Escape Trajectories from the L_2 Point of the Earth-Moon System

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To use the L_2 point of the Earth-Moon (EM) system as a space transportation hub, it is absolutely necessary to know the available orbital energy of a spacecraft departing from EM L_2 and how to obtain required excess velocity to the target planet. This paper presents a successful acceleration strategy using the combination of an impulsive maneuver and the Sun gravitational perturbation. Applying this strategy enables the spacecraft to efficiently convert the energy of the maneuver into its orbital energy.

Key Words: Escape Trajectories, L_2 Point of the Earth-Moon System, Sun-Earth-Moon Four-Body Problem

Nomenclature

a : semi-major axis
 L : Lagrangian
 G : gravitational constant
 M : mass
 r, R : distance
 U : potential
 ΔV : maneuver
 x, y, z : Cartesian coordinate
 θ : angle
 μ : mass parameter
 ω : frequency

Abbreviations

SE: Sun-Earth
EM: Earth-Moon

BCR4BP: bicircular restricted four-body problem

Subscripts

S: Sun
E: Earth
M: Moon
L: libration point

1. Introduction

1.1. Background

The locations where the gravitational and centrifugal acceleration balance each other in the restricted three-body system are known as the libration points (Euler¹⁾ and Lagrange²⁾). There exist five equilibria in the system and they are constant with regard to the rotating two bodies. Three of them, L_1 , L_2 and L_3 , are located on the line passing through the two bodies and the other two, L_4 and L_5 , complete equilateral triangles with the bodies. The L_1 and L_2 points of the Sun-Earth (SE) and Earth-Moon (EM) systems attract lots of attention for various space uses, such as obser-

vation and communication. Among them, EM L_2 has recently attracted attention again as a new base for long-duration habitation. It is considered as a desirable candidate site for extended ISS operations, which are very important to understand human space capabilities.

1.2. Problem statement

It is a major topic of astrodynamics how to gain the necessary energy to navigate a spacecraft in the interplanetary space. Over 3.0 km/s of the relative velocity with respect to the Earth is required in order to reach even the nearest outer planet, Mars. In considering the use of EM L_2 as a hub for interplanetary transfer, it is absolutely necessary to know the available orbital energy of the spacecraft departing from this point and the acceleration strategies to obtain prescribed relative velocity.

1.3. Related researches

Gobet and Doll³⁾ surveyed various types of escape maneuvers in the two-body problem. They summarized the available escape speed of one-/two-/three-impulsive transfer from circular orbit as a function of the magnitude of the impulse. Their results formed the basis of this study.

As to the escape from L_2 of the Sun-Earth system, Matsumoto and Kawaguchi⁴⁾ investigated one-impulse modes and derived effective acceleration patterns. They conducted an exhaustive search on different magnitude- and different direction-impulsive trajectories and showed desirable control strategy. They also showed trajectories with multi-impulse or low continuous acceleration. Nakamiya and Yamakawa⁵⁾ conducted optimization calculations on escape trajectories from SE L_2 and calculated how the control should be to attain a certain energy at the boundary of the Earth gravitational sphere.

One common explicit solution from their studies was that the maneuver works most efficiently when it is applied at the position where the spacecraft has highest velocity. This property held true with regard to the escape trajectories from EM L_2 .

Fig. 1. Sun-EM rotating coordinate. The origin is located at the barycenter of the Earth-Moon system. The x axis lies along the line connecting the Sun with the EM barycenter. The Earth and Moon revolve counter-clockwise about the origin.

1.4. Contribution

In this study, the escape trajectories from the L₂ point of the Earth-Moon system to the interplanetary space are considered. The paper begins with the direct escape schemes, which are the easiest to achieve. The spacecraft leaves the libration point with an impulsive ΔV and directly arrives at the boundary of the Earth gravitational region. Next, the technique of the trajectory-correction using the Sun perturbation is proposed. The spacecraft is guided to pass close to the Earth by the Sun gravity and experiences the Earth gravitational assist with an impulsive acceleration, realizing the high escape energy. The ΔV of 400 m/s at the perigee can generate escape velocity of 3.0 km/s at the boundary of the Earth gravity.

