Synchronizing of Multiple Time-of-Arrivals for Pulsar-Based Navigation

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Spacecraft can observe multiple sources to achieve the goal of absolute position determination in pulsar-based navigation. In this paper, we draw attention to the problems associated with synchronizing of multiple time-of-arrival (TOA) measurements. The importance of this for navigation performance is not fully appreciated. Based on high-precision pulsar timing models, we establish the linear pulse phase measurement equation for near-Earth spacecraft and deduce the formulae of apparent pulse frequency (APF). We also develop the TOA synchronization model that uses APF to propagate pulse phases. We apply this model in numerical simulations that implement the extended Kalman filter (EKF) to estimate navigation states for geostationary (GEO) satellites. The results show that our model can effectively control navigation errors after TOAs are synchronized properly. We expect that the TOA synchronization technique presented in this paper may be useful for performance improvement of pulsar-based navigation.

Key Words: Pulsar-Based Navigation, Time-of-Arrival Synchronization, TOA, Pulse Phase Propagation

Nomenclature

- $a$: norm of an arbitrary vector $a$
- $\hat{a}$: normalized vector ($= a / a$)
- $a_1$: spacecraft’s acceleration ($= \frac{\text{d}v_1}{\text{d}t}$)
- $a_E$: Earth’s acceleration ($= \frac{\text{d}v_E}{\text{d}t}$)
- $c$: vacuum speed of light
- $D_{TC}$: direction cosine matrix transforming vectors from the ICRS to the ITRS
- $E_{FRQ}$: epoch of frequency
- $E_{POS}$: epoch of position
- $G$: gravitational constant
- $I_3$: 3-by-3 identity matrix
- $m$: cycle ambiguity
- $m_0$: mass of the Sun
- $m_1$: mass of the Earth
- $m_j$: mass of the $j$th body in the solar system
- $l$: pulsar’s displacement relative to its initial position
- $R$: spacecraft-to-pulsar vector
- $R_0$: initial pulsar positional vector (SSB-to-pulsar) at $E_{POS}$
- $r_E$: SSB-to-Earth vector
- $r_1$: Earth-to-spacecraft vector
- $r_j$: vector from the solar system’s $j$th body to the spacecraft
- $r_0$: Sun-to-spacecraft vector
- $r$: SSB-to-spacecraft vector
- $t$: barycentric coordinate time (TCB)
- $T$: pulsar’s proper time
- $u_1$: spacecraft’s orbital eccentric anomaly
- $U_E$: geopotential evaluated at the location of the spacecraft
- $U_{SS-E}$: gravitational potential evaluated at the geocenter due to all solar system bodies except the Earth
- $v_B$: pulsar’s velocity ($= \frac{\text{d}l}{\text{d}T}$)
- $v_E$: Earth’s velocity ($= \frac{\text{d}r_E}{\text{d}t}$)
- $v_1$: spacecraft’s velocity ($= \frac{\text{d}r_1}{\text{d}t}$)
- $v_x, v_y, v_z$: $v_1$ projected in the ITRS
- $x, y, z$: $r_1$ projected in the ITRS
- $\alpha, \delta$: pulsar’s right ascension and declination in the ICRS at $E_{POS}$
- $\delta T$: elapsed time from $E_{POS}$ ($= T - E_{POS}$)
- $\Delta \tau, \Delta \tau_x, \Delta T$: elapsed time from $E_{FRQ}$ ($= \tau - E_{FRQ}$, e.g. for $\tau$)
- $A$: forward time delay
- $\hat{A}$: inverse time delay
- $e$: pulse phase error
- $\mu_0, \mu_1$: pulsar’s proper motion along right ascension and along declination
- $\nu, \dot{\nu}, \ddot{\nu}$: zeroth, first and second order time derivatives (with respect to $T$) of pulsar’s spin frequency
- $\nu^X, \dot{\nu}^X$: zeroth and first order time derivatives (with respect to $T$) of the APF
- $\tau$: spacecraft’s proper time (or terrestrial time in simplification)
- $\tau_1$: pulse TOA at the spacecraft
- $\phi$: phase’s fractional part (zero to one in value)
- $\Phi$: total phase
- $\phi_0^p$: pulsar’s spin phase at $E_{FRQ}$
- $\omega_E$: Earth’s rotation speed

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Subscripts
- P: parallel component on $\mathbf{R}_0$ (= $a \cdot \hat{R}_0$ for an arbitrary vector $a$)
- V: vector’s vertical component against $\mathbf{R}_0$ (= $a - a_\parallel \hat{R}_0$ for an arbitrary vector $a$)
- $a$: evaluated at $\tau_a$
- $\tau$: evaluated at $\tau$
- ITRE: projected in the ITRS

Superscripts
- P: of the pulsar
- X: at the spacecraft
- E: at the Earth
- $\sim$: measured

Acronyms
- APF: apparent pulse frequency
- BCRS: barycentric celestial reference system
- EKF: extended Kalman filter
- GCRS: geocentric celestial reference system
- ICRS: international celestial reference system
- ITRS: international terrestrial reference system
- MJD: modified Julian date
- RMSE: root-mean-squared error
- SD: standard deviations
- SSB: solar system barycenter
- SNR: signal-to-noise ratio
- TCB: barycentric coordinate time
- TCG: geocentric coordinate time
- TOA: time-of-arrival
- TOE: time-of-emission
- TT: terrestrial time

1. Introduction

Although people have been observing pulsars for almost half a century since pulsar’s discovery in 1967, pulsar-based navigation is a newly developed technology for spacecraft. This navigation system uses accurate pulsar timing data observed in the X-ray band to determine spacecraft’s position and velocity. Notably, pulsar-based navigation requires spacecraft to observe multiple sources to produce accurate measurements for a navigation solution.

