BANDO-CALABI-FUTAKI CHARACTER OF COMPACT TORIC MANIFOLDS

Dedicated to Professor Tadao Oda on his sixtieth birthday

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Abstract. The Bando-Calabi-Futaki character of a compact Kähler manifold is an obstruction to the existence of Kähler metrics with constant scalar curvature, which is a generalization of the Futaki character of a Fano manifold. In this paper, we study the Bando-Calabi-Futaki character of a compact toric manifold. In particular, we shall prove that the Bando-Calabi-Futaki character of a compact toric manifold vanishes on the Lie algebra of the unipotent radical of the automorphism group.

1. Introduction. Let $X$ be a compact connected $r$-dimensional complex manifold and $\eta \in H^2(X; \mathbb{R})$. We assume that $\eta$ is positive, that is, there exists a Kähler metric $g$ on $X$ such that its Kähler form

$$\omega_g := \sqrt{-1} \sum_{i,j=1}^{r} g_{ij} dz^i \wedge d\overline{z}^j$$

represents $2\pi \eta$ in the de Rham cohomology group $H^2_{dR}(X; \mathbb{R})$, where $(z^1, z^2, \ldots, z^r)$ is a local holomorphic coordinate on $X$. We denote by $\text{Aut}^0(X)$ the identity component of the group $\text{Aut}(X)$ of holomorphic automorphisms of $X$, whose Lie algebra is identified with the Lie algebra $H^0(X; \mathcal{O}(T^{1,0}X))$ of holomorphic vector fields on $X$. Here $T^{1,0}X$ is the holomorphic vector bundle of tangent vectors of type $(1,0)$ on $X$. Recall that the Albanese map of $X$ to the Albanese variety $\text{Alb}(X)$ naturally induces a Lie group homomorphism $\alpha_X : \text{Aut}^0(X) \to \text{Aut}^0(\text{Alb}(X)) \cong \text{Alb}(X)$.

Let $G_X$ be the identity component of the kernel of the homomorphism $\alpha_X$, and $\mathfrak{g}_X$ the corresponding Lie subalgebra of $H^0(X; \mathcal{O}(T^{1,0}X))$. Then, by a theorem of Fujiki [6], $G_X$ has a natural structure of a linear algebraic group (defined over $\mathbb{C}$). We denote by $U_X$ the unipotent radical of $G_X$. More generally, we consider a linear algebraic group $G$ (defined over $\mathbb{C}$) and a homomorphism $\rho : G \to \text{Aut}(X)$ of algebraic groups. By $\rho_* : \mathfrak{g} \to H^0(X; \mathcal{O}(T^{1,0}X))$, we denote the Lie algebra homomorphism induced from $\rho$, where $\mathfrak{g} := \text{Lie}(G)$ is the Lie algebra of $G$. 

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REMARK 1.1. (i) If $X$ is Fano, i.e., the first Chern class $c_1(X)$ of $X$ is positive, then $G_X = \text{Aut}^c(X)$.

(ii) If $X$ is an $r$-dimensional compact toric manifold, that is, $X$ is an $r$-dimensional compact irreducible non-singular variety defined over $C$ with an almost-homogeneous algebraic action of an $r$-dimensional algebraic torus $T_r := (C^*)^r$, then $G_X = \text{Aut}^c(X)$.

By $\text{Ric}_g$ and $s_g$, we denote the Ricci form and the scalar curvature of $g$, respectively, namely, we put

$$\text{Ric}_g = \sqrt{-1} \sum_{i,j=1}^r R_{i\overline{j}} dz^i \wedge d\overline{z}^j := -\sqrt{-1} \partial \overline{\partial} \log \det(g_{i\overline{j}}),$$

$$s_g := \sum_{i,j=1}^r g^{i\overline{j}} R_{i\overline{j}},$$

where $(g^{i\overline{j}})$ is the inverse matrix of $(g_{i\overline{j}})$. By means of the harmonic integration theory, there exists a real-valued $C^\infty$ function $f \in C^\infty(X)$ such that

$$s_g - r \mu_\eta = \Box_g f_g,$$

where $\Box_g := \sum_{i,j=1}^r g^{i\overline{j}} (\partial^2 / \partial z^i \partial \overline{z}^j)$ is the complex Laplacian for functions on the Kähler manifold $(X, g)$, and $\mu_\eta \in R$ is the constant defined by

$$\mu_\eta := \frac{(c_1(X) \cup \eta^{r-1})[X]}{\eta^r [X]} = \int_X s_g \left( \frac{\omega_g}{2\pi} \right)^r \in R.$$  

(1.2)

Bando [2], Calabi [4] and Futaki [9] defined an obstruction to the existence of Kähler metrics with constant scalar curvature as follows:

DEFINITION 1.3 (Bando [2], Calabi [4] and Futaki [9]). A linear functional $F^n_X : H^0(X; \mathcal{O}(T^{1,0}X)) \to C$ defined by

$$F^n_X(V) := \frac{1}{\sqrt{-1}} \int_X (V f_g) \left( \frac{\omega_g}{2\pi} \right)^r, \quad V \in H^0(X; \mathcal{O}(T^{1,0}X)),$$

is called the Bando-Calabi-Futaki character of $(X, \eta)$.

