq. (23), the electric current densities \(i_x, i_y,\) and \(i_z\) are obtained:

\[
i_x = \frac{I}{4\pi \sqrt{\sigma_x \sigma_y \sigma_z}} \left( \frac{x}{\sigma_x} \right)^{3/2},
\]

\[
i_y = \frac{I}{4\pi \sqrt{\sigma_x \sigma_y \sigma_z}} \left( \frac{y}{\sigma_y} \right)^{3/2},
\]

\[
i_z = \frac{I}{4\pi \sqrt{\sigma_x \sigma_y \sigma_z}} \left( \frac{z}{\sigma_z} \right)^{3/2}.
\]

### 2.4. Electric current in an infinite body

Using Eq. (24), the flow of a pair of a source and sink in an infinite body can be obtained. Let us consider the case that there is a source of electric current at \((-a, 0, 0)\) and sink (zero voltage) at \((a, 0, 0)\) in \(x-y-z\) coordinates. In 3D analysis, the analytical result for a semi-infinite body with a surface is more important than that for an infinite body because the electric current is usually applied at a surface. Therefore, the result for the semi-infinite body is obtained here by replacing \(I\) with \(2I\). The result obtained is given as Eq. (25). This is the exact analytical result of the electric current between two electrodes of a unidirectional thick CFRP plate. The result is
2.5. Finite-thickness plate

The above analyses of the electric current density are all for an infinite or semi-infinite plate or body. Practical aircraft composite structures have finite thickness. We can obtain exact results employing a well-known reflected-image analysis method with perfect-fluid potential analysis. Usually, a laminated CFRP has quite low electric conductance in the thickness direction because of the resin-rich layer between plies. The low conductance in the thickness direction results in electric current near the surface layer. In the case of practical laminated CFRP structures, the thickness is a few millimeters to 20 mm or so. For these thick laminated CFRPs, the electric current does not flow deeply in the thickness direction.5) This means that the analyses of an infinite or semi-infinite plate or body can be applied to these thick laminated CFRPs without the correction of reflected-image analysis.

2.6. Application to laminated CFRPs

Usually, laminated CFRP structures are made by stacking CFRP plies at several fiber angles. A normal laminated CFRP has not only 0° and 90° plies but also plies at angles such as ±45°. These angled plies make the analysis of electric current flow difficult because they are not orthotropic materials but perfectly anisotropic materials with respect to the 0° ply. The formula for the angled plies, therefore, cannot be transformed into a Laplacian equation. The problem makes it difficult to solve the electric current of normal laminated CFRPs including angled plies. In the present study, a cross-ply laminate comprised only of 0° plies and 90° plies is used as a simple example to avoid this difficulty.

Electric current usually flows in the direction that has highest electric conductance. Thus, let us consider the case such as in the previous unidirectional CFRP. The electric current density in the z-direction (i_z) can be calculated as for the previous unidirectional CFRP. The electric current density in the ξ-direction, however, is not easily calculated because the electric conductance in the ξ-direction depends on the distance from the surface in the ξ-direction.

The electric density in the x-direction (i_x) can be calculated from the partial differentiation of ϕ with respect to ξ. The result is the same as in Eq. (10). The strength of the source q is, however, dependent on radius r because of the dependence of σ_x on the z-location. Thus, the exact solution cannot be obtained here.

We assume that the partial differential of the potential does not change even for the cross-ply laminated CFRP from that of the unidirectional CFRP. This gives us

$$\frac{\partial \phi}{\partial \xi} = -\frac{I}{\pi \sigma_y \sqrt{\sigma_x \sigma_z}} \frac{x}{\sqrt{x^2 + z^2 \sigma_z}}.$$

(28)

The difference in the electric current density between the 0° ply and 90° ply is due to the difference in the electric conductance in the x-direction between the 0° ply and 90° ply. This gives us rough estimations of the electric current density in the x-direction. The electric current density in the 0° ply can be obtained by multiplying the electric conductance of the 0° ply with the partial differential in Eq. (28), and the electric current density in the 90° ply can be obtained by multiplying the electric current conductance of the 90° ply with the partial differential in Eq. (28).