The measured quantity for pulsar-navigation is the pulse time-of-arrival (TOA). Multiple TOA measurements from different pulsars are produced asynchronously for two reasons: (1) The TOA is discrete, for pulses arrive only when the radiation beam of the pulsar points at the detector. (2) If different X-ray detectors work separately, the observation time spans may be misaligned. Therefore, Sheikh has suggested that methods must be employed to adjust observed TOAs of different time to the same epoch for navigation. Unfortunately, to our knowledge, there is little information available in the literature about how to synchronize multiple TOAs.

The purpose of this paper is to study the problems associated with synchronizing of multiple TOAs. We wish to suggest a model of TOA synchronization. The model should be accurate and practical and it has the following characteristics: (1) It will be based on the high-precision pulsar timing model. (2) It will not comprise complicated algorithms.

We use fruitful results in pulsar timing as the basis of our study. Development of high-precision pulsar timing benefits greatly from observing millisecond pulsars, especially binary ones. Time delays in the solar system have been carefully modeled and those in binary systems have also been formulated as such as the Blandford and Teukolsky (BT) model or the Damour and Deruelle (DD) model. Time delay formulae as accurate as 1 ns were summarized by the staff of the well-known pulsar timing software TEMPO2. We also avail ourselves of the useful concept of forward and inverse timing of the DD model for clarity in discussion. To simply the measurement equation, Sheikh has proposed a linearization technique called “pulse phase single difference” for near-Earth spacecraft. We use this idea and continue Sheikh’s work to provide exact expressions and the linearization error analyses.

We investigate how to calculate the apparent pulse frequency (APF) and how to use it to synchronize multiple TOAs. Some provided methods of measuring pulse phases instead of TOAs. We call this process “pulse phase propagation.” We use it to synchronize multiple TOAs to obtain pulse phase measurements at a common epoch. Instead of direct detection, we calculate the APF and its time derivative from not only the spacecraft velocity and acceleration but also the pulsar’s parameters to include higher orders of the Doppler effect. The spacecraft velocity and acceleration are iterated from initial estimate until convergence.

Some studies provided methods of measuring pulse phases instead of TOAs. The multiple pulse phase measurements are also required to be synchronized if their epochs are not aligned. The pulse TOA is essentially the zero-phase time of the pulse. From pulse phase propagation, we can easily make the conversion between the TOA and the phase. Therefore, we identify the problem “phase synchronization” and only mention the latter in this paper.

We make two assumptions in this paper: (1) The spacecraft is near-Earth. Without this assumption, the TOA synchronization technique makes no difference, and the linearization method can still be used when the spacecraft is close to any known body. (2) The pulsar is isolated. Although time delays in the binary systems can be included in the timing models only those for isolated pulsars are considered to reduce the complexity.

This paper is separated into the following sections. The forward and inverse timing models are described in section...
2. The linear pulse phase measurement equation is presented in section 3. The TOA synchronization model is interpreted in section 4. Simulation scenarios and results are given in section 5. Finally, the conclusions are made in section 6.

2. Forward and Inverse Timing Models

Pulsar timing efforts in pulsar astronomy are devoted to characterizing of pulsars. In contrast, pulsar-based navigation uses pulsars’ known characterizations to extract navigation information. Pulsar timing models are used to describe the time delay between the TOA and the time-of-emission (TOE). The terms “forward” and “inverse” indicate the conversion direction. Converting of the TOE into the TOA is “forward,” and the reversal is “inverse.” The forward timing model is convenient for the describing of time delays in the light travel course, and the inverse timing model is necessary for tracing the observation to the pulsar’s proper time. The forward and inverse timing models are the basis of our discussion.

The geometric relationship between the pulsar and the spacecraft is exhibited in Fig. 1. The vectors except for \( \mathbf{r} \) are defined in the barycentric celestial reference system (BCRS) that is oriented according to the international celestial reference system (ICRS). The \( \mathbf{r} \) is defined in the geocentric celestial reference system (GCRS). However, according to the four-dimensional transformation between the BCRS and the GCRS, consideration of \( \mathbf{r} \) in the BCRS brings a positional error of less than 1 m, and as a result of this precision, it can still be said that \( \mathbf{r} = \mathbf{r}_E + \mathbf{r}_1 \).

