THE ORDER OF PERIODIC ELEMENTS OF TEICHMÜLLER MODULAR GROUPS

EGE FUJIKAWA

(Received May 2, 2003, revised March 10, 2004)

Abstract. We consider a quasiconformal automorphism of a Riemann surface, which fixes the homotopy class of a simple closed geodesic. Under certain conditions on the injectivity radius of the surface and bounds on the dilatation of the map, the automorphism induces a periodic element of the Teichmüller modular group. We may also estimate the order of the period.

1. Introduction. Let $R$ be an arbitrary Riemann surface with possibly infinitely generated fundamental group. An element $\chi$ of the Teichmüller modular group $\text{Mod}(R)$ is induced by a quasiconformal automorphism $f$ of $R$. We would like to determine when the order of $\chi$ is finite. When $f$ is a conformal automorphism of $R$, then the element $\chi$ of $\text{Mod}(R)$ induced by $f$ fixes the base point of the Teichmüller space $T(R)$. In [3], we proved that, for a Riemann surface $R$ with non-abelian fundamental group, a conformal automorphism $f$ of $R$ has finite order if and only if $f$ fixes either a simple closed geodesic, a puncture or a point on $R$. In each case, we obtained a concrete estimate for the order of $f$ in terms of the injectivity radius on $R$. One of our results is the following. For the definition of the upper bound condition, see the next section.

**Theorem 1.1 ([3]).** Let $R$ be a hyperbolic Riemann surface with non-abelian fundamental group. Suppose that $R$ satisfies the upper bound condition for a constant $M > 0$ and a connected component $R^*_M$ of $R_M$. Let $f$ be a conformal automorphism of $R$ such that $f(c) = c$ for a simple closed geodesic $c$ on $R$ with $c \subset R^*_M$ and $l(c) = l > 0$. Then the order $n$ of $f$ satisfies

$$n < (e^M - 1) \cosh(l/2).$$

The purpose of this paper is to extend Theorem 1.1 to a quasiconformal automorphism $f$. One of the difficulties that arise is that the element $\chi \in \text{Mod}(R)$ induced by $f$ need not have a fixed point on $T(R)$. However, we will show that if the maximal dilatation of $f$ is smaller than some constant, then $\chi$ is periodic.

The author would like to express her gratitude to Professor Katsuhiko Matsuzaki for his valuable suggestions.

2. Statement of theorem. Let $H$ be the upper half-plane equipped with the hyperbolic metric $|dz|/\text{Im} z$. Throughout this paper, we assume that a Riemann surface $R$ is hyperbolic. Namely, it is represented as $H/\Gamma$ for some torsion-free Fuchsian group $\Gamma$ acting on $H$.

2000 Mathematics Subject Classification. Primary 30F60; Secondary 30C62.
Furthermore, we also assume that $R$ has a non-abelian fundamental group. The hyperbolic distance on $H$ is denoted by $d$, and the hyperbolic length of a curve $c$ on $R$ by $l(c)$. For the axis $L$ of a hyperbolic element of the Fuchsian group $\Gamma$, we denote by $\pi_{\Gamma}(L)$ the projection of $L$ to $H/\Gamma$. When there is no fear of confusion, we denote this simply by $\pi(L)$. Also, for a quasiconformal automorphism $f$ of $H$, we denote by $\tilde{f}(L)$ the geodesic having the same end points as those of $\tilde{f}(L)$.

We recall the definition of Teichmüller spaces and Teichmüller modular groups. Fix a Riemann surface $R$. We say that two quasiconformal maps $f_1$ and $f_2$ on $R$ are equivalent if $f_2 \circ f_1^{-1}$ is homotopic to a conformal map of $f_1(R)$ onto $f_2(R)$. The reduced Teichmüller space $T(R)$ with the base Riemann surface $R$ is the set of all equivalence classes $[f]$ of quasiconformal maps $f$ on $R$. The Teichmüller distance $d_T$ on $T(R)$ is defined by $d_T([f_1], [f_2]) = \log K(g)$, where $g$ is an extremal quasiconformal map in the sense that its maximal dilatation $K(g)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. This is a complete metric on $T(R)$. The reduced Teichmüller modular group $\text{Mod}(R)$ of $R$ is a group of the homotopy classes $[h]$ of quasiconformal automorphisms $h$ of $R$. Each element $[h]$ of $\text{Mod}(R)$ induces an automorphism of $T(R)$ by $[f] \mapsto [f \circ h^{-1}]$, which is an isometry with respect to $d_T$.

