On the Approximation of a Function of two Variables by Polynomials,

by

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Prof. Landau's proof of Weierstrass' theorem depends upon the following (1):

If \( F(x) \) is limited and continuous except at a finite number of points in the interval \( 0 < x \leq a \leq b < 1 \), then the sequence of polynomials

\[
\int_0^1 F(u) \frac{[1 - (u - x)^n]}{2 \int_0^1 (1 - u^n) \, du}
\]

converges uniformly to \( F(x) \), at every point where \( F(x) \) is continuous, as \( n \) tends to infinity.

And he extends this to the polynomials of many variables.

The result of Landau was extended by Tonelli to a different direction in the case of more than two variables, i.e., he proved that under certain conditions

\[
\int_0^{z_{0.5}} \int_0^{z_{0.5}} F(u, v) \frac{[1 - (u - x)^2 - (v - y)^2]^n \, dv}{4 \int_0^{z_{0.5}} \int_0^{z_{0.5}} [1 - u^2 - v^2]^n \, dv}
\]

is also uniformly convergent to \( F(x, y) \) as \( n \) tends to infinity (2).

The principal object of this paper is to prove that if \( F(x, y) \) is limited and continuous except at a finite number of points in the domain \( (0 < x, y < 1) \), the sequence of polynomials

\[
\int_0^1 \int_0^1 F(u, v) \frac{[1 - (u - x)(v - y)^2]^n \, dv}{4 \int_0^1 \int_0^1 [1 - u^2 \cdot v^2]^n \, dv}
\]


converges uniformly to \( F(x, y) \) at every point where it is continuous as \( n \) tends to infinity.

I.

Before dealing with our subject, we shall prove some lemmas. Throughout this paper, we express \([1-u^2 v^2]^n\) by \( W_n(u, v)\).

(I) Firstly let \( a \) and \( b \) be any two fixed numbers such that

\[
0 < a < 1, \quad 0 < b < 1.
\]

Then

\[
\lim_{n \to \infty} \frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(a, v) \, dv} = 0,
\]

where \( \varepsilon \) is any fixed number such that \( 0 < \varepsilon < 1 \).

For

\[
\int_0^1 W_n(b, v) \, dv < \int_0^1 W_n(b, \varepsilon) \, dv < [1 - b^2 \varepsilon^2]^n
\]

and also

\[
\int_0^1 W_n(a, v) \, dv \geq \int_0^{n^{-\frac{1}{2}}} W_n(a, v) \, dv > \int_0^{n^{-\frac{1}{2}}} W_n(a, n^{-\frac{1}{2}}) \, dv
\]

\[
= n^{-\frac{1}{2}} [1 - a^2/n].
\]

Hence we have the inequality

\[
\frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(a, v) \, dv} < n^{\frac{1}{2}} [1 - b^2 \varepsilon^2]^n [1 - a^2/n]^{-n},
\]

from which the theorem follows immediately (1).

(Ia) Especially if \( a = b \), then

\[
(1) \text{ Even when } a \text{ and } b \text{ are variables, the above theorem holds good, if they are independent of } v, \text{ and } b \text{ satisfies the inequalities}
\]

\[
\frac{n b^2}{\log n} \geq \frac{1}{2 \varepsilon^2}
\]

for all values of \( n \) greater than \( n_0 \).

For in this case the last inequality becomes

\[
\frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(a, v) \, dv} < n^{\frac{1}{2}} [1 - \frac{G \varepsilon^2}{n \log n}]^n (1 - a^2/n) - n,
\]

and hence

\[
0 \leq \lim_{n \to \infty} \frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(a, v) \, dv} \leq \varepsilon a^2 \lim_{n \to \infty} \frac{1}{n} = 0.
\]
Therefore we have

\[ \lim_{n \to \infty} \frac{\int_0^1 W_n(a, v) \, dv}{\int_0^1 W_n(a, v) \, dv} = 0. \]

Therefore we have

\[ \lim_{n \to \infty} \frac{\int_0^1 W_n(a, v) \, dv}{\int_0^1 W_n(a, v) \, dv} = 1. \]