We now recall the following fundamental facts about the Bando-Calabi-Futaki characters:

FACT 1.4 (Bando [2], Calabi [4] and Futaki [9]). Let $X$ and $\eta$ be as above. Then we have the following:

(i) $F^n_X$ does not depend on the choice of $g$ satisfying $[\omega_g] = 2\pi \eta$.

(ii) If $X$ admits a Kähler metric $g$ with constant scalar curvature satisfying $[\omega_g] = 2\pi \eta$, then $F^n_X$ vanishes.
(iii) $F^u_X$ is a Lie algebra character of $H^0(X; \mathcal{O}(T^{1,0}X))$, that is,
$$F^u_X|_{[H^0(X; \mathcal{O}(T^{1,0}X)), H^0(X; \mathcal{O}(T^{1,0}X))]} \equiv 0.$$

**Remark 1.5.** If $\eta$ is the first Chern class $c_1(X)$ of $X$, then the Bando-Calabi-Futaki character $F_{c_1}(X)$ coincides with the original Futaki character, which was introduced in [8] as an obstruction to the existence of Einstein-Kähler metrics.

**Definition 1.6.** Let $\pi_E : E \to X$ be a holomorphic vector bundle of rank $k$ over $X$. We say that $E$ is $(G, \rho)$-linearized if $G$ acts on $E$ birationally in such a way that
(i) $\pi_E \circ \gamma = \rho(\gamma) \circ \pi_E$ for any $\gamma \in G$;
(ii) for any $\gamma \in G$ and $p \in X$,
$$\gamma|_{E_p} : E_p \to E_{\rho(\gamma)(p)}$$
is a $C$-linear map, where $E_p := \pi_E^{-1}(p)$ is the fiber of $\pi_E$ at $p \in X$.

Furthermore, if $G$ is a subgroup of $\text{Aut}(X)$ and $\rho$ is the inclusion map, then we simply say that $E$ is $G$-linearized.

In [15], the author proved the following:

**Fact 1.7 (Nakagawa [15]).** Let $X$ and $\eta$ be as above. We assume that there exists a holomorphic line bundle $L$ over $X$ such that $L$ is $G_X$-linearized and $c_1(L) = \eta$, where $c_1(L)$ is the first Chern class of $L$. Then
$$F^u_X|_{u_X} \equiv 0,$$
where $u_X := \text{Lie}(U_X)$ is the Lie algebra of $U_X$.

The main purpose of this paper is to generalize this fact to the case of a more general situation, that is, we shall prove the following theorem:

**Theorem 1.8.** Let $X$, $\eta$, $G$ and $\rho$ be as above. We assume that there exists a holomorphic line bundle $L$ over $X$ such that $L$ is $(G, \rho)$-linearized and $c_1(L) = \eta$. Then
$$(F^u_X \circ \rho_*)|_u \equiv 0$$
for any unipotent subgroup $U \subseteq G$ with Lie algebra $u := \text{Lie}(U)$.

As an application of this theorem, we shall also prove the following theorem:

**Theorem 1.9.** Let $X$ be an $r$-dimensional compact toric manifold. By definition, an $r$-dimensional algebraic torus $T_r := (\mathbb{C}^*)^r$ acts on $X$ holomorphically; hence the Lie algebra $t_r := \text{Lie}(T_r)$ of $T_r$ is regarded as a Lie subalgebra of $H^0(X; \mathcal{O}(T^{1,0}X))$. If $\eta \in H^2(X; \mathbb{Z})$ is positive, then the following are equivalent, without any assumptions concerning a linearization of the natural action of $\text{Aut}(X)$ on $X$:
(i) $F^u_X$ vanishes identically on $H^0(X; \mathcal{O}(T^{1,0}X))$.
(ii) $F^u_X$ vanishes on $t_r$.

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2. Bando-Calabi-Futaki characters as holomorphic invariants (Proof of Theorem 1.8). Throughout this section, we fix a compact connected $r$-dimensional complex manifold $X$, a positive class $\eta \in H^2(X; \mathbb{R})$, a linear algebraic group $G$ (defined over $\mathbb{C}$) and a homomorphism $\rho: G \rightarrow \text{Aut}(X)$ of algebraic groups. Let $\pi_E: E \rightarrow X$ be a holomorphic vector bundle of rank $k$ over $X$. We assume that $E$ is $(G, \rho)$-linearized. Then, for any $V \in \mathfrak{g}$, a holomorphic action (see [3])

$$A^E_V: A^0(E) \rightarrow A^0(E),$$

of $V$ on $E$ is induced, that is, $A^E_V$ satisfies the following properties:

(i) $A^E_V$ is a $\mathbb{C}$-linear map.

