The present paper sheds some light on simplified mechanical models in order to discuss the dynamic stability of slender flight bodies subjected to aerodynamic loads. Firstly, we discuss the nature of a suitable simplified mechanical model to demonstrate body divergence and flutter. A method leading to a rational mechanical model is presented. It is shown that a simplified model composed of three bars connected together by two elastic rotational hinges, having four degrees-of-freedom, is preferable to discuss both body divergence and flutter. Secondly, after having a duly simplified mechanical model, effects of mass distribution, stiffness distribution and location of a stabilizer fin are discussed. A perspective of body divergence and flutter of slender flight bodies is demonstrated on the basis of the proposed simplified mechanical model.

Key Words: Flutter, Divergence, Flight-Body, Mechanical Model, Aerodynamic Load

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>$C_{L_0}$</td>
<td>lift curve coefficient</td>
</tr>
<tr>
<td>$k$</td>
<td>rotational spring constant</td>
</tr>
<tr>
<td>$l$</td>
<td>length of rigid bar</td>
</tr>
<tr>
<td>$l_q$</td>
<td>location of stabilizer fin apart from tail</td>
</tr>
<tr>
<td>$L$</td>
<td>concentrated aerodynamic load</td>
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<tr>
<td>$M$</td>
<td>concentrated mass</td>
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<td>$S$</td>
<td>reference cross sectional area of bar</td>
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<tr>
<td>$T$</td>
<td>kinetic energy</td>
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<tr>
<td>$t$</td>
<td>time</td>
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<td>$U$</td>
<td>flight velocity</td>
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<td>$u$</td>
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<td>$x, y$</td>
<td>coordinates</td>
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<tr>
<td>$\alpha$</td>
<td>angle of attack</td>
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<tr>
<td>$\xi$</td>
<td>dimensionless location of a stabilizer fin</td>
</tr>
<tr>
<td>$\phi, \phi$</td>
<td>inclination of a bar to horizontal axis $x$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>ratio of fluid mass to concentrated mass</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of fluid</td>
</tr>
<tr>
<td>$\tau$</td>
<td>dimensionless time</td>
</tr>
<tr>
<td>$\xi, \eta$</td>
<td>dimensionless coordinates</td>
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Subscripts

<table>
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<th>Subscript</th>
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<tr>
<td>$F$</td>
<td>fin</td>
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<tr>
<td>$N$</td>
<td>nose</td>
</tr>
<tr>
<td>$T$</td>
<td>tail</td>
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1. Introduction

Aerospace structures, being extremely lightweighted and thus inevitably very flexible, are prone to dynamic instability under aerodynamic loads. Many comprehensive textbooks on aeroelasticity have been published, say by Bisplinghoff et al., Bisplinghoff and Ashley, and Fung. Recent development of supersonic aircraft and high-performance missiles has presented many new aeroelastic problems. This situation has resulted in a modern course in aeroelasticity.

The present space age has revealed new topics of aeroelasticity of rocket-vehicles. Structural bending and dynamic stability of slender rocket-propelled flying bodies may lead to structural failure and/or trajectory errors. In this context, in place of aircraft wing divergence and flutter, rocket-body divergence and flutter are to be discussed in the present paper.

Ikeda studied the dynamic stability of flight/rocket bodies due to aerodynamic loads by discussing a mechanical model with three degrees-of-freedom (3DOF). The model consists of two rigid bars connected together by a rotational elastic hinge. He concluded that only body divergence could take place. Tomita discussed failure of a small test rocket and concluded that the failure of the rocket could be attributable to rocket-body divergence. Kobayashi conducted experiments of a model space plane and suggested that space planes may be susceptible to body flutter. Bending body flutter of a rocket-propelled missile was reported first by Farrer in 1956. Under these circumstances, questions have been raised regarding whether the 3DOF mechanical model proposed by Ikeda is suitable for discussing rocket body instability, and whether rocket bodies are susceptible only to divergence, but not to flutter. The intended aims of the paper are first to discuss what is a suitable simplified mechanical model of slender rocket bodies, and secondly to grasp a perspective of body divergence and flutter of slender flight bodies on the basis of a proposed duly simplified model.
2. Simplified Aerodynamic Loads and Models

2.1. Simplified aerodynamic loads

It is assumed here that a flight body is subjected to aerodynamic loads in steady flight, but not to any other load, say end thrust load. The loads are assumed to be concentrated loads that act on the rocket body at its nose, tail and stabilizer fin. Aerodynamic loads on other portion except at the nose, tail and fin, are negligible, since it is assumed that they are small enough relative to those at the nose, tail, and fin.