Considering an arbitrary spacecraft’s proper time \( t \) and the corresponding pulsar’s proper time \( T \), the relationship between \( t \) and \( T \) can be depicted by the forward timing model

\[
T + \Delta(T, r, r_1) = t
\]

where the forward time delay \( \Delta \) is a function of \( T, r \) and \( r_1 \), which is divided into six parts

\[
\Delta(T, r, r_1) = \Delta_{RS}(T, r) + \Delta_{PS}(T, r) + \Delta_{SS}(T, r)
+ \Delta_{ES}(T, r_1) + \Delta_{VP}(T) + \Delta_{EI}(T) \tag{2}
\]

where \( \Delta_{RS} \) is the solar system Roemer delay, \( \Delta_{PS} \) is the parallax delay, \( \Delta_{SS} \) denotes the solar system Shapiro delay, \( \Delta_{ES} \) represents the solar system Einstein delay, \( \Delta_{VP} \) means the vacuum propagation delay, and \( \Delta_{EI} \) stands for the interstellar Einstein delay. These delays are discussed in detail below, and all terms greater than 1 ns (except the second-order Shapiro delay) are retained in their expressions.

The solar system Roemer delay is divided into two parts as \( \Delta_{RS} = \Delta_{RS0} + \Delta_{RS2} \), where \( \Delta_{RS0} \) is the basic Roemer delay caused by the initial spacecraft radial displacement and \( \Delta_{RS2} \) is the second-order Roemer delay induced by the proper motion

\[
\Delta_{RS0} = -c^{-1}r_p \tag{3}
\]

\[
\Delta_{RS2} = -c^{-1}R_0^{-1}I_V \cdot r_V + 1/2c^{-1}R_0^{-2}r_V^2
+ c^{-1}R_0^{-2}l_0I_V \cdot r_V \tag{4}
\]

The parallax delay is due to the displacement of the distance along the initial line of sight and the current one (except the second-order Shapiro delay)

\[
\Delta_{PS} = 1/2c^{-1}R_0^{-1}r_V^2 \tag{5}
\]

The solar system Shapiro delay is a relativistic effect due to the curved space-time of the solar system. If the second-order Shapiro delay that reaches the maximum of 18 ns is neglected, this delay is expressed as

\[
\Delta_{SS} = -2 \sum_j Gm_jc^{-3} \ln (r_p + r_j) \tag{6}
\]

The solar system Einstein delay arises from the transformation between \( t \) and the barycentric coordinate time (TCB, \( T \)). For a near-Earth Keplerian delay from the transformation between \( t \) and \( T \) is the same as the forward time delay in value, except that the inverse time delay is calculated from \( T \). Here, Eq. (1) becomes

\[
\tau - \tilde{\Delta}(\tau, r, r_1) = T \tag{10}
\]

Then, Eq. (2) is rewritten as

\[
\tilde{\Delta}(\tau, r, r_1) = \tilde{\Delta}_{RS0}(\tau) + \tilde{\Delta}_{PS0}(\tau, r) + \tilde{\Delta}_{PS}(\tau, r)
+ \tilde{\Delta}_{ES}(\tau, r_1) + \tilde{\Delta}_{VP}(\tau) + \tilde{\Delta}_{EI}(\tau) \tag{11}
\]

Assuming that the event at the spacecraft is timed by the terrestrial time (TT) (still symbolized as “\( t \)”), the solar
system Einstein delay is thus a function of only one independent variable \( \tau \) and can be calculated from the TT and TCB conversion algorithm\(^6\)

\[
\hat{\Delta}_{\text{ES}}(\tau) = \tau - \text{TCB}(\tau)
\]

(12)

Therefore, the inverse time delay can be written as \( \hat{\Delta}(\tau, r) \) with two independent variables \( \tau \) and \( r \).

Let us define the “extra” delay \( \Delta_{\text{EX}}(\tau, r) \) as

\[
\Delta_{\text{EX}}(\tau, r) = \hat{\Delta}_{\text{RS}}(\tau, r) + \hat{\Delta}_{\text{PS}}(r) + \hat{\Delta}_{\text{SS}}(\tau, r) + \Delta_{\text{ES}}(\tau) + \Delta_{\text{EP}}(r) + \Delta_{\text{EI}}(\tau)
\]

(13)

The inverse time delay is finally written as

\[
\hat{\Delta}(\tau, r) = \hat{\Delta}_{\text{RS}}(r) + \Delta_{\text{EX}}(\tau, r)
\]

(14)

3. Linear Pulse Phase Measurement Equation

3.1. Equation establishment and linearization

In multiple observation, the use of the pulse phase measurement equation is suitable for navigation because pulse phases can be evaluated at a common navigation epoch. Synchronizing of multiple TOAs is intended to generate pulse phase measurements at this epoch. On the basis of the pulsar timing model, the linear pulse phase measurement equation is established as below.