(ii) For all $\psi \in C^\infty(X)_\mathbb{C}$ and $s \in A^0(E)$,

$$A^E_V(\psi s) = ((\rho_\ast V) \psi) s + \psi A^E_V s.$$

(iii) $A^E_V$ commutes with $\partial$, i.e., $\partial A^E_V = A^E_V \partial$.

Here we denote by $A^p(E)$ the space of $E$-valued $p$-forms on $X$ for $p = 0, 1, \ldots, r$.

**Example 2.1.** $E = T^1,0 X$ is canonically $\text{Aut}(X)$-linearized. In this case, $\Lambda^{T^1,0 X} V$ is the Lie differentiation $LV$ of vector fields with respect to a holomorphic vector field $V \in H^0(X; \mathcal{O}(T^1,0 X))$ on $X$.

Let $h$ be a Hermitian metric on $E$ and $\nabla^h: A^0(E) \rightarrow A^1(E)$ the Hermitian connection of $h$ (see for instance [12, p. 12]). We define the curvature $\Theta^h$ of $\nabla^h$ by

$$\Theta^h = \partial(h^{-1} \partial h) \in A^2(\text{End}(E)).$$

where $\text{End}(E)$ is the endomorphism bundle of $E$ over $X$. For each $V \in \mathfrak{g}$, we put $\mathcal{L}^{(E, h)}_V := \nabla^h - A^E_V \in A^0(\text{End}(E))$. Let $l \in \mathbb{Z}_{\geq 0}$ be a non-negative integer and $s$ a $GL(k, \mathbb{C})$-invariant symmetric polynomial of degree $r + l$ on $\mathfrak{gl}(k, \mathbb{C})$ (see [10, p. 21]). For example, $c^{r+l}_1 := ((\sqrt{-1}/2\pi) tr)^{r+l}$ is a $GL(k, \mathbb{C})$-invariant symmetric polynomial of degree $r + l$ on $\mathfrak{gl}(k, \mathbb{C})$. We now define a map $C^E_{\phi}: \mathfrak{g} \rightarrow \mathbb{C}$ by

$$C^E_{\phi}(V) := \int_X \phi(\mathcal{L}^{(E, h)}_V + \Theta^h), \quad V \in \mathfrak{g}.$$

For this map $C^E_{\phi}$, we can prove the following facts:

**Fact 2.2 (cf. Futaki and Morita [11]).** Let $X, (E, h)$ and $\phi$ be as above. Then we have the following:

(i) $C^E_{\phi}$ dose not depend on the choice of a Hermitian metric $h$ on $E$, i.e., $C^E_{\phi}$ is a holomorphic invariant of $(X, E)$.

(ii) $C^E_{\phi}$ is a $G$-invariant symmetric polynomial of degree $l$ on $\mathfrak{g}$. In particular, if $l = 1$, then $C^E_{\phi}$ is a character of the Lie algebra $\mathfrak{g}$.
(iii) For any \( V \in H^0(X; \mathcal{O}(T^{1,0}X)) \),

\[
F_{X}^{c_1} (V) = - \frac{2\pi}{r+1} c_1^{r+1}_{T^{1,0}X} (V) = - \frac{2\pi}{r+1} c_1^{r+1}_{K_X^{-1}} (V),
\]

where \( T^{1,0}X \) and \( K_X^{-1} := \det T^{1,0}X = \bigwedge^r T^{1,0}X \) are regarded as \( \text{Aut}(X) \)-linearized bundles over \( X \) in terms of the canonical \( \text{Aut}(X) \)-actions on them.

Let \( g' \) be an arbitrary Hermitian metric on \( X \). For \( V \in \mathfrak{g} \), if a point \( p \in X \) is a zero point of \( \rho_* V \in H^0(X; \mathcal{O}(T^{1,0}X)) \), then \( \mathcal{L}_{\rho_* V}^{(T^{1,0}X,g')} \) induces the linear map

\[
\mathcal{L}_{\rho_* V, p}^{(T^{1,0}X,g')} : T^{1,0}_p X \to T^{1,0}_p X.
\]

\( V \in \mathfrak{g} \) is said to be non-degenerate if the following two conditions hold:

(i) The zero set \( \text{Zero}(\rho_* V) \) of \( \rho_* V \) is finite.

(ii) For each zero point \( p \in \text{Zero}(\rho_* V) \) of \( \rho_* V \), the linear map

\[
\mathcal{L}_{\rho_* V, p}^{(T^{1,0}X,g')} : T^{1,0}_p X \to T^{1,0}_p X
\]

is non-singular.

The following localization formula for \( \mathcal{C}_E^\phi \) allows us to calculate explicitly the Bando-Calabi-Futaki character of a compact toric manifold (see Corollary 4.6):

**Fact 2.3** (Bott [3]). Let \( X, (E, h) \) and \( \phi \) be as above, and \( V \in \mathfrak{g} \) a non-degenerate element. Then we have

\[
\mathcal{C}_E^\phi (V) = \sum_{p \in \text{Zero}(V)} \frac{\phi (\mathcal{L}_{(E,h), p}^{(T^{1,0}X,g')})}{\sqrt{-1} \det \frac{\mathcal{L}_{\rho_* V, p}^{(T^{1,0}X,g')}}{2\pi}},
\]

where \( g' \) is an arbitrary Hermitian metric on \( X \).