The aerodynamic loads can be written as

\[ L_N = \frac{1}{2} \rho U^2 S N, \]
\[ L_T = \frac{1}{2} \rho U^2 S T, \]
\[ L_F = \alpha_F CL F \frac{1}{2} \rho U^2 S. \] (1)

2.2. Simplified model with 3DOF

Figure 1 shows the simplified model with 3DOF. It is assumed that simplified model of a flexible rocket body is accommodated with a stabilizer fin. The model is composed of two rigid and uniform bars and one massless elastic joint (the 3DOF model; two-bar model). Each bar has the same length \( l \). The dimensionless location of the stabilizer fin \( \varepsilon \) is given by

\[ \varepsilon = \frac{l_F}{l}. \] (2)

Concentrated masses \( M_i \) (\( i = 1, 2, 3 \)) are attached to each bar at both its ends. The modeling process of mass distribution is illustrated in Fig. 2.

The mass ratio \( \beta \) is given by

\[ \frac{M_3}{M_1} = \beta, \quad \frac{M_2}{M_1} = 1 + \beta. \] (3)

Inclinations of the bars to the horizontal axis are given, assuming that they are small enough, by

\[ \phi_1 = \frac{y_2 - y_1}{l}, \quad \phi_2 = \frac{y_3 - y_2}{l}, \] (4)

where \( y_i \) (\( i = 1, 2, 3 \)) are the displacements of the concentrated masses on the body.

The angles of attack can be given by

\[ \alpha_N = \phi_2 - \frac{1}{U} \frac{dy_3}{dr}, \quad \alpha_T = \phi_1 - \frac{1}{U} \frac{dy_1}{dr}, \]
\[ \alpha_F = \frac{1}{U} \left( 1 - \epsilon \right) \frac{dy_1}{dr} + \frac{\epsilon}{U} \frac{dy_2}{dr}, \] for \( 0 \leq \epsilon < 1 \)
\[ \alpha_F = \frac{1}{U} \left( 2 - \epsilon \right) \frac{dy_3}{dr} + \left( \epsilon - 1 \right) \frac{dy_3}{dr}, \] for \( 1 \leq \epsilon \leq 2 \) (5)

Note that the effect of the rigid body motions of the models, translation and rotation, on aerodynamic load are taken into account. This means that aerodynamic damping is taken into account in stability analysis.

2.3. Simplified model with 4DOF

Figure 3 shows a simplified model with 4DOF. The model is composed of three rigid bars and two massless elastic joints (the 4DOF model; three-bar model). The first and second joints have restoring moment coefficient \( k_1, k_2 \), respectively. The restoring coefficient ratio is given by

\[ \kappa = \frac{k_2}{k_1}. \] (6)

A stabilizer fin is attached to the bar \( l_F \) apart from the tail. The dimensionless fin location \( \varepsilon \) is given by the same form as Eq. (2).
Concentrated masses \( M_i \) (\( i = 1, 2, 3, 4 \)) are attached to each bar at its both ends. The mass ratio \( \beta_i \) (\( i = 1, 2 \)) is given by
\[
\frac{M_2}{M_1} = \beta_1, \quad \frac{M_3}{M_1} = \beta_2.
\]
A dimensionless form of Eq. (5) is given by
\[
\frac{\beta}{\alpha} \frac{d^2 \eta_2}{dt^2} + \frac{1}{2} \frac{d^2 \eta_3}{dt^2} = \frac{C_{eAe}}{2} \mu u^2 \left( \eta_2 - \eta_1 - \left( \frac{1}{u} \frac{d \eta_1}{dt} + \frac{1}{u} \frac{d \eta_2}{dt} \right) \right),
\]
where \( \eta \) are the displacements of the concentrated masses on the model. The angles of attack can be given by
\[
\alpha_N = \frac{1}{U} \frac{dy_3}{dt}, \quad \alpha_T = \frac{1}{U} \frac{dy_1}{dt}, \quad \alpha_F = \frac{1}{U} \left( \frac{1}{u} \frac{dy_1}{dt} + \frac{1}{u} \frac{dy_3}{dt} \right),
\]
where \( \epsilon = \frac{1}{U} (2 - \epsilon) \frac{dy_2}{dt} + (\epsilon - 1) \frac{dy_3}{dt} \), for
\[
\begin{align*}
0 & \leq \epsilon < 1, \\
1 & \leq \epsilon < 2, \\
2 & \leq \epsilon < 3.
\end{align*}
\]