(Ib) More generally we have

\[ \lim_{n \to \infty} \frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(a, v) \, dv} = \frac{a}{b}. \]

For in the case \( a > b \), let us put \( bv = au \) in the numerator. Then

\[ \frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(a, v) \, dv} = \frac{a}{b} \frac{\int_0^1 W_n(a, u) \, du}{\int_0^1 W_n(a, v) \, dv}. \]

Next in the case \( a < b \), putting \( av = bu \) in the denominator, we obtain

\[ \frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(a, v) \, dv} = \frac{a}{b} \frac{\int_0^1 W_n(b, v) \, dv}{\int_0^1 W_n(b, u) \, du}. \]

(II) Secondly the sequence

\[ \lambda_n = \frac{\int_0^1 W_n(a, v) \, dv}{\int_0^1 W_n(a, v) \, dv} \]

is monotonously increasing with \( n \).

To prove this, let

\[ B_n = \int_0^1 W_n(a, v) \, dv, \]

\[ A_n = \int_0^1 W_n(a, v) \, dv. \]

Then

\[ A_n - A_{n+1} = \int_0^1 W_n(a, v) \, dv - \int_0^1 W_{n+1}(a, v) \, dv = \alpha \int_0^1 v^3 W_n(a, v) \, dv. \]

But

\[ \frac{d}{dv} [v W_{n+1}(a, v)] = W_{n+1}(a, v) + 2 (n + 1) \alpha^2 v^3 W_n(a, v). \]

Integrating and using the above equations we have

\[ (1 - \alpha^2)^{n+1} = A_{n+1} - 2 (n + 1) (A_n - A_{n+1}), \]

or

\[ A_n + \frac{(1 - \alpha^2)^{n+1}}{2 (n + 1)} = \frac{(2n + 3)}{2 (n + 1)} \cdot A_{n+1}. \]

Similarly
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\[ B_n + \frac{\varepsilon(1-a^2 \varepsilon^2)^{n+1}}{2(n+1)} = \frac{2n+3}{2(n+1)} \cdot B_{n+1}. \]

Therefore we have

\[ B_n + \frac{\varepsilon(1-a^2 \varepsilon^2)^{n+1}}{2(n+1)} = \frac{B_{n+1}}{A_{n+1}}. \]

But since

\[ 0 < \frac{1-a^2}{1-a^2 \varepsilon^2} < 1, \]

we can take a positive integer \( n_0 \) for which

\[ \varepsilon > \left( \frac{1-a^2}{1-a^2 \varepsilon^2} \right)^{n+1}, \quad \text{or} \quad \varepsilon (1-a^2 \varepsilon^2)^{n+1} > (1-a^2)^{n+1}, \quad n \geq n_0. \]

Consequently to deduce \( \lambda_{n+1} \) from \( \lambda_n \) we must add a larger number to the numerator than to the denominator, but \( \lambda_n \) is positive and smaller than unity for all values of \( n \); so that \( \lambda_n \) increases with \( n \) monotonously.

(IIa) The sequence

\[ \lambda'_n = \frac{\int_0^\varepsilon W_n(b, v) \, dv}{\int_0^\lambda W_n(a, v) \, dv} \]

is also monotonously increasing with \( n \).

(IIb)

\[ \mu_n = (1-\lambda_n) = \frac{\int_0^\varepsilon W_n(a, v) \, dv}{\int_0^\lambda W_n(a, v) \, dv} \]

is a sequence monotonously decreasing with \( n \).

(IIc)

\[ \mu'_n = (1-\lambda'_n) = \frac{\int_0^\varepsilon W_n(b, v) \, dv}{\int_0^\lambda W_n(a, v) \, dv} \]

is also a sequence monotonously decreasing with \( n \).

(III) Thirdly we will show that the sequence

\[ \frac{\int_0^u \int_0^v W_n(u, v) \, dv}{\int_0^u \int_0^v W_n(u, v) \, dv} \]

converges to zero.