2. Preliminary Development

2.1. Equations of motion

In the Sun-Earth-Moon bicircular restricted four-body problem (BCR4BP), three bodies are assumed to move in a circular motion in the same plane. The Earth and Moon revolve around their barycenter and the Earth-Moon barycenter revolves around the barycenter of all the system. Here is a brief description of the development of the equations of the motion of the BCR4BP.

The potential of the Sun-Earth-Moon BCR4BP is expressed in the form

$$U = -GM_S \left(\frac{1}{|r_S - r|} - \frac{r_S \cdot r}{|r_S|^3} \right) - \frac{GM_E}{|r_E - r|} - \frac{GM_M}{|r_M - r|} \quad (1)$$

where G is the gravitational constant. r is the position vector with respect to the Earth-Moon barycenter and M_i stands for the mass of the body i . The subscripts, S, E and M, indicate that the value relates to the Sun, Earth and Moon, respectively.

For simplicity, the system is normalized with the following values. The unit of mass and length are selected as the sum of the masses of the Sun, Earth and Moon and the distance between the Sun and the Earth-Moon barycenter. The

unit of time is selected such that the orbital period of the Sun and the Earth-Moon barycenter becomes 2π and the orbital mean motion is 1. In addition, the following mass parameters are defined.

$$\begin{aligned} \mu_{SE} &= \frac{M_E + M_M}{M_S + M_E + M_M} \\ \mu_{EM} &= \frac{M_M}{M_E + M_M} \end{aligned} \quad (2)$$

μ_{SE} is a mass ratio of the Earth-Moon system with regard to the Sun-Earth-Moon system and μ_{EM} is a mass ratio between the Moon and the Earth-Moon system.

In the normalized rotating coordinate where the origin is located at the Earth-Moon barycenter and the negative x axis lies along the line toward the Sun as shown in Fig. 1, the position vectors can be expressed as

$$r_S = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad r_E = \begin{pmatrix} R_E \cos \theta_E \\ R_E \sin \theta_E \\ 0 \end{pmatrix}, \quad r_M = \begin{pmatrix} R_M \cos \theta_M \\ R_M \sin \theta_M \\ 0 \end{pmatrix} \quad (3)$$

with

$$\begin{aligned} R_M &= (1 - \mu_{EM})a_M \\ R_E &= \mu_{EM}a_M \\ \theta_M &= \omega_M t + \theta_{M0} \\ \theta_E &= \theta_M + \pi \end{aligned} \quad (4)$$

where a_M denotes the distance between the Earth and Moon. ω_M is the mean motion of the Earth-Moon system in the rotating frame. Normalizing Eq. (1) and substituting Eq. (3) into Eq. (1) yield the simplified form of U as

$$U = -(1 - \mu_{SE}) \left(\frac{1}{r_S} + x \right) - \frac{\mu_{SE}(1 - \mu_{EM})}{r_E} - \frac{\mu_{SE}\mu_{EM}}{r_M} \quad (5)$$

with

$$\begin{aligned} r_S^2 &= (x + 1)^2 + y^2 + z^2 \\ r_E^2 &= (x - R_E \cos \theta_E)^2 + (y - R_E \sin \theta_E)^2 + z^2 \\ r_M^2 &= (x - R_M \cos \theta_M)^2 + (y - R_M \sin \theta_M)^2 + z^2 \end{aligned} \quad (6)$$

Then the Lagrangian L in the rotating frame is expressed as

$$L = \frac{1}{2} ((\dot{x} - y)^2 + (\dot{y} + x)^2 + \dot{z}^2) - U \quad (7)$$