### 3. Equations of Motion

#### 3.1. Basic equations for the 3DOF model

Lagrangian equations of motion are written in the form
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}_n} \right) - \frac{\partial T}{\partial y_n} + \frac{\partial V}{\partial y_n} = Q_n, \quad (n = 1, 2, 3).
\]
where \( T \) is kinetic energy, \( V \) potential energy, and \( Q_n \) external generalized forces. The kinetic energy of the 3DOF model is expressed by
\[
T = \frac{1}{2} M_1 \dot{y}_1^2 + \frac{1}{2} M_2 \dot{y}_2^2 + \frac{1}{2} M_3 \dot{y}_3^2.
\]
where \( \dot{y}_1, \dot{y}_2, \dot{y}_3 \) are the generalized velocities.

The potential energy of the model is given by
\[
V = \frac{1}{2} k (\phi_2 - \phi_1)^2.
\]

The external generalized forces are given in the form
\[
Q_1 = L_T + (1 - \epsilon)L_F, \quad Q_2 = \epsilon L_F, \quad Q_3 = L_N,
\]
where \( L_T, L_F, L_N \) are the forces for the 3DOF model.

Substitution of Eqs. (1)–(5) and (11)–(13) to Eq. (10) leads to the equations for a small motion of the 3DOF model in the dimensionless forms
\[
\begin{align*}
(1 + \beta) \frac{d^2 \eta_1}{dt^2} + (-2 \eta_1 + 4 \eta_2 - 2 \eta_3) &= \frac{C_{eAe}}{2} \mu u^2 \left( \eta_2 - \eta_1 - \left( \frac{1}{u} \frac{d \eta_1}{dt} + \frac{1}{u} \frac{d \eta_3}{dt} \right) \right), \\
(1 + \beta) \frac{d^2 \eta_2}{dt^2} + (-2 \eta_1 + 4 \eta_2 - 2 \eta_3) &= \frac{C_{eAe}}{2} \mu u^2 \left( \eta_2 - \eta_1 - \left( \frac{1}{u} \frac{d \eta_1}{dt} + \frac{1}{u} \frac{d \eta_2}{dt} \right) \right), \\
(1 + \beta) \frac{d^2 \eta_3}{dt^2} + (-2 \eta_1 + 4 \eta_2 - 2 \eta_3) &= \frac{C_{eAe}}{2} \mu u^2 \left( \eta_2 - \eta_1 - \left( \frac{1}{u} \frac{d \eta_2}{dt} \right) \right), \\
(1 + \beta) \frac{d^2 \eta_2}{dt^2} + (-2 \eta_1 + 4 \eta_2 - 2 \eta_3) &= (2 - \epsilon) \frac{C_{eAe}}{2} \mu u^2 \left( \eta_3 - \eta_2 - \left( \frac{1}{u} \frac{d \eta_2}{dt} + \frac{1}{u} \frac{d \eta_3}{dt} \right) \right), \\
(1 + \beta) \frac{d^2 \eta_3}{dt^2} + (-2 \eta_1 + 4 \eta_2 - 2 \eta_3) &= (2 - \epsilon) \frac{C_{eAe}}{2} \mu u^2 \left( \eta_3 - \eta_2 - \left( \frac{1}{u} \frac{d \eta_2}{dt} + \frac{1}{u} \frac{d \eta_3}{dt} \right) \right).
\end{align*}
\]
where the dimensionless parameters are as follows

\[ \eta_n = \frac{y_n}{l}, \quad (n = 1, 2, 3), \]

\[ u = \sqrt{\frac{M_1}{k}} U, \quad \mu = \frac{\rho l}{M_1}, \quad \tau = \frac{1}{l} \sqrt{\frac{k}{M_1}}. \]  