3. Comparison with FEM Analysis

As mentioned previously, FEM analyses of finite-thickness unidirectional CFRP are carried out in this work and the FEM results are compared with the analytical results. The limitation of the approximation method is also discussed.

Four types of FEM models are used to investigate the effectiveness of the infinite-body approximation. Model A is a 2D plate model that has two electrodes on a thin unidirectional CFRP plate surface: the two electrodes are used as an electric current source and ground (0 V) electrode. This model is used to investigate the effect of orthotropic conductance on the electric current density.

Model B is a 2D cross-section model for investigating the effect of the thickness; the fiber direction and the thickness direction are the two selected axes. Model C is a 3D thick CFRP plate model for investigating the effectiveness of the 3D analysis.
Model D is a 2D cross-section model of a cross-ply laminate for investigating the effect of a stack of plies. For these analyses, the commercially available FEM code ANSYS Ver. 11 is adopted here.

3.1. Electric current density of 2D square CFRP

As shown in Fig. 1, a square with lengths of 40 mm (and thickness of 1 mm) is prepared with $x$-$y$ rectangular coordinates; the origin is at the center of the square. The fiber direction of the square is the $x$-direction. Three ratios ($\sigma_x/\sigma_x$) of the electric conductance are selected: $\sigma_x/\sigma_x = 1, 0.1$ and 0.01. For the FEM calculations, square elements with lengths of 0.5 mm are adopted; the total number of nodes is 6,561 and the total number of elements is 6,400. Direct current of 1 A/mm is applied to the dot at ($0, 0$) and the dot at ($40, 0$) is used as the ground; the voltage is set to 0 V.

Figure 2 compares the analytical results of the electric current density of the isotropic material ($\sigma_x/\sigma_x = 1$) with the FEM results. The triangular symbols indicate the FEM results, and the solid curve indicates the analytical results obtained using the infinite-body approximation. Although the plate is wide (40 mm) compared with the spacing between electrodes, the results show that the analysis with infinite-body approximation has large errors when the electric conductance ratio is isotropic ($\sigma_x/\sigma_x = 1$).

Figure 3 shows the results of the electric current density $i_y$ of the weakly orthotropic CFRP ($\sigma_x/\sigma_x = 0.1$). The triangular symbols indicate the FEM results and the solid curve indicates the analytical result obtained using the infinite-body approximation. The comparison shows that the analytical results agree well with the FEM results. The electric current density decreases with an increase in the distance from the $x$-axis. At $y=15$ mm, the electric current density is almost 0 A/m. This indicates that the width of the CFRP plate is large enough for an approximation of infinite plates. This enables us to use the infinite-body approximation for the actual finite CFRP plate. Even for finite plates of weakly orthotropic material, the infinite-body approximation is shown to be effective.

Figure 4 compares the electric current density $i_y$ between the analytical results and the FEM results of the strongly orthotropic CFRP ($\sigma_x/\sigma_x = 0.01$). The triangular symbols indicate the FEM results and the solid curve indicates the analytical result. The comparison shows that the analysis of infinite plates agrees very well with the FEM results. The electric current density at $y = 5$ mm is 0 A/mm$^2$, indicating that the infinite-body approximation agrees well with the actual FEM results. As shown in this figure, the infinite-body approximation is in good agreement with the finite CFRP plate because of the orthotropic electric conductance.

3.2. Effect of thickness (2D analyses)

To investigate the effect of the thickness of a thin CFRP plate, a cross-section of a CFRP beam of 40 mm length that has a couple of electrodes with 8 mm spacing is analyzed as shown in Fig. 5. Four thicknesses of plates are analyzed here: $t = 1, 2, 3$ and 4 mm. For the thin CFRP beams with thickness of 1 and 2 mm, square elements with length of 0.1 mm are used in the FEM analyses. For the CFRP plates with thickness of 3 and 4 mm, square elements with length of 0.2 mm are used. The total number of nodes of a beam
specimen of thickness \( t = 1 \) mm is 441, and the total number of elements is 4,000. For the beam specimen with thickness of 2 mm, the total number of nodes is 8,421 and the total number of elements is 8,000. For the beam specimen with thickness of 3 mm, the total number of nodes is 3,216 and the total number of elements is 300. For the beam specimen with thickness of 4 mm, the total number of nodes is 4,221 and the total number of elements is 4,000. In the present analysis, the conductance in the fiber direction \( \sigma_z \) at \( x = 0 \) is completely uniform. The results agree with the FEM results for the results of beam-type CFRP model B (\( t = 1 \) mm).