We now make a couple of definitions given in terms of the hyperbolic geometry of Riemann surfaces.

**Definition.** For a constant $M > 0$, we define $R_M$ to be the set of points $p \in R$ for which there exists a non-trivial simple closed curve $c_p$ passing through $p$ with $l(c_p) < M$. The set $R_\varepsilon$ is called the $\varepsilon$-thin part of $R$ if $\varepsilon > 0$ is smaller than the Margulis constant. Furthermore, a connected component of the $\varepsilon$-thin part corresponding to a puncture is called the cusp neighborhood.

**Remark.** The injectivity radius at a point $p \in R$ is the supremum of radii of embedded hyperbolic discs centered at $p$. Note that $R_M$ coincides with the set of those points having the injectivity radius less than $M/2$.

**Definition.** We say that $R$ satisfies the lower bound condition if there exists a constant $\varepsilon > 0$ such that $\varepsilon$-thin part of $R$ consists of only cusp neighborhoods or neighborhoods of geodesics which are homotopic to boundary components. We also say that $R$ satisfies the upper bound condition if there exist a constant $M > 0$ and a connected component $R_M^*$ of $R_M$ such that the homomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ induced by the inclusion map of $R_M^*$ into $R$ is surjective.

**Remark.** The lower and upper bound conditions are quasiconformally invariant notions (see [5, Lemma 8]).

We shall obtain a range of maximal dilatations of quasiconformal automorphisms $f$ inducing periodic elements $\chi \in \text{Mod}(R)$. Moreover, we get a concrete estimate for the order of $\chi$. 
**Theorem 2.1.** Let $R$ be a Riemann surface satisfying the lower bound condition for a constant $\varepsilon > 0$ as well as the upper bound condition for a constant $M > 0$ and a connected component $R^s_M$ of $R_M$. For a given constant $l > 0$, there exists a constant $K_0 = K_0(\varepsilon, M, l) > 1$ depending only on $\varepsilon, M$, and $l$ that satisfies the following: Let $f$ be a quasiconformal automorphism of $R$ such that $f(c)$ is homotopic to $c$ for a simple closed geodesic $c$ on $R$ with $c \subset R^s_M$ and $l(c) = l$. Suppose $K(f) < K_0$. Then there exists a positive integer $n \leq N_0$ such that $f^n$ is homotopic to the identity. Here

$$N_0 = N_0(M, l) = -\frac{l}{\log(\tanh(D + 13.5))},$$

$$D = D(M, l) = \begin{cases} 2 \arccosh \left( \frac{\sinh(M/2)}{\sinh(l/2)} \right) + M & \text{if } l \leq M, \\ M & \text{if } l \geq M. \end{cases}$$

In particular, when $K(f) = 1$, we have the following:

**Theorem 2.2.** Let $R$ be a Riemann surface satisfying the upper bound condition for a constant $M > 0$ and a connected component $R^s_M$ of $R_M$ as well as the lower bound condition. Let $f$ be a conformal automorphism of $R$ such that $f(c) = c$ for a simple closed geodesic $c$ on $R$ with $c \subset R^s_M$ and $l(c) = l > 0$. Then the order $n$ of $f$ satisfies

$$n \leq -\frac{l}{\log(\tanh(D/2))},$$

where $D = D(M, l)$ is the same constant as in Theorem 2.1.