(15)

The solution to Eqs. (14a) and (14b) is now cast in the form

\[ y_n = Y_n e^{\lambda t}, \quad (n = 1, 2, 3). \]  

(16)

Substituting Eq. (16) into Eqs. (14a) and (14b), one obtains the characteristic equation

\[ \lambda^2 (a_1 \lambda^4 + a_2 \lambda^3 + a_3 \lambda^2 + a_4 \lambda + a_5) = 0. \]  

(17)

The root of Eq. (17) may be expressed by a complex eigenvalue

\[ \lambda = \sigma \pm i \omega, \]  

(18)

where \( \sigma \) is a real part, \( \omega \) is an imaginary part, and \( i \) is the imaginary unit. The system is stable when \( \sigma < 0 \). The system is dynamically unstable (flutter) when \( \sigma > 0 \) and \( \omega \neq 0 \), while it is statically unstable (divergence) when \( \sigma > 0 \) and \( \omega = 0 \). The boundary for body flutter is determined by the Routh-Hurwitz condition

\[ a_2 a_3 a_4 - a_1 a_3^2 - a_2^2 a_5 = 0. \]  

(19)

The limit for static instability (body-divergence) is determined by the condition of vanishing characteristic root

\begin{align*}
\beta_1 \frac{d^2 \eta_1}{dt^2} + (\eta_1 - 2 \eta_2 + \eta_3) &= \frac{C_{LaT}}{2} \mu u^2 \left( \eta_2 - \eta_1 - \frac{1}{u} \frac{d \eta_1}{dt} \right) + (1 - \varepsilon) \frac{C_{LaE}}{2} \mu u^2 \left[ \eta_2 - \eta_1 - \frac{1}{u} \left( (1 - \varepsilon) \frac{d \eta_1}{dt} + \varepsilon \frac{d \eta_2}{dt} \right) \right], \\
\beta_2 \frac{d^2 \eta_2}{dt^2} + (\eta_1 - 2 \eta_2 + \eta_3) &= \frac{C_{LaN}}{2} \mu u^2 \left( \eta_2 - \eta_1 - \frac{1}{u} \frac{d \eta_1}{dt} \right), \quad \text{for } 0 \leq \varepsilon < 1
\end{align*}

(24a)

\begin{align*}
(1 + \beta_1) \frac{d^2 \eta_2}{dt^2} + (-2 \eta_1 + 4 \eta_2 - 2 \eta_3) + \kappa(\eta_2 - 2 \eta_3 + \eta_4) &= \varepsilon \frac{C_{LaE}}{2} \mu u^2 \left[ \eta_2 - \eta_1 - \frac{1}{u} \left( (1 - \varepsilon) \frac{d \eta_1}{dt} + \varepsilon \frac{d \eta_2}{dt} \right) \right], \\
(1 + \beta_2) \frac{d^2 \eta_3}{dt^2} + (\eta_1 - 2 \eta_2 + \eta_3) &= \frac{C_{LaT}}{2} \mu u^2 \left( \eta_3 - \eta_2 - \frac{1}{u} \frac{d \eta_2}{dt} \right) + (2 - \varepsilon) \frac{C_{LaE}}{2} \mu u^2 \left[ \eta_3 - \eta_2 - \frac{1}{u} \left( (2 - \varepsilon) \frac{d \eta_2}{dt} + \varepsilon \frac{d \eta_3}{dt} \right) \right], \\
(1 + \beta_1) \frac{d^2 \eta_3}{dt^2} + (-2 \eta_1 + 4 \eta_2 - 2 \eta_3) + \kappa(\eta_2 - 2 \eta_3 + \eta_4) &= (2 - \varepsilon) \frac{C_{LaE}}{2} \mu u^2 \left[ \eta_3 - \eta_2 - \frac{1}{u} \left( (2 - \varepsilon) \frac{d \eta_2}{dt} + \varepsilon \frac{d \eta_3}{dt} \right) \right], \\
(1 + \beta_2) \frac{d^2 \eta_4}{dt^2} + (\eta_1 - 2 \eta_2 + \eta_3) + \kappa(-2 \eta_2 + 4 \eta_3 - 2 \eta_4) &= (\varepsilon - 1) \frac{C_{LaE}}{2} \mu u^2 \left[ \eta_4 - \eta_3 - \frac{1}{u} \frac{d \eta_3}{dt} \right], \quad \text{for } 0 \leq \varepsilon < 2
\end{align*}