The present study deals with general orthotropic CFRP for which \( \sigma_z/\sigma_y = 0.01 \) as presented in Ref. 1. The electric current density \( i_z \) at \( x = 0 \) is completely uniform. The results agree with the FEM results.

Figure 7 shows the results for the beam with \( t = 2 \) mm. The analytical results agree quite well with the FEM results for \( t = 3 \) mm. Figure 8 shows the results for \( t = 4 \) mm; the thickness of \( t = 4 \) mm is half the length of the spacing \( (a) \) of electrodes. In this figure, the analytical results agree with the FEM results throughout the entire region. The thickness seems sufficient for the infinite-body approximation.

The minimum thickness limit for the infinite-body approximation depends on the spacing \( (2a) \) of the electrodes. The infinite-body approximation can be used to obtain the limitation of the infinite-body approximation comparing with the FEM results. In Eq. (16), the input electric current \( I \) is replaced with \( 2I \) and the \( y \)-coordinate is replaced by the \( z \)-coordinate. This gives the electric current density of beam-type specimens. Since the line \( x = 0 \) is midway between the source and sink, the electric current density \( i_z \) on the line \( x = 0 \) is adopted here as the reference electric current. The electric current density \( i_z \) at \( x = 0 \) is

\[
i_z = \frac{2Ia\lambda}{\pi(z^2 + (a\lambda)^2)}.
\]  

(29)

Here, \( \lambda \) is the square root of the ratio of the electric conductance:

\[
\lambda = \sqrt{\frac{\sigma_z}{\sigma_y}}.
\]

(30)

The integral \( \delta \) of Eq. (29) with respect to \( z \) from 0 to \( z \) is equal to total electric current that flows through the cross-section from \( z = 0 \) to \( z = z' \).
When the integral is divided by the total electric current \( I \) that is input in the beam specimen at the source, the value \( \delta_t \) becomes the ratio of the electric current that flows in the cross-section from \( z = 0 \) to \( z = z_0 \):

\[
\delta_t = \frac{2}{\pi} \tan^{-1} \left( \frac{z}{aL} \right). \tag{31}
\]

Since the integral value of Eq. (31) from \( z = 0 \) to \( z = \infty \) is 1, a decrease from 1 is a cut-off value, and it is the error in the analysis of the infinite-body approximation. When \( \lambda \) is 0.1, \( \delta_t = 0.87 \) for the beam specimen with \( t = 2 \) mm, and \( \delta_t = 0.91 \) for the beam specimen with \( t = 3 \) mm. This implies that the infinite-body approximation is effective when \( \delta_t \) exceeds 0.9. To simplify the effective limitation of the infinite-body approximation, the calculation of Eq. (29) with \( \delta = 0.9 \) gives the limitation value as

\[
\frac{I}{aL} \geq 6.3. \tag{33}
\]

### 3.3. Electric current density of a 3D body

Figure 10 shows the 3D unidirectional square CFRP plate model C with length of 40 mm and thickness of 5 mm. The origin of the orthogonal \( x \)-\( y \)-\( z \) coordinates is placed in the middle of the surface of the square CFRP plate as shown in Fig. 10. The spacing of the electrodes is 8 mm and both electrodes are on the \( x \)-axis. The electric conductance of the orthotropic CFRP plate is \( \sigma_x = 1 \) A/mm², \( \sigma_y = 0.1 \) A/mm² and \( \sigma_z = 0.01 \) A/mm². Although the conductance values differ from the actual conductance of the CFRP, these values are adopted because only the ratios between the conductance values are important in the analysis.