Note that for $M \geq \arcsinh(2/\sqrt{3}) = 0.98 \cdots$ and every $l > 0$, we have

$$-\frac{l}{\log(\tanh(M/2))} < (e^M - 1) \cosh(l/2).$$

Here the constant $\arcsinh(2/\sqrt{3})$ is the smallest possible value of $M$ for which $R$ satisfies the upper bound condition (see [6]). Hence when $l \geq M$, the upper bound of the order of $f$ obtained in Theorem 2.2 is smaller than that in Theorem 1.1. However, when $l < M$, the estimate in Theorem 1.1 is still better than that in Theorem 2.2 for all sufficiently small $l$. In fact, $(e^M - 1) \cosh(l/2)$ converges to $e^M - 1$ as $l \to 0$, while $-l/(\log(\tanh(D/2)))$ diverges to $+\infty$.

In connection with Theorems 2.1 and 2.2, we would like to mention the result about the discreteness of the orbit of a point in the Teichmüller space under the action of a certain subgroup of the Teichmüller modular group.

**Proposition 2.3 ([5]).** Let $R$ be a Riemann surface satisfying the lower and upper bound conditions. For a simple closed geodesic $c$ on $R$, let $G$ be a subgroup of $\text{Mod}(R)$ such that $g(c)$ is homotopic to $c$ for every $[g] \in G$. Then for every point $p \in T(R)$, the orbit $G(p)$ of $p$ is a discrete subset in $T(R)$. Furthermore, for any point $p \in T(R)$, there exist only finitely many elements $[g]$ in $G$ that fix $p$. 

3. Proof of theorems. For a proof of these theorems, we first prove some properties on the hyperbolic geometry of Riemann surfaces.

**Proposition 3.1.** Let \( R = H/\Gamma \) be a Riemann surface satisfying the upper bound condition for a constant \( M > 0 \) and a connected component \( \Gamma_M^* \) of \( R_M \). Suppose that \( L \) is the axis of a hyperbolic element of \( \Gamma \) such that the projection \( \pi(L) \) is a simple closed geodesic \( c \) on \( R \) with \( c \subset \Gamma_M^* \). Suppose also that there exists an axis \( L' \) of a hyperbolic element of \( \Gamma \) such that \( L \cap L' = \emptyset, d(L, L') \leq D \) and \( \pi(L') = \pi(L) \). Here \( D = D(M, l) \) is the same constant as in Theorem 2.1.

**Proof.** First we assume that \( l > M \). Since \( c \subset \Gamma_M^* \), there exists a non-trivial simple closed curve \( \alpha \) passing through \( p \in c \) with \( l(\alpha) < M \). It follows from the assumption \( l > M \) that \( \alpha \) is not homotopic to \( c \), which implies that there exists an axis \( L' \) \((\neq L)\) such that \( \pi(L') = c \) and \( d(L, L') < M \).

Next we assume that \( l \leq M \). We further assume that there exists an annular neighborhood \( A(c) \) of \( c \) with width \( \omega(c) \), where

\[
\omega(c) = \arccosh \left( \frac{\sinh(M/2)}{\sinh(l/2)} \right).
\]

Then, for any \( q \in \partial A(c) \), the boundary of \( A(c) \), the shortest simple closed curve \( \gamma \) passing through \( q \) and homotopic to \( c \) has length \( M \).

Indeed, we may assume that \( L = \{ iy \mid y > 0 \} \), and \( \tilde{q} = e^{i\theta} \) and \( \tilde{q}' = e^{l+ i\theta} \) are lifts of \( q \) to \( H \). Then, by the equality (7.20.3) in [2], we have

\[
\frac{1}{\sin \theta} = \frac{1}{\cos(\pi/2 - \theta)} = \cosh d(\tilde{q}, L) = \cosh \omega(c) = \frac{\sinh(M/2)}{\sinh(l/2)}.
\]

Thus, by Theorem 7.2.1 in [2], we see that

\[
\sinh \frac{1}{2} d(\tilde{q}, \tilde{q}') = \frac{|\tilde{q} - \tilde{q}'|}{2 \Im \tilde{q} \Im \tilde{q}'}^{1/2} = \frac{e^l - 1}{2e^{l/2} \sin \theta} = \frac{\sinh(l/2)}{\sin \theta} = \sinh \frac{M}{2},
\]

which implies that \( l(\gamma) = d(\tilde{q}, \tilde{q}') = M \).