(24b)
\[
\frac{d^2 \eta_1}{dt^2} + (\eta_1 - 2\eta_2 + \eta_3) = \frac{C_{L\alpha T}}{2} \mu u^2 \left( \eta_2 - \eta_1 - \frac{1}{u} \frac{d\eta_1}{dt} \right),
\]

\[
(1 + \beta_1) \frac{d^2 \eta_2}{dt^2} + (-2\eta_1 + 4\eta_2 - 2\eta_3) + \kappa (\eta_2 - 2\eta_3 + \eta_4) = 0,
\]

\[
(\beta_1 + \beta_2) \frac{d^2 \eta_3}{dt^2} + (\eta_1 - 2\eta_2 + \eta_3) + \kappa (-2\eta_2 + 4\eta_3 - 2\eta_4) = (3 - \varepsilon) \frac{C_{L\alpha F}}{2} \mu u^2 \left[ \eta_4 - \eta_3 - \frac{1}{u} \left( (3 - \varepsilon) \frac{d\eta_3}{dt} + (\varepsilon - 2) \frac{d\eta_4}{dt} \right) \right],
\]

\[
\beta_2 \frac{d^2 \eta_4}{dt^2} + \kappa (\eta_2 - 2\eta_3 + \eta_4) = \frac{C_{L\alpha N}}{2} \mu u^2 \left[ \eta_4 - \eta_3 - \frac{1}{u} \left( (3 - \varepsilon) \frac{d\eta_3}{dt} + (\varepsilon - 2) \frac{d\eta_4}{dt} \right) \right] + (\varepsilon - 2) \frac{C_{L\alpha F}}{2} \mu u^2 \left[ \eta_4 - \eta_3 - \frac{1}{u} \left( (3 - \varepsilon) \frac{d\eta_3}{dt} + (\varepsilon - 2) \frac{d\eta_4}{dt} \right) \right].
\]

for \( 2 \leq \varepsilon \leq 3 \),

\[ \tag{24c} \]

with dimensionless parameters

\[
\eta_n = \frac{y_n}{l}, \quad (n = 1, 2, 3, 4).
\]

The solution to Eqs. (24a)–(24c) is now put in the form

\[
y_n = Y_n e^{\lambda t}, \quad (n = 1, 2, 3, 4).
\]

Substituting Eq. (26) into Eqs. (24a)–(24c), one obtains the characteristic equation

\[
\lambda^2 (a_1 \lambda^6 + a_2 \lambda^5 + a_3 \lambda^4 + a_4 \lambda^3 + a_5 \lambda^2 + a_6 \lambda + a_7) = 0.
\]

The root of Eq. (27) may be expressed by a complex eigenvalue as shown in Eq. (18). The flutter bound is determined by the Routh-Hurwitz condition. The limit for body divergence is determined by the condition of vanishing characteristic root

\[
a_7 = 0.
\]

\[ \tag{28} \]

4. Results and Discussions

4.1. Effect of DOF

It is assumed that a slender rocket is accommodated with a stabilizer fin. It is assumed, just for case study, that \( C_{L\alpha N} = 2.0 \), \( C_{L\alpha T} = 0.0 \), \( \mu = 0.002 \), and \( \beta_1 = \beta_2 = 1.0 \). Figure 4 shows the dimensionless critical flight velocity \( u_c \) as a function of the dimensionless location of the stabilizer fin, \( \varepsilon \), for the 3DOF model. It is confirmed that the 3DOF model loses its stability only by divergence.