Finally, the Euler-Lagrange equation gives the equations of motion of the BCR4BP

$$\begin{aligned} \ddot{x} - 2\dot{y} - x &= -U_x \\ \ddot{y} + 2\dot{x} - y &= -U_y \\ \ddot{z} &= -U_z \end{aligned} \quad (8)$$

2.2. Libration points

The position of the collinear libration point is represented as a relative position with regard to the primary bodies. The distance γ between the smaller mass and L₂ is obtained by Szebehely⁶⁾ and is expressed in the form

$$\gamma_{L2} = r_h \left(1 + \frac{1}{3} r_h - \frac{1}{9} r_h^2 - \frac{31}{81} r_h^3 + \dots \right) \quad (9)$$

with the Hill radius r_h defined by

$$r_h = \left(\frac{\mu}{3(1-\mu)} \right)^{\frac{1}{3}} \quad (10)$$

Thus the position of EM L_2 in the Earth-Moon rotating frame is obtained as

$$x_{L2} = (x_M + \gamma_{L2}, 0, 0)^T \quad (11)$$

where x_M denotes the position of the Moon in the x direction.

2.3. Earth gravitational boundary

On an ordinary planning of the interplanetary missions, the method of the patched conics is employed. A trajectory is divided into several parts depending the mission phase. The sphere of influence gives an estimation of the range where the gravity of the subject planet is dominant over other gravitational perturbations. The radius of it is generally defined in terms of the ratio between the Keplerian force and the perturbation of the planets and has the form of

$$r_{SOI} = \left(\frac{\mu}{1-\mu} \right)^{\frac{2}{5}} \quad (12)$$

According to this equation, the radius of the Earth gravitational sphere is calculated as 925,000 km. V-infinity, an incoming or outgoing velocity with respect to the subject planet, is generally defined as the velocity at the crossing of the sphere of the dominant planet.

It is, however, inconvenient to adopt this definition of the gravitational boundary on the escape problem with low orbital energy. This is because the Sun perturbation cannot be neglected and its influence works at least within the Hill's region of the Sun-Earth system of which radius is about 1,500,000 km. In this paper, the boundary of the Hill region is adopted as a new gravitational boundary of the Earth as shown in Eq. (13) and in Fig. 2.

$$r_{SOI} = r_h \quad (13)$$

2.4. Moon age

EM L_2 rotates around the barycenter of the EM system synchronizing the motion of the Moon. Thus the position

Fig. 2. Definition of the gravitational boundary of the Earth. It is represented as a spherical shell centered on the Earth of which radius is equal to the Hill radius.

of EM L_2 can be represented by that of the Moon as an angle measured from the Sun-EM line θ_M as shown in Fig. 3. In the following discussion, the term “initial Moon age” shall indicate the position of the Moon when the spacecraft departs from EM L_2 .

3. Escape from EM L_2

3.1. Natural escape

The trajectories from EM L_1 require maneuvers to open up the energetic barrier around EM L_2 and to reach the boundary of the Earth gravitational sphere, but this is not the case with the trajectories from EM L_2 . The potential of L_2 is higher than that of L_1 and thus a spacecraft leaving there can escape the Earth gravity field without maneuvers. In this section, the trajectories which can reach the exterior region of the Earth gravity from EM L_2 without maneuvers are discussed.

The following results are obtained by employing the Sun-Earth-Moon BCR4BP of Eq. (8). The initial position of the spacecraft, which is identical with that of EM L_2 , is obtained from Eq. (11) as the value in the Earth-Moon rotating frame and is transformed to the values in the Sun-EM rotating coordinate.

Figure 4 shows the escape position of the trajectories from EM L_2 as a function of the initial Moon age. The position is expressed as the angle measured from the x axis in the Sun-EM rotating frame. These non-maneuver transfers can be divided into two categories depending on their escape directions. Trajectories of one group escape to around -30 degrees (i.e., in the direction of the anti-Sun as shown in Fig. 5) and the others are symmetrical about the Earth as shown in Fig. 6. These two families of trajectories are generated by the effect of the Sun gravitational perturbation, which accelerate the spacecraft along the x axis in the direction of leaving from the Earth.