We can take a point \( q_0 \in \partial A(c) \) such that \( q_0 \in \Gamma_M^* \). Indeed, otherwise, \( \partial A(c) \cap \Gamma_M^* = \emptyset \).

Since \( c \subset \Gamma_M^* \), this means that \( \Gamma_M^* \) is an annular neighborhood of \( c \), contradicting the upper bound condition.

By the definition of \( R_M \), there exists a non-trivial simple closed curve \( \beta \) passing through \( q_0 \) with \( l(\beta) < M \). By the consideration above, we see that the curve \( \beta \) is not homotopic to \( c \). Hence there exists an axis \( L' \) \((\neq L)\) such that \( \pi(L') = c \) and \( d(L, L') < 2 \omega(c) + M \).

Finally, we assume that \( l \leq M \) and that the width of the maximal annular neighborhood \( A(c) \) of \( c \) is less than \( \omega(c) \). Then there exists an axis \( L' \) \((\neq L)\) such that \( \pi(L') = c \) and \( d(L, L') < 2 \omega(c) \).

We now estimate the number of axes satisfying Proposition 3.1.

**Definition.** For an element \( \gamma \) of a Fuchsian group, we say that two axes \( L_1 \) and \( L_2 \) are \( \gamma \)-equivalent if \( \gamma^n(L_1) = L_2 \) for some \( n \in \mathbb{Z} \).
Proposition 3.2. Let \( R = \mathcal{H}/\Gamma \) be a Riemann surface and \( D_0 > 0 \) a constant. Furthermore, let \( L \) be the axis of a hyperbolic element \( \gamma \in \Gamma \) such that the projection \( \pi(L) \) is a simple closed geodesic \( c \) on \( R \) with \( l(c) = l > 0 \). Let \( S \) be the set of axes \( L' \) of hyperbolic elements of \( \Gamma \) satisfying the following: (i) \( L \cap L' = \emptyset \), (ii) \( d(L, L') \leq D_0 \), (iii) \( \pi(L') = c \) and (iv) there exists an arc \( \alpha \) connecting \( L \) and \( L' \) whose projection to \( R \) has no intersection with \( c \) except at the end points. Then the number of \( \gamma \)-equivalence classes of axes in \( S \) is dominated by

\[
-\frac{l}{\log(\tanh(D_0/2))}.
\]

Proof. We may assume that \( L = \{iy \mid y > 0\} \). We take \( \theta_0 \) \((0 < \theta_0 < \pi/2)\) so that \( \cosh D_0 = (\cos \theta_0)^{-1} \) and set \( \theta = \pi/2 - \theta_0 \). Furthermore, we set

\[
T_+ = \{re^{i\theta} \mid 1 \leq r < e^l\} \quad \text{and} \quad T_- = \{re^{i(\pi/2-\theta)} \mid 1 \leq r < e^l\}.
\]

Then \( d(L, T_+) = D_0 \) and \( d(L, T_-) = D_0 \). To estimate the number of \( \gamma \)-equivalence classes of elements in \( S \), we have only to consider the maximal number \( n \) of disjoint axes \( L' \) that are tangent to \( T_+ \) or \( T_- \).

Let \( C \) be the Euclidean circle on \( C \) that is tangent to the segment \( T_+ \) and has center \( a > 0 \) with radius \( r \). Then \( r = a \sin \theta \), and the circle \( C \) passes through two points,

\[
x_1 = (1 - \sin \theta)a \quad \text{and} \quad x_2 = (1 + \sin \theta)a.
\]

The ratio of these points is given by

\[
s = \frac{x_2}{x_1} = \frac{1 + \sin \theta}{1 - \sin \theta} = \frac{1 + \cos \theta_0}{1 - \cos \theta_0} = \frac{\cosh D_0 + 1}{\cosh D_0 - 1} = \frac{1}{(\tanh(D_0/2))^2}.
\]

Hence it is easy to see that

\[
n \leq 2 \cdot \frac{l}{\log s} = \frac{l}{\log(\tanh(D_0/2))}. \quad \square
\]

The following proposition gives a relationship between the hyperbolic distance of two axes and that of their images under a quasiconformal map.