The pulse phase of \( \tau \) at the spacecraft shares the same value of the pulsar’s spin phase of \( T \), only if \( \tau \) and \( T \) are related according to Eq. (1) or (10). Hence, the time delay formula has the equivalent phase form as

\[
\phi^X(\tau) = \phi^P[\tau - \hat{\Delta}(\tau, r)]
\]

(15)

where the equation of the pulsar’s proper spin phase \( \phi^P \) is\(^6\)

\[
\phi^P(T) = \phi^P_0 + \nu \Delta T + 1/2\nu \Delta T^2 + 1/6\bar{\nu} \Delta T^3
\]

(16)

Thus the pulse phase measurement equation is obtained.

\[
\hat{\phi}^X(\tau) = \phi^P[\tau - \hat{\Delta}(\tau, r)] - m^X + \varepsilon_{\text{W}}
\]

(17)

where \( m^X \) represents the integral phase known as the ‘cycle ambiguity,’\(^4\) and \( \varepsilon_{\text{W}} \) is the pulse phase measurement error.

For near-Earth spacecraft, the quantity “Earth pulse phase” is defined, which can be apprehended as the pulse phase’s fractional part of the same time \( \tau \) if the spacecraft is located at the center of the Earth.

\[
\phi^E(\tau) = \phi^P[\tau - \hat{\Delta}(\tau, r)] - m^E
\]

(18)

where \( m^E \) is the cycle ambiguity at the Earth. From Eqs. (16), (17) and (18), and the inverse timing model, Eq. (17) is linearized by subtracting \( \hat{\phi}^X(\tau) \) from \( \phi^E(\tau) \).

\[
\delta\hat{\phi}(\tau) \equiv \phi^E(\tau) - \hat{\phi}^X(\tau) = -c^{-1}_1 \bar{\nu} \Delta T + \nu /2 \Delta T^2 + m^X - m^E + \varepsilon_F - \varepsilon_{\text{W}}
\]

(19)

where \( \varepsilon_F \) represents the linearization error.

3.2. Linearization error analysis

The linearization error \( \varepsilon_F \) consists of the \( \nu \)-induced error \( \varepsilon_{\text{F0}} \), the \( \nu \)-induced error \( \varepsilon_{\text{F1}} \) and the \( \nu \)-induced error \( \varepsilon_{\text{F2}} \) as \( \varepsilon_F = \varepsilon_{\text{F0}} + \varepsilon_{\text{F1}} + \varepsilon_{\text{F2}} \). The formulae for the three parts of \( \varepsilon_F \) are obtained as follows.

\[
\delta\Delta_{\text{EX}} \equiv \hat{\Delta}_{\text{EX}}(\tau, r) - \Delta_{\text{EX}}(\tau, r_E)
\]

(20)

\[
\varepsilon_{\text{F0}} = \nu \delta\Delta_{\text{EX}}
\]

(21)

\[
\varepsilon_{\text{F1}} = \nu \Delta T \delta\Delta_{\text{EX}} + \nu \Delta T \delta\Delta_{\text{EX}} \left[ (\Delta T - \hat{\Delta}(\tau, r)) (c^{-1}_1 \bar{\nu} \Delta T - \delta\Delta_{\text{EX}}) \right]
\]

(22)

\[
\varepsilon_{\text{F2}} = \nu \Delta T c^{-1}_1 \bar{\nu} \Delta T \left[ (\Delta T - \hat{\Delta}(\tau, r)) (c^{-1}_1 \bar{\nu} \Delta T - \delta\Delta_{\text{EX}}) \right]
\]

(23)

The \( \delta\Delta_{\text{EX}} \) can be expanded to the first order through the position partial derivative.

\[
\delta\Delta_{\text{EX}} = \left[ \delta\Delta_{\text{EX}}(\tau, r)/\partial r \right] \cdot r
\]

(24)

So the \( \nu \)-induced error can be divided into three position derivative terms as \( \varepsilon_{\text{F0}} = \varepsilon_{\text{F0,R0}} + \varepsilon_{\text{F0,PS}} + \varepsilon_{\text{F0,SS}} \) where

\[
\varepsilon_{\text{F0,R0}} = \nu \delta\Delta_{\text{RS}}(\tau, r)/\partial r \cdot r
\]

(25)

\[
\varepsilon_{\text{F0,PS}} = \nu \delta\Delta_{\text{PS}}(r)/\partial r \cdot r
\]

(26)

\[
\varepsilon_{\text{F0,SS}} = \nu \delta\Delta_{\text{SS}}(\tau, r)/\partial r \cdot r
\]

(27)

Expanding of the position derivatives in Eqs. (25) and (26) yields

\[
v^{-1}\varepsilon_{\text{F0,R0}} = -c^{-1}_1 \bar{\nu} \Delta T \cdot r + O(C_0^{-2})
\]

(28)

\[
v^{-1}\varepsilon_{\text{F0,PS}} = c^{-1}_1 \bar{\nu} \Delta T \cdot r + O(C_0^{-2})
\]

(29)

Then, if the effect of only the Sun in the Shapiro delay is considered and \( r \) is used to approximate \( r_0 \), there is

\[
v^{-1}\varepsilon_{\text{F0,SS}} = -2GM_0 c^{-3} \bar{\nu} \bar{R}/(r_0 + r)
\]

(30)

Seven isolated pulsars (see Table 1) from the top ten rotation-powered pulsars ranked by the range accuracy in Tables 3–5 of Ref. 4) are selected to demonstrate the linearization error. The pulsar parameters are chosen to be five spin parameters: \( \phi^P_0, E_{\text{FQO}}, \nu, \bar{\nu} \) and six astronomical parameters: \( R_0, E_{\text{PS}}, \alpha, \delta, \mu_0, \mu_1 \). The radial proper motion and acceleration parameters are considered to be absorbed into other parameters and are ignored in our parameter set. The values of the parameters are provided by the ANTF pulsar catalogue\(^17,18\).