Figure 5 shows the same stability map for the 4DOF model. The 4DOF model, however, implies that the flight body may lose its stability by divergence, and by flutter as well, if the stabilizer fin is fixed to the rocket body ahead of the tail end. It is suggested that a simplified mechanical model for discussing the aeroelasticity of the rocket body should have at least 4DOF.

Note that the 3DOF in the 3DOF model consist of two rigid body motions, translational and rotational, and one bending motion. The two rigid body motions are related to attitude stability of the rocket body and free from structural failure. However the bending motion is responsible for structural stability, that is divergence. So it is obvious that the 3DOF model can be responsible for divergence, but not flutter. The 4DOF model consists of the first and second bending modes. The second bending mode is prone to cause bending flutter.

4.2. Effect of mass distribution

It is assumed here that no stabilizer fin is attached to a flight body in order to discuss the effect of a mass distribution alone. Thus \( C_{L\alpha F} = 0 \). It is assumed, just for case study, that \( C_{L\alpha N} = 2.0 \), \( C_{L\alpha T} = 4.0 \), \( \mu = 0.002 \) and \( \beta_1 = 1.0 \). Figure 6 shows the dimensionless critical flight velocity
as a function of the mass ratio \( \beta_2 \) for the 4DOF model. The rocket with no fin loses its stability only by divergence. The divergence velocity increases as the mass ratio increases. This means that a nose-heavy body is less susceptible to divergence. As a mass at the nose is resistible to motion of the nose, then the inclination of the nose bar, \( \phi_3 \), is prone to decrease. This leads to a reduction of the aerodynamic load at the nose.

4.3. Effect of stiffness distribution

It is assumed here again that a flight body is not accommodated with a stabilizer fin in order to discuss the pure effect of a stiffness distribution, that is, \( C_{LaF} = 0 \). It is assumed that \( C_{LaN} = 2.0, C_{LaT} = 4.0, \mu = 0.002 \) and \( \beta_1 = 1.0 \). Figure 7 shows the dimensionless critical flight velocity \( \alpha_c \) as a function of the restoring spring ratio \( \kappa \) for the 4DOF model. The body loses its stability only through divergence. The divergence velocity increases as the restoring moment coefficient increases. Thus a nose-stiff flight body is less susceptible to divergence. The physical reason for this seems to be the same as it is in the case of mass distribution. The inclination of the third/nose bar, \( \phi_3 \), may be suppressed relatively smaller than the inclination of the first/tail bar, \( \phi_1 \). This may result in a smaller aerodynamic load at the nose.

4.4. Effect of fin location

Now let us consider a flight body accommodated with a stabilizer fin, that is \( C_{LaF} = 4.0 \). It is assumed that \( C_{LaN} = 2.0, C_{LaT} = 4.0, \mu = 0.002 \) and \( \beta_1 = \beta_2 = 1.0 \). Figures 8–10 show the dimensionless critical flight velocity \( \alpha_c \) as a function of the dimensionless location of the stabilizer fin \( \varepsilon \) for the 4DOF model. Figure 8 shows the effect of the fin when it is fixed on the first bar, while Figs. 9 and 10 demonstrate the effect when it is mounted on the second and third bar, respectively. The flight body loses its stability by divergence in the range of \( 0 \leq \varepsilon \leq 2.26 \). However, flutter instability can take place in the range of \( 2.26 < \varepsilon \leq 3.0 \). Thus, it is confirmed that a fin attached on the flight body between the tail and the center of the whole body may lead to divergence. The stabilizer fin on the third bar may cause instability by flutter.

5. Concluding Remarks

This paper proposed a duly simplified mechanical model in order to provide a perspective of body divergence and
flutter. It was demonstrated that the simplified mechanical model should have at least 4DOF in discussing aeroelastic stability of flight bodies. The present model confirmed that the mechanical models with no fin lose their stability solely by divergence, regardless of mass and stiffness distribution. It was confirmed by the present model that a stabilizer fin fixed near to the nose may lead to dynamic instability by flutter.

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References