Figure 7 shows the magnitude of the relative velocity with respect to the Earth at the boundary in the Earth-centered inertial coordinate. It reaches a maximum, 0.69 km/s, for the initial Moon age around 40 degrees and 220 degrees. The paths of the two have their apogee in the fourth or second quadrant respectively in the Sun-EM rotating frame and then escape. The Sun gravitational perturbation works to accelerate the path along the x axis and that is why the escape velocity increases.

Fig. 3. Definition of the initial Moon age.

Fig. 4. Passing position of the escape trajectory at the boundary as a function of the initial Moon age.

Fig. 5. Family of the natural escape trajectories heading for the $+x$ direction in the Sun-EM rotating frame. The initial Moon age is between 40 and 210 degrees.

Fig. 6. Family of the natural escape trajectories heading for the $-x$ direction in the Sun-EM rotating frame. The initial Moon age is between 220 and 30 degrees.

Fig. 7. Magnitude of the escape velocity in the Earth-centered inertial frame as a function of the initial Moon age.

3.2. Escape with an initial impulsive ΔV

The next scheme is an extended version of the natural escape discussed above. The difference is to add an initial impulsive ΔV at $EM L_2$. This is categorized as a one-impulsive trajectory and can be called a direct escape. Here, the word “direct” means that the spacecraft leaves the vicinity of the Earth-Moon system without encountering either the Earth or Moon.

The initial impulsive ΔV at $EM L_2$ is defined by its magnitude and direction as shown in Fig. 8. In the two-body problem, a tangential acceleration can most efficiently increase the orbital energy of the spacecraft. On the other hand, it is not the case in the three- or four-body problem, where the influence of the perturbations of the bodies other than the dominant one cannot be neglected. In addition, the start position also affects the results. Therefore, the direct escape trajectory has three independent parameters to select; i.e., the initial Moon age θ_M , the magnitude of the impulse $|\Delta V|$ and its direction $\theta_{\Delta V}$.

Figure 9 shows plots of trajectories with an initial impulsive ΔV when the initial Moon age is 270 degrees. The ΔV

Fig. 8. Geometry of the initial impulsive ΔV at $EM L_2$.

is supposed to be tangentially applied and its magnitude is selected from the values between 100 m/s and 500 m/s. This figure indicates the escape velocity and position at the boundary change depending on the magnitude of the initial ΔV . Figure 9 also shows the typical orbital profiles of this escape mode. A trajectory which has almost the same shape can be obtained under the condition of the other initial Moon age.

Fig. 9. Escape trajectories with an initial impulsive ΔV at EM L_2 in the Sun-EM rotating frame. The initial Moon age is 270 degrees and the impulsive ΔV is tangentially applied.

Fig. 11. Direction of the initial impulsive ΔV to achieve the escape velocity shown in Fig. 10.

Fig. 10. Magnitude of the escape velocity in the Earth-centered inertial frame as a function of the initial Moon age.

Figures 10 and 11 show the magnitude of the escape velocity at the boundary as a function of the initial Moon age and the corresponding injection angle of the impulsive ΔV , respectively. Five curves of each graph denote the values calculated under the different ΔV condition. We can see from Fig. 10 that the available escape velocity bears a proportionate relationship to the magnitude of the initial ΔV when it is large enough. This means that the more initial ΔV can be available, the more escape velocity the spacecraft can obtain at the boundary. For example, the spacecraft obtains an escape velocity as much as 1.25 km/s at a maximum when the magnitude of the initial ΔV is 500 m/s. On the other hand, when it is small, the kinetic energy of the spacecraft remains small and the following motion is governed by the high nonlinearity of the multi-body system. As a result, the escape velocity loses the proportional relationship to the magnitude of the initial ΔV .