Provided the spacecraft is a geostationary (GEO) satellite, different parts of the linearization error are computed and

Table 1. Different parts of the pulse phase measurement equation linearization error.

<table>
<thead>
<tr>
<th>PSR name</th>
<th>( \nu^{-1}\varepsilon_{\text{F0,R0}} ) (ns)</th>
<th>( \nu^{-1}\varepsilon_{\text{F0,PS}} ) (ns)</th>
<th>( \nu^{-1}\varepsilon_{\text{F0,SS}} ) (ns)</th>
<th>( \nu^{-1}\varepsilon_{\text{F1}} ) (ns)</th>
<th>( \nu^{-1}\varepsilon_{\text{F2}} ) (ns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0531+21</td>
<td>3.8 \times 10^{-3}</td>
<td>0.40</td>
<td>160</td>
<td>2.6</td>
<td>0.068</td>
</tr>
<tr>
<td>B1821–24</td>
<td>5.5 \times 10^{-8}</td>
<td>0.16</td>
<td>130</td>
<td>4.4 \times 10^{-3}</td>
<td>1.3 \times 10^{-9}</td>
</tr>
<tr>
<td>B1937+21</td>
<td>1.1 \times 10^{-8}</td>
<td>0.082</td>
<td>5.1</td>
<td>3.5 \times 10^{-6}</td>
<td>2.2 \times 10^{-8}</td>
</tr>
<tr>
<td>B1509–58</td>
<td>0</td>
<td>0.12</td>
<td>6.0</td>
<td>0.32</td>
<td>6 \times 10^{-3}</td>
</tr>
<tr>
<td>B0540–69</td>
<td>0</td>
<td>0.014</td>
<td>3.8</td>
<td>2.2 \times 10^{-2}</td>
<td>1.6 \times 10^{-4}</td>
</tr>
<tr>
<td>B1823–13</td>
<td>5.0 \times 10^{-7}</td>
<td>0.17</td>
<td>19</td>
<td>8.2 \times 10^{-3}</td>
<td>4.7 \times 10^{-4}</td>
</tr>
<tr>
<td>J1124–5916</td>
<td>0</td>
<td>0.063</td>
<td>4.8</td>
<td>0.12</td>
<td>3.0 \times 10^{-4}</td>
</tr>
</tbody>
</table>
the maximum absolute values in the computation time span of one year beginning at MJD 55742 are listed in Table 1. Notably, the value of $v^{-1} \varepsilon_{f0,SS}$ for PSR B0531+21 versus the time is plotted in Fig. 2.

According to Eq. (30), it is deduced that $v^{-1} \varepsilon_{f0,SS}$ will not be greater than 600 ns for near-Earth spacecraft and reaches the maximum value when the spacecraft lies just behind the solar limb along the line of sight. The two maximal values of $v^{-1} \varepsilon_{f0,SS}$: 160 ns (see Fig. 2 and Table 1) and 130 ns (see Table 1) are due to the fact that PSR B0531+21 and PSR B1821−24 are close to the ecliptic with the ecliptic elevation of $-1.3^\circ$ and $-1.5^\circ$, respectively.

The results show that the $f_{0,SS}$ term (less than 600 ns) and the $f_{0,RS}$ term (mainly induced by the proper motion) dominates the $v$-induced and the whole linearization error, and the $v$-induced and the $v$-induced errors are scarcely greater 1 ns.

4. TOA Synchronization Model

4.1. APF and its time derivative

The APF and its time derivative are required to propagate the pulse phase from one time to another. The APF differs from the pulsar’s inherent spin frequency. It is known as the Doppler effect of a wave receiver in motion to the wave source. In this paper, not only the spacecraft velocity effect but also higher-order terms are considered to acquire satisfactory accuracy. We again use the inverse timing model to deduce the full expressions.