Figure 11 shows the distribution of \( i_t \) in the thickness direction at \( x = 0 \) and \( y = 0 \). The abscissa is the distance from the surface (\( z \)) and the ordinate is the electric current density in the \( x \)-direction. The triangular symbols indicate the FEM results, and the solid curve indicates the analytical results of the infinite-body approximation. The figure shows that the infinite-body approximation gives excellent agreement with the FEM results.

Reference 2) shows that the highly toughened CFRP has a resin-rich layer, which is responsible for a very low \( \lambda \) value of approximately of \( 10^{-4} \). When the plate thickness of the CFRP is 10 mm, Eq. (33) gives the limit spacing of electrodes of \( 2a = 0.3 \) m. If users accept the error of the same order as in Fig. 7, the limit spacing expands to \( 2a = 0.4 \) m. Practical CFRP structures usually have rivets in a square block with side length of 1 m. This means that the infinite-body approximation may give a good approximation for actual CFRP structures.

### 3.4. Electric current density of cross-ply laminates

In this section, the cross-section of a beam similar to that in Fig. 5 is adopted for analysis of the cross-ply laminate (model D). The thickness of the CFRP is 6.4 mm. The surface ply is \( 0^\circ \) ply of 0.8 mm thickness; the second ply is a \( 90^\circ \) ply of 0.8 mm thickness; the third ply is a \( 0^\circ \) ply of 1.4 mm thickness; the fourth ply is a \( 90^\circ \) ply of 0.8 mm thickness; and the fifth ply is a \( 0^\circ \) ply of 0.8 mm thickness. The total number of nodes is 6,633, and the total number of elements is 6,400. The side length of the square element is 0.2 mm.

Figure 12 shows the results of the analysis. The ordinate is the voltage in the \( x \)-direction and the abscissa is the distance from the surface. The triangular symbols with a solid curve indicate the FEM results. As shown in this figure, the results agree well with the infinite-body approximation. Although the curve for the second ply appears to decrease linearly compared with the other plies, the agreement means the hypothesis used in section 2.6 is rational.

For analysis of the infinite-body approximation, the electric current density analysis of a plate with 6.4 mm thickness

![Fig. 10. Unidirectional 3D CFRP of model C.](image)

![Fig. 11. Comparison of \( i_t \) of 3D plate model C \((x = 0, y = 0)\).](image)

![Fig. 12. FEM results of the voltage distribution of model D (cross-ply laminate).](image)
was performed. The electric current density was then divided by the conductance in the fiber direction ($G_2$) and the voltage distribution was obtained. The voltage of each ply multiplied by the conductance in the $x$-direction gives the electric current density of each ply. The results are shown in Fig. 13.

In Fig. 13, the ordinate is the electric current density in the $x$-direction at $x = 0$ and the abscissa is the distance from the surface ($z$). This figure shows that the analysis is only an approximate estimation. When the electric current density in the second ply (90° ply) is magnified as shown in Fig. 14, it is seen that the estimation is not good.

Figure 14 shows the estimations are half the FEM results. This implies that the infinite-body approximation is not exact although it gives a rough estimation for the cross-ply laminate. The detail of the approximation must be investigated in future work. When angled plies such as a 45° ply or −45° ply are used for the thick CFRP laminate, the electric conductance equation is not a Laplacian equation. This case requires further investigation and is also a focus of our future work.

4. Conclusions

Analysis of the potential flow of a perfect fluid was applied to CFRP with orthotropic electric conductance employing coordinate conversion. The infinite-body approximation was compared with the FEM results. The results obtained are as follows.

1) The electric current density of an orthotropic CFRP plate was obtained from the potential flow analysis of a perfect fluid with a coordinate conversion.

2) Comparisons of the analytical results with FEM results showed that the infinite-body approximation is effective for a strongly orthotropic CFRP plate.

3) The analysis provided the limit of the effectiveness of the infinite-body approximation.

4) The infinite-body approximation was extended to a cross-ply laminate by hypothesizing a similar distribution of voltage to that in unidirectional CFRP. The method provided rough estimations but requires further investigation.

References