Figure 11 also shows the desirable ΔV direction at EM L_2 . When the initial ΔV is large, the magnitude of the escape velocity reaches the maximum around 90 degrees, which means the tangential ΔV can accelerate the space-

craft most efficiently. Meanwhile, when it is small, there is a maximum of the escape velocity at the ΔV angle other than 90 degrees due to the non-linearity of the system. In addition, when the inappropriate ΔV angle is selected, the spacecraft could proceed in the direction of the Earth ending up trapped by its gravity, which means the failure to escape to the interplanetary space.

3.3. Escape with the Earth powered swing-by

The so-called three-impulsive escape normally requires three independent maneuvers at the initial, apoapsis and periapsis. The Sun-Earth-Moon four-body system, however, can remove the first and second impulse by the effective use of the perturbations of the bodies instead. The following discussion begins by designing trajectories which can pass nearby the Earth and then constructs an escape mode with the perigee impulsive burn.

3.3.1. Earth swing-by trajectories

Plots of the Earth swing-by trajectories from EM L_2 are shown in Figs. 12 and 13. The spacecraft departs from L_2 of the Earth-Moon system and moves along the unstable manifold associated with EM L_2 . Then it is slowed down by the Sun gravitational perturbation and finally arrives at the Earth and its vicinity. To obtain a trajectory which returns to the Earth, it should be allowed to cross the boundary once. These types of trajectories are very sensitive to the initial Moon age and need to choose an appropriate departure date. A few degrees difference of the initial Moon age results in different approaches to the Earth. Figures 12 and 13 clearly show its sensitivity. When the initial Moon age is 204.21108 degrees, the spacecraft approaches the Earth from the left, and when it is 206.83193 degrees, it does from the right. They both get a perigee of 1,000 km height from the surface of the Earth and have approximately the same value of the perigee speed, 11.1 km/s. If the spacecraft departs from EM L_2 at the time between the two cases, it ends up the colliding with the Earth.

3.3.2. Earth powered swing-by trajectories

Applying an impulsive ΔV at the perigee can increase the orbital energy. A plot of the trajectory with the Earth pow-

Fig. 12. Escape trajectory with the Earth swing-by in the Sun-EM rotating frame. The initial Moon age is 204.21108 degrees and the altitude of the perigee is 1,000 km high.

Fig. 14. Escape trajectory with the Earth powered swing-by in the Sun-EM rotating frame. The initial Moon age is 204.21108 degrees. The spacecraft passes by the Earth at an altitude of 1,000 km with 100 m/s impulsive ΔV .

Fig. 13. Escape trajectory with the Earth swing-by in the Sun-EM rotating frame. The initial Moon age is 206.83193 degrees and the altitude of the perigee is 1,000 km high.

Fig. 15. Magnitude of the escape velocity in the Earth-centered inertial frame as a function of the impulsive ΔV at the perigee. The initial Moon age is 204.21108 degrees.

ered swing-by is shown in Fig. 14. The path from EM L_2 to the pre-swing-by is all the same as the trajectory represented before. Applying the tangential impulsive ΔV of different magnitudes at the perigee can generate various escape trajectories.

There is clearly a major difference of the magnitude of the escape velocity when one compares the trajectories of Figs. 12 and 14. The result insists that applying the impulsive ΔV of only 100 m/s at the perigee generates the difference of the escape velocity of about 1,000 m/s at the boundary.