The APF at the spacecraft ($v^X$) and its time derivative ($\dot{v}^X$) (with respect to $t$) can be derived from the first-order and the second-order time derivatives of the pulse phase respectively:

$$v^X(t) = d\Phi^X(t)/dt$$

$$\dot{v}^X(t) = d^2\Phi^X(t)/dt^2$$

According to Eq. (15), the derivatives yield

$$\dot{v}^X(t) = [1 - d\Delta(t, r)/dt]\dot{\Phi}(T)$$

$$\ddot{v}^X(t) = [d^2\Delta(t, r)/dt^2]\ddot{\Phi}(T)$$

where

$$\Phi^X(T) = v + \Delta T + 1/2 \dot{v}\Delta T^2$$

The time derivatives of $\dot{v}(t, r)$ in Eqs. (33) and (34) are discussed below. From Eq. (14), there is

$$\frac{d\Delta(t, r)}{dt} = -c^{-1}(v_{EP} + v_{IP})\frac{dt}{d\tau} + \frac{d\Delta_{EX}(t, r)}{d\tau}$$

where

$$\frac{d\Delta_{EX}(t, r)}{d\tau} = \frac{dt}{d\tau} \left( \frac{d\Delta_{RS2}}{dt} + \frac{d\Delta_{RS}}{dt} + \frac{d\Delta_{SS}}{dt} \right)$$

$$+ \frac{d\Delta_{ES}}{d\tau} + \frac{d\Delta_{VP}}{d\tau} + \frac{d\Delta_{EL}}{d\tau}$$

The following expressions are derived to resolve Eq. (39).

$$\frac{dt}{d\tau} = 1 + L_G + c^{-2}(U_{SS,E} + v_E^2/2)$$

$$\frac{dT}{d\tau} = 1 - 1/2c^2 R_0^2 \mu_V$$

$$\frac{d\Delta_{RS2}}{dt} = -c^{-1} R_0^{-1} I_V \cdot (v_E + v_1V)$$

$$\frac{d\Delta_{PS}}{dt} = c^{-1} R_0^{-1} [r \cdot (v_E + v_1) - r_p(v_{EP} + v_{IP})]$$

$$\frac{d\Delta_{SS}}{dt} = -2Gm_0c^{-1} [v_E + v_{IP} + \dot{P} \cdot (v_E + v_1)]/(r_p + r)$$

$$\frac{d\Delta_{ES}}{d\tau} = 1 - d\tau/d\tau$$

$$\frac{d\Delta_{VP}}{dt} = c^{-1} R_0^2 \mu_V^2 \delta T(d\tau/d\tau)(dT/d\tau)$$

$$\frac{d\Delta_{EL}}{d\tau} = 1/2c^{-2} R_0^2 \mu_V^2 (dT/d\tau)$$

where $\mu_V^2 = \mu_a^2 + \mu_z^2$, and $L_G$ equals 6.969290134 $\times 10^{-10}$.

The second-order time derivative of $\dot{v}(t, r)$ cannot be ignored. It can be written with respect to the Earth’s and spacecraft’s acceleration as

$$d^2\dot{\Delta}(t, r)/dt^2 = -c^{-1}(a_{EP} + a_{IP})(d\tau/d\tau)$$

where the accelerations are approximated by $a_{EP} = -Gm_0r_{E}^{-2} \dot{r}_E$ and $a_{IP} = -Gm_1r_{I}^{-2} \dot{r}_I$, respectively.

The Earth’s position and velocity in the above equations are available from the solar system ephemerides. The spacecraft’s position and velocity relative to the Earth can be iterated from zero to current best-estimated values. Equations (33) to (48) are all the formulae required to calculate the APF and its time derivative.

4.2. Pulse phase propagation

The pulse phase propagation is a phase transfer course from $\tau_a$ to $\tau$ like the map $\Phi^X \rightarrow \Phi^X$. The spacecraft is supposed to move in a smooth trajectory so that $\Phi^X$ can be expanded as a Taylor series. The expansion to the order of $\Phi^X$ (namely $\dot{v}^X$) is precise enough for the pulse phase to be propagated for a short time span, so the following equation is acquired.

$$\dot{v}^X = \dot{v}^X(t - \tau_a) + 1/2 \ddot{v}^X(t - \tau_a)^2$$
Analogously, the inverse propagation equation is

$$\Phi^X_a = \Phi^X_t + v_t^X(\tau_a - \tau) + 1/2 \nu_t^X(\tau_a - \tau)^2$$

(50)

The sum of Eqs. (49) and (50) yields

$$\left(\nu_t^X - \nu_a^X\right) / (\tau - \tau_a) = 1/2(\nu_t^X + \nu_a^X)$$

(51)

From Eqs. (49) and (51), the pulse phase propagation equation is established as

$$\Phi^X_t = \Phi^X_a + v_a^X(\tau - \tau_a) - 1/2 \nu_a^X(\tau - \tau_a)^2 - \varepsilon_p$$

(52)

where $-\varepsilon_p$ stands for the propagation error. Equation (52) indicates that we can apply the APF and its time derivative of $\tau$ instead of those of $\tau_a$ into Eq. (49) to make the propagation only if the opposite sign of the APF time derivative is used. From this trick, the solar ephemerides only needs to be referred to once when more than one pulse phases are propagated.