Proposition 3.3 ([1]). Let \( f \) be a \( K \)-quasiconformal automorphism of \( \mathcal{H} \). Then there exists a constant \( C = C(K) > 0 \) depending only on \( K \) such that, for any two geodesics \( L_1 \) and \( L_2 \) in \( \mathcal{H} \), the inequality

\[
K^{-1} \cdot d(L_1, L_2) - C \leq d(f(L_1)_*, f(L_2)_*) \leq K \cdot d(L_1, L_2) + C
\]

holds. The constant \( C(K) \) satisfies \( C(K) \to 0 \) as \( K \to 1 \), and may be taken to be

\[
(1/2) \arccosh(2^{-(K-1)^2} e^{6(K+1)^2\sqrt{K-1}}).
\]

The following proposition gives a sufficient condition for the maximal dilatations of quasiconformal maps to be bounded away from one.
PROPOSITION 3.4 ([4]). Let $R = H/\Gamma$ be a Riemann surface. Suppose that $R$ satisfies the lower bound condition for a constant $\epsilon > 0$ as well as the upper bound condition for a constant $M > 0$ and a connected component $R^*_M$ of $R_M$. Let $B > 0$ and $l > 0$ be constants. Then there exists a constant $A_0 = A_0(\epsilon, M, B, l) > 1$ depending only on $\epsilon$, $M$, $B$, $l$ and satisfying the following conditions: Given a quasiconformal automorphism $f$ of $R$, suppose that there exist three disjoint axes $L_i$ ($i = 1, 2, 3$) of hyperbolic elements of $\Gamma$ such that

1. their projections $\pi(L_i)$ on $R$ are simple closed geodesics $c_i$ ($i = 1, 2, 3$) with $c_i \subset R^*_M$ and $l(c_i) \leq l$,
2. $d(L_1, L_2) \leq B$,
3. $\tilde{f}(L_1)_* = L_1$, $\tilde{f}(L_2)_* = L_2$, $\tilde{f}(L_3)_* \neq L_3$ for a lift $\tilde{f}$ of $f$ to $H$.

Then $K(f) \geq A_0$.

We now prove our theorems.

PROOF OF THEOREM 2.1. We set $B := D = D(M, l)$ in Proposition 3.4 and let $A_0 = A_0(\epsilon, M, l) > 1$ be a constant depending only on $\epsilon$, $M$ and $l$ obtained in Proposition 3.4. Setting $A = \min(A_0, 2)$, we prove the statement for $K_0 = A^{1/(N_0 + 1)}$. Namely, we show that, if $K(f) < K_0$, then there exists an integer $n \leq N_0$ such that $f^n$ is homotopic to the identity.

Let $\Gamma$ be a Fuchsian model of $R$. Furthermore let $L_1$ be an axis such that $\pi(L_1) = c$ and $\gamma_1$ the primitive hyperbolic element of $\Gamma$ with axis $L_1$. By applying Proposition 3.1 to $L_1$, we see that there exists an axis $L_2$ of a hyperbolic element $\gamma_2$ of $\Gamma$ such that $L_1 \cap L_2 = \emptyset$, $d(L_1, L_2) \leq D$ and $\pi(L_1) = \pi(L_2)$.

Let $\tilde{f}$ be a lift of $f$ to $H$ satisfying $\tilde{f}(L_1)_* = L_1$. Since $K(f) < K_0 = A^{1/(N_0 + 1)}$, we have $K(f^k) < A$ for $k \leq N_0 + 1$. Then, by Proposition 3.3,

$$d(L_1, f^k(L_2)_*) = d(\tilde{f}(L_1)_*, \tilde{f}(L_2)_*) \leq A \cdot d(L_1, L_2) + C(A)$$

for all $k \leq N_0 + 1$.