Now, we assume that there exists a holomorphic line bundle \( L \) over \( X \) such that \( L \) is \( (G, \rho) \)-linearized and \( c_1(L) = \eta \). Under this assumption, an argument similar to that in [17, Section 6] allows us to prove the following Tian’s formula for the Bando-Calabi-Futaki character (see also [15, Section 3]):

**Theorem 2.4** (Tian [17]). Let \( X, \eta, G, \rho \) and \( L \) be as above. Then, for any integer \( \delta \in \mathbb{Z} \) and \( V \in \mathfrak{g} \), we have

\[
F_{X}^{\delta} (\rho_* V) = - \frac{2\pi}{2^r(r+1)!} \sum_{j=0}^{r} (-1)^j \binom{r}{j} c_1^{r+1}_{K_X^{-1} \otimes L^{\delta+r-2j}} (V)
+ 2\pi \left( \delta + \frac{r \mu_\eta}{r+1} \right) c_1^{r+1}_L (V),
\]

where \( L^{\delta+r-2j} := L^{\delta+r-2j} \) is the \( (\delta + r - 2j) \)-th tensor power of \( L \). Here we regard \( K_X^{-1} \otimes L^{\delta+r-2j} \), \( j = 0, 1, \ldots, r \), as \( (G, \rho) \)-linearized line bundles by the canonical \( \text{Aut}(X) \)-action on \( K_X^{-1} \).
Together with this Tian’s formula, the following fact implies Theorem 1.8 by the same argument as that in [15, Section 4]:

**FACT 2.5 (Mabuchi [13]).** Let $X, G, \rho$ and $L$ be as above. Then, for any unipotent subgroup $U$ of $G$, $c_L^{r+1}$ vanishes on the Lie algebra $\mathfrak{u} := \text{Lie}(U)$ of $U$.

### 3. Bando-Calabi-Futaki character of compact toric manifolds (Proof of Theorem 1.9)

First, we recall some basic notions and facts concerning toric manifolds (see [16] for more details). Let $\mathfrak{T}_r := (\mathbb{C}^\times)^r$ be an $r$-dimensional algebraic torus. We put $N := \mathbb{Z}^r$ and $M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z}) \equiv \mathbb{Z}^r$, where we regard elements of $N$ and $M$ as $r$-dimensional column vectors and row vectors, respectively. Let $\Sigma$ be a complete non-singular fan in $N$ (see [16] for the definition of a complete non-singular fan) and $\Sigma(i)$ the set of $i$-dimensional cones in $\Sigma$ for $i = 0, 1, \ldots, r$. We denote by $X_\Sigma$ the $r$-dimensional compact toric manifold associated with $\Sigma$. Then $\mathfrak{T}_r$ acts on $X_\Sigma$ biholomorphically, and $X_\Sigma$ has an open dense $\mathfrak{T}_r$-orbit $O_\Sigma$ isomorphic to $\mathfrak{T}_r$.

**FACT 3.1 (Cox [5]).** Let $\Sigma$ be a complete non-singular fan in $N$ and $d_\Sigma := \# \Sigma(1)$ the number of the one-dimensional cones in $\Sigma$. Then:

(i) There exists a $(d_\Sigma - r)$-dimensional algebraic subtorus $H_\Sigma$ of $(\mathbb{C}^\times)^{d_\Sigma}$ and an $H_\Sigma$-invariant open subset $W_\Sigma$ of $\mathbb{C}^{d_\Sigma}$ such that $H_\Sigma$ acts freely on $W_\Sigma$ and

$$X_\Sigma = W_\Sigma / H_\Sigma.$$ 

Here the $H_\Sigma$-action on $\mathbb{C}^{d_\Sigma}$ is induced from the canonical $(\mathbb{C}^\times)^{d_\Sigma}$-action on $\mathbb{C}^{d_\Sigma}$.

(ii) Let $\tilde{G}_\Sigma$ be the centralizer of $H_\Sigma$ in $\text{Aut}(W_\Sigma)$. Then

$$\text{Aut}^0(X_\Sigma) \cong \tilde{G}_\Sigma / H_\Sigma.$$ 

(iii) $\tilde{G}_\Sigma$ and $\text{Aut}^0(X_\Sigma)$ are connected linear algebraic groups (defined over $\mathbb{C}$). Let $\tilde{U}_\Sigma$ and $U_\Sigma$ be the unipotent radicals of $\tilde{G}_\Sigma$ and $\text{Aut}^0(X_\Sigma)$, respectively. Then