We advise avoiding use of the pulsar’s inherent spin frequency to propagate the phase even for a short time span (say 1 s). This improper way is expressed as

$$\Phi^X_t = \Phi^X_a + v_a(\tau - \tau_a) + 1/2 \nu_a(\tau - \tau_a)^2$$

(53)

where

$$v_a = \nu + \nu \Delta \tau_a$$

and

$$\nu_a = \nu + \nu \Delta \tau_a$$

For GEO satellites, the propagation time errors (phase error multiplied by $v^{-1}$) of PSR B0531+21 as the propagation time span increases from 0 to 1,000 s are illustrated in Fig. 3, and the propagation time errors of the seven pulsars in Table 1 against different propagation time spans are compared in Table 2 for Eqs. (52) and (53).

The results show that the pulse phase propagation using Eq. (52) is much more accurate than that using Eq. (53). The propagation time span of 100 s or less will bring a time error of less than 1 $\mu$s for Eq. (52), but much greater for Eq. (53). If Eq. (53) is used, the propagation error increases linearly against the propagation time span (see left of Fig. 3), since the velocities of the Earth and the spacecraft can be considered invariable for a short period. The noise of the curve on the right of Fig. 3 is approximately 0.5 $\mu$s. This may be because we use a computer system only supporting 64-bit floating-point numbers and the timing in MID makes the computation incur a precision loss up to 1 $\mu$s. If a system supporting 96-bit or 128-bit floating-point numbers were used, the propagation time error for Eq. (52) might be smaller.

### 4.3. TOA synchronization equation

On the basis of former discussions, we can provide the TOA synchronization equation and rewrite the linear phase measurement equation to a simpler form. Provided $N$ pulsars are observed, and for the $j$th measured TOA ($\tau_{aj}$) the measurement error is $-(\nu_{aj}^{-1})\varepsilon_{wj}$, the pulse phase of $\tau_{aj}$ will be

$$\Phi^X(\tau_{aj}) = \Phi^X \left[ \tau_{aj} - \left(\nu_{aj}^{-1}\varepsilon_{wj}\right) \right]$$

(54)

which can be expanded as the following according to Eq. (52).

$$\Phi^X(\tau_{aj}) = m_{aj}^X - \varepsilon_{wj}$$

(55)

where $m_{aj}^X$ is the pulse phase of the actual TOA ($\tau_{aj}$) (just an integer).

Multiple TOAs are required to be synchronized to yield the pulse phase counterparts $\Phi^X_{aj}$ of a common epoch $\tau$ to acquire the map like $\tilde{\tau}_{aj} \rightarrow \Phi^X_{aj}$. According to Eqs. (52) and (55), the pulse phase of $\tau$ is

$$\Phi^X(\tilde{\tau}_{aj}) = m_{aj}^X + v_{aj}^X(\tau - \tilde{\tau}_{aj}) - 1/2 \nu_{aj}^X(\tau - \tilde{\tau}_{aj})^2 - \varepsilon_{pj} - \varepsilon_{wj}$$

(56)

Hence, the TOA synchronization equation is provided as the following.
where $\bar{\phi}_{ij}^X$ is the fractional part of $\phi_{ij}^X$ and $m_{ij}^X$ is the integral part of $\phi_{ij}^X$. The measurement error of Eq. (57) is now $\varepsilon_{ip} + \varepsilon_{ij}$.

It is defined that

$$\delta \phi_{ij} \equiv \phi_{ij}^E - \phi_{ij}^X,$$

$$v_{ij} \equiv \left( v_j + \dot{v}_i \Delta \tau_j + 1/2 \ddot{v}_i \Delta \tau_j^2 \right),$$

$$\varepsilon_{ij} \equiv cv_{ij}^{-1} (\varepsilon_{ij} - \varepsilon_{iw} - \varepsilon_{pj})$$

and

$$Z_j \equiv cv_{ij}^{-1} \left( \delta \phi_{ij} - m_{ij}^X + m_{ij}^E \right).$$

Given this, the linear phase measurement equation (Eq. (19)) can be changed to the simpler form with a range-like measurement $Z_j$.

$$Z_j = -r_{1p} + \varepsilon_{ij}$$

To sum up, the TOA synchronization model is characterized by Eq. (57) that represents the map $\tau_{ij} \rightarrow \bar{\phi}_{ij}^X$. The linear measurement equation has the simpler form of Eq. (58) with the measurement error composed of the TOA error, the linearization error and the pulse phase propagation error. The linearization error and the pulse phase propagation error are of the order of 0.1 $\mu$s at given conditions as discussed in previous sections. They are comparatively small because the TOA error estimated by the signal-to-noise ratio (SNR) for the brightest pulsar is of $1$ $\mu$s order.4,3

5. Simulation and Results

The numerical simulation is designed to demonstrate the effect of the TOA synchronization technique. The spacecraft (GEO satellite) is assumed to be navigated by observing three pulsars. The orbit dynamics is established in the Earth-fixed reference system; i.e., the international terrestrial reference system (ITRS).16 so that the dynamics model is weakly non-linear.19 The TOA measurements are simulated according to Eq. (17) and the forward timing model. The initial position of the spacecraft is assumed to be coarsely known so that the cycle ambiguity is easily acquired. The EKF is used to best estimate the navigation states. Multiple TOAs are incorporated every 1,000 s into the EKF after being synchronized.