Proceeding as in (I) we can prove the following inequalities
and hence we have

\[ \frac{\int_0^1 du \int_0^1 W_n(u, v) \, dv \cdot \int_0^1 dv}{\int_0^1 du \int_0^1 W_n(u, v) \, dv} < n (1 - \delta^2 \varepsilon^2)^n (1 - \eta^2)^n, \]

which proves our theorem.

(IV) Lastly the sequence

\[ \frac{\int_0^1 du \int_0^1 W_n(u, v) \, dv}{\int_0^1 du \int_0^1 W_n(u, v) \, dv} \]

converges uniformly to zero as \( n \) tends to infinity.

Let us consider

\[ P_n = \frac{\int_0^1 du \int_0^1 W_n(u, v) \, dv}{\int_0^1 du \int_0^1 W_n(u, v) \, dv} = \frac{B'_n}{A'_n}. \]

Then \( P_n \) is a monotonously increasing sequence for sufficiently large values of \( n \). For as in (II)

\[ \int_0^1 W_n(u, v) \, dv + \frac{\varepsilon (1 - u^2 \varepsilon^2)^{n+1}}{2(n+1)} = \frac{2n+3}{2(n+1)} \int_0^1 W_{n+1}(u, v) \, dv, \]

\[ \int_0^1 W_n(u, v) \, dv + \frac{(1 - u^2)^{n+1}}{2(n+1)} = \frac{2n+3}{2(n+1)} \int_0^1 W_{n+1}(u, v) \, dv. \]

Integrating

\[ B'_n + \frac{\varepsilon}{2(n+1)} \int_0^1 W_{n+1}(\varepsilon, u) \, du = \frac{2n+3}{2(n+1)} B'_{n+1}, \]

\[ A'_n + \frac{1}{2(n+1)} \int_0^1 W_{n+1}(1, u) \, du = \frac{2n+3}{2(n+1)} A'_{n+1}. \]

But for a sufficiently large \( n_0 \)

\[ \frac{\varepsilon}{2(n+1)} \int_0^1 W_n(\varepsilon, u) \, du > \frac{1}{2(n+1)} \int_0^1 W_n(1, u) \, du, \]

and the similar consideration as in (II) shows us that the sequence \( P_n \) increases monotonously with \( n \geq n_0 \).

By the consequence of this result

\[ Q_n = \frac{\int_0^1 du \int_0^1 W_n(u, v) \, dv}{\int_0^1 du \int_0^1 W_n(u, v) \, dv} \]
is a sequence monotonously decreasing with \( n \geq n_0 \). As \( Q_n \) is obviously positive, it must converge to a finite limit \( l \), i.e.,

\[
\lim_{n \to \infty} Q_n = l \geq 0.
\]

To prove our theorem, it is sufficient to show that \( l = 0 \).

Suppose \( l \) is not zero; then for a sufficiently large number \( N \) we have

\[
Q_n > l, \quad n \geq N;
\]

hence at least for one value \( u \) of \( u \) in \((0,1)\), it must be

\[
\int_0^1 W_n(u,v) \, dv > l \int_0^1 W_n(u,v) \, dv, \quad n \geq N,
\]

which contradicts the result of (I).

Therefore

\[
l = 0,
\]

i.e.,

\[
\lim_{n \to \infty} \frac{\int_0^1 du \int_0^1 W_n(u,v) \, dv}{\int_0^1 du \int_0^1 W_n(u,v) \, dv} = 0 \quad (1).
\]

(IV) Similarly

\[
\lim_{n \to \infty} \frac{\int_0^1 du \int_0^1 W_n(u,v) \, dv}{\int_0^1 du \int_0^1 W_n(u,v) \, dv} = 0.
\]

Consequently

\[
\lim_{n \to \infty} \frac{\int_0^1 du \int_0^1 W_n(u,v) \, dv}{\int_0^1 du \int_0^1 W_n(u,v) \, dv} = 1.
\]

(1) By the first mean value theorem

\[
Q_n = \frac{\int_0^1 W_n(\theta_n,v) \, dv}{\int_0^1 W_n(\theta_n,v) \, dv}, \quad 0 < \theta_n < 1.
\]

Then it can be proved that \( \theta_n \) is monotonously decreasing with \( n \), and \( \lim_{n \to \infty} \theta_n = 0 \).