Figure 15 shows how the magnitude of the escape velocity increases with the magnitude of the impulsive ΔV at the perigee. Two lines denote the values under a different swing-by altitude; i.e., 1,000 km and 10,000 km from the surface of the Earth. When the spacecraft swings by the Earth without the ΔV , the gained escape velocity at the boundary is small and the corresponding orbital energy with regard to the Earth becomes negative. The negative energy in the two-body problem means the spacecraft is captured in

an elliptical orbit and cannot escape, but the multi-body system can accept low energy and escapes rather than being captured in a hyperbolic-type orbit. In this figure, the monotonically increasing of the escape velocity with respect to the magnitude of the impulsive ΔV at the perigee can be found. If the spacecraft obtains the ΔV of over 400 m/s at the perigee of 1,000 km altitude, the following escape velocity reaches nearly 3.0 km/s. However, the larger the swing-by radius is, the less the efficiency of the acceleration becomes.

A plot of the Earth powered swing-by trajectory of $\theta_M = 206.83193$ degrees is shown in Fig. 16 and its escape property is shown in Fig. 17. The escape velocity and flight time at the crossing of the boundary almost keep the same level of the previous escape trajectory. The difference between the two lies in the position of the escape. Because they have a different approach direction to the Earth, the post-swing-by trajectories proceed in a different direction. In the previous case, the angle of escape monotonically increases with the magnitude of the ΔV at the perigee, but in this case the angle reaches a maximum for the ΔV near 30 m/s and

Fig. 16. Escape trajectory with the Earth powered swing-by in the Sun-EM rotating frame. The initial Moon age is 206.83193 degrees. The spacecraft passes by the Earth at an altitude of 1,000 km with 100 m/s impulsive ΔV .

Fig. 17. Magnitude of the escape velocity in the Earth-centered inertial frame as a function of the impulsive ΔV at the perigee. The initial Moon age is 206.83193 degrees.

gradually decreases. For larger value of the ΔV , the escape direction is almost parallel to the x axis.

4. Summary

4.1. What this paper showed

In this study, the escape trajectories from L_2 of the Earth-Moon system to the interplanetary field were investigated. The natural escape mode from EM L_2 had a merit of requiring no maneuver to arrive at the boundary of the Earth gravitational region but its available escape velocity was small as shown in Fig. 7. Since the orbital energy of the spacecraft was not high, the escaping path was limited to the two types: one headed for the $+x$ direction with respect to the Earth and the other for $-x$ direction (see Figs. 5 and 6).

The escape with an initial impulsive ΔV was easy to achieve and could get higher energy than the natural escape mode as shown in Figs. 9 and 10. When the magnitude of the initial ΔV was large enough, it could be applied in the tangential direction to the orbit according to the theory of

the two-body problem. On the other hand, when the ΔV was small (around a few hundreds m/s), the optimal launch direction was not always tangential as shown in Fig. 11 and it also depended on the initial Moon age. Thus it had no choice but to be numerically calculated.

The third option, the indirect escape mode, referred to as the Earth powered swing-by trajectory in this paper, had a great advantage of the ΔV efficiency. It had a similar shape to three-impulsive escape of the two-body system, but could be achieved with one-impulsive ΔV by utilizing the planets' gravity. This strategy could generate a larger escape velocity than the former two options. Figures 14 and 16 showed plots of such trajectories which successfully gained large escape velocity at the boundary by amplifying the energy through the perigee- ΔV . Figures 15 and 17 indicated that the velocity was accelerated up to nearly 3.0 km/s by applying the 400 m/s impulsive ΔV at the perigee of 1,000 km altitude. Thus, this strategy was recommended for missions aiming at the deep interplanetary space from EM L_2 .

4.2. Significance of the results

The result of this paper supported the possibility of EM L_2 as a space hub. By adopting Earth powered swing-by, a spacecraft could efficiently obtain large energy. This transfer scheme can be also combined with other orbital manipulation strategies, such as EDVEGA (Kawaguchi⁷).

4.3. Extension

This study proved that the Earth powered swing-by escape trajectory could generate large energy enough to reach other planets. However, in a real case, it is necessary to consider not only the available energy but the orientation of the Earth, Moon and target body. Thus the next step of this study is the trajectory design including the position investigation.

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