The navigation states are chosen to be the spacecraft’s position and velocity in the ITRS; i.e., $X = [r_{1,ITRS}, v_{1,ITRS}]^T$, where $r_{1,ITRS} = [x, y, z]^T$ and $v_{1,ITRS} = [v_x, v_y, v_z]^T$. If the onboard clock error dominates the measurements, they can be modeled by two states: the clock bias and the clock drift. However, the clock is assumed to be precise in this paper so the clock error states are omitted, which makes no difference for the analysis of the synchronization performance. The dynamics equation is established as the following.

$$\dot{X} = AX + f(X) + Ga_{d,ITRS}$$

where $a_{d,ITRS}$ stands for the disturbance acceleration, and

$$f(X) = \begin{bmatrix} 0, 0, 0, \frac{\partial U_0}{\partial x}, \frac{\partial U_0}{\partial y}, \frac{\partial U_0}{\partial z} \end{bmatrix}^T$$

$$A = \begin{bmatrix} \theta_{3 \times 3} & I_3 \\ \omega_0^2 & 0 \\ 0 & \omega_0^2 & 2\omega_0 & 0 \\ 0 & 0 & -2\omega_0 & 0 \end{bmatrix}$$

$$G = [\theta_{3 \times 3}, I_3]^T$$

From Eq. (58), the measurement equation can be written as (with $N = 3$)

$$Z = HX + V$$

where

$$Z = [Z_j]_{N \times 1}, \quad V = [\varepsilon_{ij}]_{N \times 1},$$

$$H = \begin{bmatrix} \dot{\mathbf{R}}_{0,ICRS}^T, D_{1C}^T, \theta_{N \times 3} \end{bmatrix},$$

and

$$\dot{\mathbf{R}}_{0,ICRS} = \begin{bmatrix} \cos \delta_j \cos \alpha_j, \cos \delta_j \sin \alpha_j, \sin \delta_j \end{bmatrix}^T$$

The three pulsars chosen to be the top three from

![Fig. 4. The navigation estimate SDs with the left figure being the range estimate SD and the right one being the speed estimate SD.](Note: The subplot is an enlarged view of the second-day values).
Tables 3–5 of Ref. 4) are PSR B0531+21, PSR B1821–24 and PSR B1937+21. Their range estimate errors (1σ) are respectively 328 m, 241 m and 250 m at conditions of 1 m² detector, 1,000 s integration time and SNR limited to 1,000.⁴¹

The simulation time span is two days from MJD 55742 to MJD 55744. The standard deviations (SD) of the range estimates and the speed estimates are plotted in Fig. 4. The SDs are given by extracting square roots of diagonal values of the EKF variance matrices. The results of Fig. 4 show that the SDs take one day to converge at a steadily low level.

The performance of navigation with multiple TOAs synchronized is simulated using the Monte-Carlo technique. The TOAs are the arrival times nearest the middle of the observation interval; i.e., the navigation epoch. The misalignment between the TOAs and the navigation epoch will be no more than 1 s because the pulse periods of most normal pulsars are of the order of seconds. Thus, to simulate a worse condition, the misalignment is set to be approximately 1 s. Before being incorporated into the EKF, multiple TOAs are synchronized. The results are presented as the root-mean squared errors (RMSE) of range and speed estimates (see Fig. 5). Each RMSE point is calculated from 100 independent estimate samples. The range estimate RMSE is finally below 150 m and the speed estimate RMSE is finally below 0.012 m/s. The trend and the value of the RMSE curves (Fig. 5) accord roughly with those of the SD curves (Fig. 4), not only because the Monte-Carlo technique uses RMSE to give a good realization of the SD, but also for the fact that the TOA synchronization technique in the paper is accurate and effective for pulsar-based navigation.

The case when multiple TOAs are badly synchronized (using Eq. (53) for pulse phase propagation) is also simulated. The navigation estimate RMSEs in this case are plotted in Fig. 6. It is seen in Fig. 6 that the RMSEs clearly deviate from the SDs, which reach 40,000 m in range and 2.5 m/s in speed. Although TOAs are misaligned for only 1 s, improper synchronizing introduces large errors.

6. Conclusions

We investigated the problems associated with synchronizing of multiple TOAs for pulsar-based navigation in this paper. Simulation results showed that the technique using the APF-based pulse phase propagation to synchronize TOAs was accurate and effective for pulsar-based navigation. Although only near-Earth spacecraft were used to demonstrate our technique, the methods of APF calculation, pulse phase propagation and TOA synchronization are applicable in the whole solar system, and the measurement linearization method can also be used when spacecraft are close to any known body. We recommend avoiding use of this technique when the spacecraft is in maneuver because a highly non-linear trajectory will increase the pulse propagation error. Our investigation is based on the assumption that initial navigation information is roughly known. Without any space apriori information, it would be interesting to learn the combined problem associated with TOA synchronization, APF detection and cycle ambiguity resolution in the future. We expect our TOA synchronization technique to be useful for performance improvement of pulsar-based navigation.
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References