But it must be

\[
\lim_{n \to \infty} n \theta_n = \infty.
\]

For if not, however large \( n \) may be, we can take at least one integer \( n \) greater than \( n \) for which

\[
n\theta_n < \theta.
\]

Therefore

\[
\int_0^1 du \int_0^1 W_n(u,v) \, dv = \int_0^1 W_n(\theta_n,v) \, dv > \int_0^1 W_n(\theta(n)^{1/2},v) \, dv > (1 - \theta(n))n;
\]

i.e.,

\[
\lim_{n \to \infty} \int_0^1 du \int_0^1 W_n(u,v) \, dv = 0.
\]

This contradicts the obvious result

\[
\lim_{n \to \infty} \int_0^1 du \int_0^1 W_n(u,v) \, dv = 0.
\]
II.

Theorem. Let \( F(x, y) \) be limited and continuous except at a finite number of points in the domain \( 0 \leq x, y \leq 1 \). Then at a point \( x_0, y_0 \) \((0 < a \leq x_0 \leq b < 1, 0 < a' \leq y_0 \leq b' < 1)\) where \( F(x, y) \) is continuous, the sequence of polynomials

\[
F_n(x_0, y_0) = \frac{\int_0^1 \int_0^1 F(x, y) \left[ 1 - (x-x_0)(y-y_0) \right]^n \, dy}{\int_0^1 \int_0^1 \left[ 1 - x^3 y^3 \right] \, dy}
\]

converges uniformly to \( F(x_0, y_0) \) as \( n \) tends to infinity.

To prove this, it is sufficient to show that for an arbitrary small positive \( \omega \), we can take \( \nu = \nu(\omega) \) such that

\[
| F_n(x_0, y_0) - F(x_0, y_0) | < \omega \quad \text{for all } n \geq \nu.
\]

Since \( F(x, y) \) is continuous at the point \((x_0, y_0)\), we can take \((x', y')\) for which

\[
| y' - y_0 | \leq \varepsilon, \quad | x' - x_0 | \leq \delta,
\]

and

\[
| F(x', y') - F(x, y) | < \frac{\omega}{2},
\]

where \( \varepsilon \) and \( \delta \) are sufficiently small positive numbers.

Secondly, as \( F(x, y) \) is limited in the domain \( 0 \leq x, y \leq 1 \), we can choose a positive constant \( G \) such that

\[
| F(x, y) | < G, \quad 0 \leq x, y \leq 1.
\]

Now let

\[
I(x, n) = \int_0^1 F(x, y) \, W_n(x-x_0, y-y_0) \, dy;
\]

then

\[
I(x, n) = \int_0^{y_0-\varepsilon} F(x, y) \, W_n(x-x_0, y-y_0) \, dy
\]

\[
+ \int_{y_0-\varepsilon}^{y_0+\varepsilon} F(x, y) \, W_n(x-x_0, y-y_0) \, dy
\]

\[
+ \int_{y_0+\varepsilon}^{y_0+\varepsilon} F(x, y) \, W_n(x-x_0, y-y_0) \, dy
\]

\[
= I_1(x) + I_2(x) + I_3(x).
\]

By an easy deduction we can show that
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\[ I_1(x) < G \int_0^1 W_n(x-x_0, v) \, dv. \]

Similarly

\[ I_3(x) < G \int_0^1 W_n(x-x_0, v) \, dv. \]

Hence

\[
\int_0^1 I_1(x) \, dx < G \int_0^1 dx \int_0^1 W_n(x-x_0, v) \, dv = G \left[ \int_0^{x_0+\delta} dx \int_0^1 W_n(x-x_0, v) \, dv \right. \\
+ \int_{x_0+\delta}^{\infty} dx \int_0^1 W_n(x-x_0, v) \, dv + \int_0^1 dx \int_{x_0+\delta}^1 W_n(x-x_0, v) \, dv \left. \right] \\
= G \left[ \int_0^{x_0} \int_0^1 + 2 \int_0^\delta \int_0^1 + \int_{x_0+\delta}^1 \int_0^1 \right],
\]

that is,

\[
\int_0^1 I_1(x) \, dx < 2 G \int_0^1 du \int_0^1 W_n(u, v) \, dv.
\]