We consider the set $S_0$ of all axes $L'$ of hyperbolic elements of $\Gamma$ satisfying the following conditions: (i) $L_1 \cap L' = \emptyset$, (ii) $d(L_1, L') \leq 2D + 27$, (iii) $\pi(L') = c$ and (iv) there exists an arc $\alpha$ connecting $L_1$ and $L'$ such that the projection of $\alpha$ to $R$ has no intersection with $c$ except at the end points. We see that the set $S' = \{f^k(L_2)_*\}_{k=1}^{N_0+1}$ is contained in $S_0$. Indeed, by the proof of Proposition 3.1, the axis $L_2$ satisfies the property (iv), and since $\tilde{f}$ is a homeomorphism, the axes $f^k(L_2)_*$ satisfy the same property. The other properties (i), (ii), (iii) are also satisfied.

By Proposition 3.2, the number of $\gamma_1$-equivalence classes of elements in $S_0$ is dominated by $N_0$. Hence there exist at least two elements in $S'$, say $f^{m_1}(L_2)_*$ and $f^{m_2}(L_2)_*$ ($1 \leq m_1 < m_2 \leq N_0 + 1$), that are $\gamma_1$-equivalent to each other. Thus there exists $j \in \mathbb{Z}$ such that $\gamma_1^j \circ f^n(L_2)_* = L_2$, where $n = m_2 - m_1$ ($\leq N_0$). With this $n$, we will prove that $f^n$ is homotopic to the identity. We set $F = \gamma_1^j \circ f^n$, which is a lift of $f^n$ to $H$. 
Suppose to the contrary that $f^n$ is not homotopic to the identity. We set $\chi(\gamma) = F \circ \gamma \circ F^{-1}$ for $\gamma \in \Gamma$. Then there exists $\gamma_3 \in \Gamma$ such that $\chi(\gamma_3) \neq \gamma_3$. Setting $\gamma_i' = \gamma_3 \circ \gamma_i \circ \gamma_3^{-1}$ for $i = 1, 2$, we claim that either $\chi(\gamma_1') \neq \gamma_1'$ or $\chi(\gamma_2') \neq \gamma_2'$ is satisfied. Suppose that both $\chi(\gamma_1') = \gamma_1'$ and $\chi(\gamma_2') = \gamma_2'$ are satisfied. Since $\chi(\gamma_i) = \gamma_i$, we have $\beta \circ \gamma_i \circ \beta^{-1} = \gamma_i$ ($i = 1, 2$), where $\beta = \gamma_3^{-1} \circ \chi(\gamma_3)$. Thus, $\beta$ fixes all fixed points of $\gamma_1$ and $\gamma_2$. Since $\gamma_1$ and $\gamma_2$ are non-commutative, the Möbius transformation $\beta$ fixes four points and must be the identity. This contradicts that $\chi(\gamma_3) \neq \gamma_3$.

Hence either $F(\gamma_3(L_1)_s) \neq \gamma_3(L_1)$ or $F(\gamma_3(L_2)_s) \neq \gamma_3(L_2)$ is satisfied. Also, we may assume without loss of generality that $F(\gamma_3(L_1)_s) \neq \gamma_3(L_1)$. Since $\pi(\gamma_3(L_1)) = \pi(L_1) = c$, we can apply Proposition 3.4 to the lift $F$ of $f^n$ and to the three axes $L_1, L_2$ and $\gamma_3(L_1)$. Then we have $K(f^n) \geq A_0$, a contradiction, since we assumed $K(f^n) < A_0$. Hence if $K(f) < A_1/(N_0+1)$, then $f^n$ is homotopic to the identity. \hfill $\Box$

**Proof of Theorem 2.2.** In the proof of Theorem 2.1, we can replace the inequality (1) with
\[ d(L_1, j^k(L_2)_s) = d(j^k(L_1)_s, j^k(L_2)_s) = d(L_1, L_2) = D. \]
Hence we have only to replace the constant $2D + 27$ with $D$ in Theorem 2.1. \hfill $\Box$

**References**