$$\rho_\Sigma|_{\tilde{U}_\Sigma} : \tilde{U}_\Sigma \to U_\Sigma$$

is an isomorphism, where $\rho_\Sigma : \tilde{G}_\Sigma \to \text{Aut}^0(X_\Sigma)$ is the natural projection induced by the isomorphism $\text{Aut}^0(X_\Sigma) \cong \tilde{G}_\Sigma / H_\Sigma$. Furthermore, there exists a reductive algebraic subgroup $R_\Sigma$ of $\text{Aut}^0(X_\Sigma)$ with $\mathfrak{T}_r$ as a maximal algebraic torus such that

$$\text{Aut}^0(X_\Sigma) = R_\Sigma \ltimes U_\Sigma.$$ 

**EXAMPLE 3.2.** A typical example of an $r$-dimensional compact toric manifold is the $r$-dimensional complex projective space $\mathbb{P}(\mathbb{C})$. If $X_\Sigma = \mathbb{P}(\mathbb{C})$, then we have:

- $d_\Sigma = r + 1$,
- $H_\Sigma = \{(t, t, \ldots, t) \in (\mathbb{C}^\times)^{r+1} ; t \in \mathbb{C}^\times \} \cong \mathbb{C}^\times$,
- $W_\Sigma = \mathbb{C}^{r+1} \setminus \{0\}$,
- $\tilde{G}_\Sigma = GL(r + 1, \mathbb{C})$,
- $\text{Aut}^0(X_\Sigma) = \text{Aut}(X_\Sigma) = \text{PGL}(r + 1, \mathbb{C})$. 

To each $v \in \Sigma(1)$, there corresponds a $T$-invariant Weil divisor $D_v$ on $X_\Sigma$. More generally, a map $\alpha: \Sigma(1) \to \mathbb{Z}$ defines a $T$-invariant Weil divisor $D(\alpha) := -\sum_{v \in \Sigma(1)} \alpha(v) D_v$, and we denote by $L_\alpha$ the $T$-linearized holomorphic line bundle over $X_\Sigma$ corresponding to $D(\alpha)$, i.e., $L_\alpha = \mathcal{O}(D(\alpha))$.

**Example 3.3.** Let $\Sigma$ be a complete non-singular fan in $\mathbb{N}$ and $X_\Sigma$ the compact toric manifold associated with $\Sigma$. Then the anti-canonical line bundle $K^{-1}_X$ of $X_\Sigma$ corresponds to the map $\alpha_0: \Sigma(1) \ni v \mapsto -1 \in \mathbb{Z}$, that is, $K^{-1}_X$ corresponds to the $T$-invariant Weil divisor $\sum_{v \in \Sigma(1)} D_v$.

If $L_\alpha$ is ample, that is, $c_1(L_\alpha) \in H^2(X_\Sigma; \mathbb{Z})$ is positive, then we say that $\alpha$ is ample. Let $\Sigma(1) = \{v_1, v_2, \ldots, v_{d_\Sigma}\}$ and put $\alpha_i := \alpha(v_i) \in \mathbb{Z}$ for $i = 1, 2, \ldots, d_\Sigma$. Then we define a character $\lambda_\alpha: (\mathbb{C}^*)^{d_\Sigma} \to \mathbb{C}^*$ of $(\mathbb{C}^*)^{d_\Sigma}$ by $\lambda_\alpha(s_1, s_2, \ldots, s_{d_\Sigma}) := s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{d_\Sigma}^{\alpha_{d_\Sigma}}$. $H_\Sigma$ acts on $W_\Sigma \times \mathbb{C}$ by $k: (z, \xi) \mapsto (k \cdot z, \lambda_\alpha(k)^{-1} \xi)$, where $k \in H_\Sigma, z \in W_\Sigma$ and $\xi \in \mathbb{C}$.

**Fact 3.4** (cf. Audin [1, Chapter VI]). The projection $W_\Sigma \to X_\Sigma$ is a principal $H_\Sigma$-bundle. Furthermore, the $T$-linearized holomorphic line bundle $L_\alpha$ over $X_\Sigma$ is given by $L_\alpha = W_\Sigma \times_{\lambda_\alpha} \mathbb{C} := (W_\Sigma \times \mathbb{C}) / H_\Sigma$.

**Proposition 3.5.** For $\alpha$ as above, $L_\alpha$ is the $(\tilde{G}_\Sigma, \rho_\Sigma)$-linearized holomorphic line bundle over $X_\Sigma$.

**Proof.** The natural $\tilde{G}_\Sigma$-action on $W_\Sigma$ commutes with the $H_\Sigma$-action on $W_\Sigma$. Then, by means of Fact 3.4, $\tilde{G}_\Sigma$ acts on $L_\alpha = W_\Sigma \times_{\lambda_\alpha} \mathbb{C}$ and $L_\alpha$ is $(\tilde{G}_\Sigma, \rho_\Sigma)$-linearized. $\square$

For any $\eta \in H^2(X_\Sigma; \mathbb{Z})$, in view of [7, Section 3.4], there exists a map $\alpha_\eta: \Sigma(1) \to \mathbb{Z}$ such that $c_1(L_{\alpha_\eta}) = \eta$. Therefore, Theorem 1.8 together with Fact 3.1 (iii) and Proposition 3.5 implies the following theorem:

**Theorem 3.6.** Let $\Sigma$ be a complete non-singular fan in $\mathbb{N}$ and $\eta \in H^2(X_\Sigma; \mathbb{Z})$ a positive class. Then the Bando-Calabi-Futaki character $F^\eta_{X_\Sigma}$ of $(X_\Sigma, \eta)$ vanishes on the Lie algebra $u_\Sigma := \text{Lie}(U_\Sigma)$ of $U_\Sigma$.