Similarly

\[
\int_0^1 I_3(x) \, dx < 2 G \int_0^1 du \int_0^1 W_n(u, v) \, dv.
\]

Next we consider \( I_2(x) \). Integrating with respect to \( x \)

\[
\int_0^1 I_2(x) \, dx \\
= \int_0^1 dx \int_{y_0-\delta}^{y_0+\epsilon} F(x, y) W_n(x-x_0, y-y_0) \, dy \\
= \left[ \int_1^{x_0-\delta} dx \int_{y_0-\delta}^{y_0+\epsilon} W_n(x-x_0, y-y_0) \, dy \right. \\
+ \int_{x_0-\delta}^{x_0+\epsilon} dx \int_{y_0-\delta}^{y_0+\epsilon} W_n(x-x_0, y-y_0) \, dy \\
\left. + \int_{x_0+\epsilon}^1 dx \int_{y_0-\delta}^{y_0+\epsilon} W_n(x-x_0, y-y_0) \, dy \right],
\]

\[ \times F(x, y) W_n(x-x_0, y-y_0) \, dy = J_1 + J_2 + J_3. \]

As in the case of \( I_1, I_3 \), we obtain

\[ J_1 < 2 G \int_0^1 du \int_0^1 W_n(u, v) \, dv, \]

\[ J_3 < 2 G \int_0^1 du \int_0^1 W_n(u, v) \, dv. \]

Lastly
\[ J = \int_{-\delta}^{+\delta} du \int_{-\varepsilon}^{+\varepsilon} F(x_0 + u, y_0 + v) W_n(u, v) \, dv \]

\[ = 4 F(x_0 + u', y_0 + v') \int_{0}^{\delta} du \int_{0}^{\varepsilon} W_n(u, v) \, dv, \quad |u'| < \delta, \quad |v'| < \varepsilon. \]

Therefore

\[ \left| \int_{0}^{\delta} I(x) \, dx - 4 F(x_0, y_0) \int_{0}^{\delta} du \int_{0}^{\varepsilon} W_n(u, v) \, dv \right| < 4 \left| F(x_0 + u', y_0 + v') - F(x_0, y_0) \right| \int_{0}^{\delta} du \int_{0}^{\varepsilon} W_n(u, v) \, dv \]

\[ + 4G \left[ \int_{0}^{\delta} \int_{0}^{\varepsilon} + \int_{0}^{\delta} \int_{\varepsilon}^{\delta} + \int_{\delta}^{\delta} \int_{0}^{\varepsilon} + \int_{\delta}^{\varepsilon} \int_{0}^{\delta} \right] \]

\[ < 2 \omega \int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv + 4G \left[ \int_{0}^{\delta} \int_{0}^{\varepsilon} + \int_{0}^{\delta} \int_{\varepsilon}^{\delta} + \int_{\delta}^{\delta} \int_{0}^{\varepsilon} + \int_{\delta}^{\varepsilon} \int_{0}^{\delta} \right] W_n(u, v) \, du \, dv \]

In consequence of these results, we obtain

\[ \left| \int_{0}^{\delta} \int_{0}^{\varepsilon} F(x, y) \, W_n(x-x_0, y-y_0) \, dy - 4 F(x_0, y_0) \int_{0}^{\delta} du \int_{0}^{\varepsilon} W_n(u, v) \, dv \right| \]

\[ < 2 \omega \int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv + 4G \left[ \int_{0}^{\delta} \int_{0}^{\varepsilon} + \int_{0}^{\delta} \int_{\varepsilon}^{\delta} + \int_{\delta}^{\delta} \int_{0}^{\varepsilon} + \int_{\delta}^{\varepsilon} \int_{0}^{\delta} \right] W_n(u, v) \, du \, dv \]

or

\[ \left| F_n(x_0, y_0) - F(x_0, y_0) \right| \]

\[ < \frac{\omega}{2} \frac{\int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv}{\int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv} \]

\[ + G \frac{\int_{0}^{\delta} \int_{0}^{\varepsilon} + \int_{0}^{\delta} \int_{\varepsilon}^{\delta} + \int_{\delta}^{\delta} \int_{0}^{\varepsilon} + \int_{\delta}^{\varepsilon} \int_{0}^{\delta}}{\int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv} \]