Recall that, for a reductive algebraic group $R$, 

$$\text{Lie}(R) = \text{Lie}(\text{Center}(R)) + [\text{Lie}(R), \text{Lie}(R)],$$ 

and $\text{Lie}(\text{Center}(R)) \subseteq \text{Lie}(T)$ for every maximal algebraic torus $T$ of $R$, where Center($R$) is the center of $R$. Since $R_\Sigma$ is reductive, Theorem 3.6 together with Fact 3.1 (iii) and (3.7) immediately implies Theorem 1.9.
4. A combinatorial formula for the Bando-Calabi-Futaki character of compact toric manifolds. In [14], the author established a combinatorial formula for the Futaki character of a toric Fano manifold. In this section, we shall also establish a combinatorial formula for the Bando-Calabi-Futaki character of a compact toric manifold by the same argument as in [14].

Throughout this section, we fix a complete non-singular fan $\Sigma$ in $N := \mathbb{Z}^r$ and a positive class $\eta \in H^2(X_\Sigma; \mathbb{Z})$. We shall use the same notation as that in Section 3.

We define a basis $\{\tau_i := t_i(\partial/\partial t_i); i = 1, 2, \ldots, r\}$ of the Lie algebra $t_r$ of $T_r$, where $(t^1, t^2, \ldots, t^r)$ is the standard coordinate for $T_r = (\mathbb{C}^*)^r$. Note that we can regard $t_r$ as a complex Lie subalgebra of $H^0(X_\Sigma; \mathcal{O}(T^1\mathbb{C}^r))$. For each $\sigma \in \Sigma(r)$ and $S \in GL(r, \mathbb{C})$, let

$$a_1(\sigma) = \begin{pmatrix} a_{11}^1(\sigma) \\ a_{21}^1(\sigma) \\ \vdots \\ a_{r1}^1(\sigma) \end{pmatrix}, \ldots, a_r(\sigma) = \begin{pmatrix} a_{1r}^1(\sigma) \\ a_{2r}^1(\sigma) \\ \vdots \\ a_{rr}^1(\sigma) \end{pmatrix} \in \mathbb{N}$$

be the generator of $\sigma$. We put

$$A(\sigma) := (a_1(\sigma), a_2(\sigma), \ldots, a_r(\sigma))$$

$$= \begin{pmatrix} a_{11}^1(\sigma) & a_{12}^1(\sigma) & \cdots & a_{1r}^1(\sigma) \\ a_{21}^1(\sigma) & a_{22}^1(\sigma) & \cdots & a_{2r}^1(\sigma) \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}^1(\sigma) & a_{r2}^1(\sigma) & \cdots & a_{rr}^1(\sigma) \end{pmatrix} \in GL(r, \mathbb{Z})$$

and $Q(S; \sigma) = (q_j^i(S; \sigma)) := A(\sigma)^{-1}S \in GL(r, \mathbb{C})$. A non-singular matrix $S \in GL(r, \mathbb{C})$ is said to be non-degenerate if $S$ satisfies $q_j^i(S; \sigma) \neq 0$ for all $i, j = 1, 2, \ldots, r$, and $\sigma \in \Sigma(r)$.

**Example 4.1.** For example, a non-singular matrix

$$S_0 := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \pi & \pi^2 & \cdots & \pi^r \\ \pi^2 & \pi^4 & \cdots & \pi^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \pi^{r-1} & \pi^{2(r-1)} & \cdots & \pi^{r(r-1)} \end{pmatrix} \in GL(r, \mathbb{C})$$

is non-degenerate.

For $S = (s_j^i) \in GL(r, \mathbb{C})$ and $i = 1, 2, \ldots, r$, we define a holomorphic vector field $V_i(S) := \sum_{j=1}^r s_j^i \tau_j$ on $X_\Sigma$. Then $\{V_i(S); i = 1, 2, \ldots, r\}$ is a basis of $t_r$. For a map $\alpha: \Sigma(1) \to \mathbb{Z}$, we define constants $\beta_i(S; \sigma, \alpha), i = 1, 2, \ldots, r$, by

$$\beta_i(S; \sigma, \alpha) := \sum_{j=1}^r \alpha((a_j(\sigma)))q_j^i(S; \sigma),$$

where $(a_i(\sigma)) \in \Sigma(1)$ is the one-dimensional cone generated by $a_i(\sigma) \in N$. We put $b_i(\sigma, \alpha) := \beta(I_r; \sigma, \alpha)$, where $I_r \in GL(r, \mathbb{C})$ is the identity matrix.
In terms of the notation as above, we can establish the following combinatorial formula for $C^{r+l}_{L_\alpha}(V_i(S))$:

**Theorem 4.2.** Let $X_\Sigma$ be an $r$-dimensional compact toric manifold associated with a complete non-singular fan $\Sigma$ and $S \in GL(r, \mathbb{C})$ a non-degenerate non-singular matrix. Then we have

$$C^{r+l}_{L_\alpha}(V_i(S)) = \left(\frac{\sqrt{-1}}{2\pi}\right)^l \sum_{\sigma \in \Sigma(r)} \beta_i(S; \sigma, \alpha)^{r+l} \prod_{j=1}^r q_j^i(S; \sigma)$$

for any $\alpha: \Sigma(1) \to \mathbb{Z}$, $l \in \mathbb{Z}_{\geq 0}$ and $i = 1, 2, \ldots, r$, where we regard $L_\alpha$ as a $T_r$-linearized holomorphic line bundle over $X_\Sigma$.

**Proof.** For each $\sigma \in \Sigma(r)$, there exists a $T_r$-invariant open subset $W_\sigma$ of $X_\Sigma$ such that $W_\sigma \cong \mathbb{C}^r$ and $X_\Sigma = \bigcup_{\sigma \in \Sigma(r)} W_\sigma$. Let $(t^1, t^2, \ldots, t^r)$ and $(z^1(\sigma), z^2(\sigma), \ldots, z^r(\sigma))$ be the coordinate systems on $D_\Sigma \cong T_r = (\mathbb{C}^*)^r$ and $W_\sigma \cong \mathbb{C}^r$, respectively. The following system of identities is the coordinate transformation between these coordinates:

$$t^i = \prod_{j=1}^r z^j(\sigma) \frac{\partial}{\partial z^j(\sigma)}, \quad i = 1, 2, \ldots, r.$$

From these identities, for every $\sigma \in \Sigma(r)$ and $i = 1, 2, \ldots, r$, we have

$$V_i(S) = \sum_{k=1}^r q_k^i(S; \sigma) \frac{\partial}{\partial z^k(\sigma)}$$

on $W_\sigma$. In view of this expression of $V_i(S)$ and the non-degeneracy of $S$, we obtain

$$\text{Zero}(V_i(S)) = \{\text{the origin } o(\sigma) \text{ of } W_\sigma \cong \mathbb{C}^r : \sigma \in \Sigma(r)\}$$

for $i = 1, 2, \ldots, r$. For each $\sigma \in \Sigma(r)$, the $T_r$-linearized holomorphic line bundle $L_\alpha$ over $X_\Sigma$ is trivialized on $W_\sigma$. In terms of this trivialization, the $T_r$-action on $L_\alpha|_{W_\sigma} = W_\sigma \times \mathbb{C}$ is given by

$$t: W_\sigma \times \mathbb{C} \ni (z, \xi) \mapsto \left(t^i \cdot z, \prod_{i=1}^r t^i - b_i(\sigma, \alpha) \frac{\partial}{\partial z^i(\sigma)}\right) \in W_\sigma \times \mathbb{C},$$

where $t = (t^1, t^2, \ldots, t^r) \in T_r$ (see [16, p. 69]). Hence, for $\sigma \in \Sigma(r)$ and $i = 1, 2, \ldots, r$, we have

$$L^{(L_\alpha, h)}_{V_i(S), o(\sigma)} = \sum_{j=1}^r b_j(\sigma, \alpha) x_j^i = b_i(S; \sigma, \alpha),$$

where $h$ is an arbitrary Hermitian metric on $L_\alpha$. Moreover we also have, for $\sigma \in \Sigma(r)$ and $i = 1, 2, \ldots, r$, 
with respect to a basis \{((\partial/\partial z^1(\sigma))_{\alpha(\sigma)}), \ldots, ((\partial/\partial z^\ell(\sigma))_{\alpha(\sigma)})\} of $T_{\alpha(\sigma)}^1 X_P$, where $g'$ is an arbitrary Hermitian metric on $X_\Sigma$. Together with (4.3) and (4.4), Fact 2.3 immediately implies the theorem.

\[ L_{\alpha(\sigma)}^\prime = \begin{pmatrix} q_1^1(S; \sigma) & q_1^2(S; \sigma) & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & q_r^r(S; \sigma) \end{pmatrix}, \]

As a corollary of this theorem, we obtain the following:

**Corollary 4.5.** Let $X_\Sigma$ be an $r$-dimensional compact toric manifold associated with a complete non-singular fan $\Sigma$, $S \in GL(r, \mathbb{C})$ a non-degenerate non-singular matrix, $\alpha : \Sigma(1) \to \mathbb{Z}$, $a = 1, 2, \ldots, k$, and $b_1, b_2, \ldots, b_k \in \mathbb{N}$ with $b_1 + b_2 + \cdots + b_k = r$. Then we have

\[
(c_1(L_{\alpha_1})^{b_1} \cup c_1(L_{\alpha_2})^{b_2} \cup \cdots \cup c_1(L_{\alpha_k})^{b_k})[X_\Sigma] = \sum_{\sigma \in \Sigma(r)} \prod_{a=1}^{k} \beta_i(S; \sigma, \alpha_a)^{b_a} \prod_{j=1}^{r} q_j^i(S; \sigma),
\]

for any $i = 1, 2, \ldots, r$. In particular, for $\alpha : \Sigma(1) \to \mathbb{Z}$ and $i = 1, 2, \ldots, r$, we have

\[
\mu_{c_1(L_\omega)} = \beta_i(S; \sigma, \omega) \sum_{\sigma \in \Sigma(r)} \prod_{j=1}^{r} q_j^i(S; \sigma),
\]
PROOF. Applying Theorem 4.2 to \( C^1_{L_1^1 \otimes L_2^2 \otimes \cdots \otimes L_k^k}(V_i(S)) \), we obtain

\[
\sum_{b_1 + \cdots + b_k = r} \frac{r!}{b_1! \cdots b_k!} \lambda_1^{b_1} \cdots \lambda_k^{b_k} (c_1(L_{a_1})^{b_1} \cup \cdots \cup c_1(L_{a_k})^{b_k})(V_i(S)) \left[ X_\Sigma \right]
\]

\[
= \sum_{\sigma \in \Sigma(r)} \left( \sum_{a=1}^k \lambda_a \beta_i(S; \sigma, a) \right)^r \prod_{j=1}^r q_j^i(S; \sigma).
\]

By comparing the coefficients of \( \lambda_1^{b_1} \lambda_2^{b_2} \cdots \lambda_k^{b_k} \) in the equation above, we obtain the formula (4.5.1). The formula (4.5.2) is straightforward from the definition (1.2) of \( \mu_{\eta} \) and the formula (4.5.1).

In view of Theorems 2.4 and 4.2 and Corollary 4.5 combined with the equalities

\[
\sum_{j=0}^r (-1)^j \binom{r}{j} (r - 2j)^k = \begin{cases} 
0 & \text{if } k = 0, 1, \ldots, r - 1, r + 1, \\
2^r r! & \text{if } k = r,
\end{cases}
\]

we can prove the following combinatorial formula for the Bando-Calabi-Futaki character of a compact toric manifold:

**Corollary 4.6.** Let \( X_\Sigma \) be an \( r \)-dimensional compact toric manifold associated with a complete non-singular fan \( \Sigma \) and \( S \in \text{GL}(r, \mathbb{C}) \) a non-degenerate non-singular matrix. If \( \alpha : \Sigma(1) \to \mathbb{Z} \) is ample, then, for \( i = 1, 2, \ldots, r \), we have

\[
\sqrt{-1} F_{X_\Sigma}^{c_1(L_{a_i})}(V_i(S)) = \sum_{\sigma \in \Sigma(r)} \frac{\beta_i(S; \sigma, \alpha)^r \sum_{j=1}^r q_j^i(S; \sigma)}{\prod_{j=1}^r q_j^i(S; \sigma)}
\]

\[
= \frac{r}{r + 1} \left( \sum_{\sigma \in \Sigma(r)} \frac{\beta_i(S; \sigma, \alpha)^{r-1} \sum_{j=1}^r q_j^i(S; \sigma)}{\prod_{j=1}^r q_j^i(S; \sigma)} \right)
\]

\[
- \sum_{\sigma \in \Sigma(r)} \frac{\beta_i(S; \sigma, \alpha)^r \sum_{j=1}^r q_j^i(S; \sigma)}{\prod_{j=1}^r q_j^i(S; \sigma)}
\]

\[
\left( \sum_{\sigma \in \Sigma(r)} \frac{\beta_i(S; \sigma, \alpha)^{r+1} \sum_{j=1}^r q_j^i(S; \sigma)}{\prod_{j=1}^r q_j^i(S; \sigma)} \right).
\]
REMARK 4.7. (i) Let $X$, $\alpha$ and $S$ be as in Corollary 4.6. Then we have

$$\left( F_{X}(L_{\alpha})(\tau_{1}), F_{X}(L_{\alpha})(\tau_{2}), \ldots, F_{X}(L_{\alpha})(\tau_{r}) \right) = \left( F_{X}(V_{1}(S)), F_{X}(V_{2}(S)), \ldots, F_{X}(V_{r}(S)) \right) S^{-1}.$$ 

Therefore, in view of Corollary 4.6, we can calculate $F_{X}(L_{\alpha})(\tau_{i})$ for all $i = 1, 2, \ldots, r$.

(ii) Let $X$, $\alpha$ be as in Corollary 4.6. Then by means of Theorem 1.9, Corollary 4.6 and the identity in (i), we can obtain the entire information about the Bando-Calabi-Futaki character $F_{X}(L_{\alpha})$ of $(X, c_{1}(L_{\alpha}))$.

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