But by our lemmas

\[ 0 < \frac{\int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv}{\int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv} < 1, \]

\[ 0 < \frac{\int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv}{\int_{0}^{\delta} \int_{0}^{\varepsilon} W_n(u, v) \, du \, dv} < \frac{\omega}{4G}, \text{ etc.}, \quad n \geq v(\omega). \]

Thus we have arrived at

\[ \left| F_n(x_0, y_0) - F(x_0, y_0) \right| < \omega, \quad n \geq v(\omega), \]

which proves our theorem.
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III.

In a special case where \( F(x, y) = 1 \), we have

\[
\lim_{n \to \infty} \lambda_n(x_0, y_0) = \lim_{n \to \infty} \frac{\int_0^1 dx \int_0^1 W_n(x-x_0, y-y_0) \, dy}{\int_0^1 du \int_0^1 W_n(u, v) \, dv} = 1 \quad (1) .
\]

Therefore if we put

\[
I_n(x_0, y_0) = \frac{\int_0^1 dx \int_0^1 F(x, y) \, W_n(x-x_0, y-y_0) \, dy}{\int_0^1 dx \int_0^1 W_n(x-x_0, y-y_0) \, dy} \lambda_n(x_0, y_0)
\]

then

\[
\lim_{n \to \infty} I_n(x_0, y_0) = I(x_0, y_0).
\]

In the other words, \( F(x_0, y_0) \) can be approximately expressed by a sequence of rational functions

\[
\int_0^1 dx \int_0^1 F(x, y) \left[ 1 - (x-x_0)^2 (y-y_0)^2 \right]^n \, dy
\]

\[
\int_0^1 dx \int_0^1 \left[ 1 - (x-x_0)^2 (y-y_0)^2 \right]^n \, dy
\]

These theorems can be extended to the case of many variables:

(I) If \( F(x_1, x_2, \ldots, x_m) \) be limited and continuous except at a finite number of points in the domain \((0 < x_1, x_2, \ldots, x_n < 1)\), then at every point where \( F(x_1, x_2, \ldots, x_m) \) is continuous the sequence of polynomials

\[
\int_0^1 \cdots \int_0^1 F(u_1, u_2, \ldots, u_m) \left[ 1 - (u_1-x_1)^2 (u_2-x_2)^2 \cdots (u_m-x_m)^2 \right]^n \, du_1 \, du_2 \cdots du_m
\]

converges uniformly to \( F(x_1, x_2, \ldots, x_m) \) as \( n \) tends to infinity.

(II) Under the similar conditions as above the sequence of rational functions

\[
\int_0^1 \cdots \int_0^1 F(u_1, u_2, \ldots, u_m) \left[ 1 - (u_1-x_1)^2 (u_2-x_2)^2 \cdots (u_m-x_m)^2 \right]^n \, du_1 \, du_2 \cdots du_m
\]

\[
\int_0^1 \cdots \int_0^1 \left[ 1 - (u_1-x_1)^2 (u_2-x_2)^2 \cdots (u_m-x_m)^2 \right]^n \, du_1 \, du_2 \cdots du_m
\]

is also uniformly convergent to \( F(x_1, x_2, \ldots, x_m) \) as \( n \) tends to infinity.

Ikeda near Osaka, Oct. 1918.

\(^{(1)}\) It is easy to prove that \( \lambda_n \) decreases to zero monotonously with \( n \geq n_0 \), where \( n_0 \) is a fixed positive integer.