On Projective Generalization of Some Theorems on Algebraic Curves and Surfaces,

by

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In one of my foregoing essays\(^1\), I have given some noneuclidean metric theorems as necessary consequences of the corresponding euclidean ones. Now noneuclidean geometry is a generalization of euclidean. Thus it is natural to expect such theorems as were given there to be in reality of a purely projective nature. In this paper, I will give some projective generalizations of some metric theorems on algebraic curves and surfaces and thence deduce some consequences.

1. In my former essay\(^2\), I have proved the following theorems and have shown that they are projective generalizations of the theorems of Ceva and Menelaus:—

Theorem I. The necessary and sufficient condition that a general point $K$ and a general line $PQR$ may stand in an anharmonic pole and polar relation with respect to a triangle $ABC$, is

$$\rho_1\rho_2\rho_3 = -1,$$

where

$$(AK, BC) = P_1, \ (BK, CA) = Q_1, \ (CK, AB) = R_1,$$

$$(BC, PP_1) = \rho_1, \ (CA, QQ_1) = \rho_2, \ (AB, RR_1) = \rho_3.$$

Theorem I'.\(^3\) The necessary and sufficient condition that a general point $K$ and a general plane $P_{23}P_{24}P_{13}P_{14} (P_{ij}$ lying on the edges $A_iA_j$) may stand in an anharmonic pole and polar relation with respect to a tetrahedron $A_1A_2A_3A_4$, is that any three of

$$\rho_{12}\rho_{14}\rho_{13} + \rho_{23}\rho_{21}\rho_{24} = 0, \ \rho_{12}\rho_{14}\rho_{13} + \rho_{23}\rho_{21}\rho_{24} = 0,$$

$$\rho_{12}\rho_{14}\rho_{13} + \rho_{23}\rho_{21}\rho_{24} = 0, \ \rho_{12}\rho_{14}\rho_{13} + \rho_{23}\rho_{21}\rho_{24} = 0.$$

\(^1\) Ota, Some Noneuclidean Metric Theorems as Necessary Consequences of the Corresponding Euclidean Theorems. These Journals, vol. 17, Nos. 3, 4, p. 282.


\(^3\) Ota, ibid. p. 244.
hold, where

\([A_k A_i K \overline{A_i A_j}] \equiv P_{ij}\)
\((A_2 A_3, P_{43}P'_{32}) \equiv \rho_{32}^\Omega / \rho_{32}^\Omega\),
\((A_2 A_4, P_{42}P'_{24}) \equiv \rho_{42}^\Omega / \rho_{42}^\Omega\),
\((A_1 A_2, P_{12}P'_{21}) \equiv \rho_{12}^\Omega / \rho_{12}^\Omega\),
\((A_1 A_3, P_{13}P'_{31}) \equiv \rho_{13}^\Omega / \rho_{13}^\Omega\),
\((A_1 A_4, P_{14}P'_{41}) \equiv \rho_{14}^\Omega / \rho_{14}^\Omega\).

From this we may deduce the following

**Theorem I.** If a plane cuts the sides \(A_i A_j\) of a gauche quadrilateral \(A_1 A_2 A_3 A_4\) in the points \(P_{ij}\), then

\(\rho_{ij} \rho_{23} \rho_{34} \rho_{41} = 1\),

where

\(\rho_{ij} \equiv (A_i A_j, P_{ij}P'_{ij})\),
\([A_k A_i K \overline{A_i A_j}] \equiv P_{ij}\),
\(K\) being an arbitrary point not coincident with any of \(A_1, A_2, A_3, A_4\).

**Proof.**

\(\rho_{12} \equiv \rho_{12}^{(1)} / \rho_{12}^{(1)}\), etc.

\(= \prod_{s=1}^n \rho_{ij} \rho_{kl} \rho_{lm} \rho_{mn} = (-1)^{n} \)

by the first and the third conditions in I.

**2. Theorem II.** If a plane curve of \(n\)-th order meet the sides \(A_i A_j\) of a triangle \(A_1 A_2 A_3\) in \(P_{ij}^q\), \((q=1, 2, \ldots, n)\) respectively, then

\(\prod_{s=1}^n \rho_{ij}^q \cdot \rho_{kl}^q \cdot \rho_{lm}^q = (-1)^n\),

where

\(\rho_{ij}^q \equiv A_i(A_j, EP_{ij}^q)\),
\(E\) being an arbitrary point in the plane of \(A_1 A_2 A_3\), and not coincident with any of \(A_1, A_2, A_3\).

**Theorem II'.** If the tangents drawn from the vertices \(a_1 a_2 a_3\) of a triangle \(a_1 a_2 a_3\) to a curve of \(n\)-th class be \(p_i^q\), \((q=1, 2, \ldots, n)\), then

\(\prod_{s=1}^n \rho_{ij}^q \cdot \rho_{kl}^q \cdot \rho_{lm}^q = (-1)^n\),

where

\(\rho_{ij}^q \equiv a_k(a_i a_j, ep_{ij}^q)\),
e being an arbitrary straight line not coincident with any of \(a_1, a_2, a_3\).
ON PROJECTIVE GENERALIZATION, ETC.

Proof. It is evidently sufficient to prove the left side only. If the coordinates of \( P_i^0 \) referred to \( A_i A_j A_k \) be \( \lambda_i^0 : \lambda_j^0 : \lambda_k^0 : 0 \), then

\[
\prod_{s=1}^{n} \frac{\lambda_s^0}{\lambda_j^0} \cdot \frac{\lambda_j^0}{\lambda_k^0} = (-1)^n.
\]

For, if the equation to the curve of \( n \)-th order be

\[
a_k x_k^n + a_j x_j^n + a_j x_j^n + \cdots = 0,
\]

putting \( x_k = 0 \), we have

\[
a_i x_i^n + \cdots + a_j x_j^n = 0,
\]

whose roots are

\[
x_i/x_j = \lambda_i^0/\lambda_j^0, \quad (s = 1, 2, \ldots, n),
\]

so that

\[
\prod_{s=1}^{n} \frac{\lambda_s^0}{\lambda_j^0} = (-1)^n \frac{a_i}{a_j},
\]

and therefore

\[
\prod_{s=1}^{n} \frac{\lambda_s^0}{\lambda_j^0} \cdot \frac{\lambda_j^0}{\lambda_k^0} \cdot \frac{\lambda_k^0}{\lambda_i^0} = (-1)^n \frac{a_j a_k a_i}{a_i a_j a_k} = (-1)^n(1).
\]

Since for the unit-point of the coordinates, we might have taken \( E \), we have

\[
\frac{i^0}{j^0} = A_k(A_i A_j, EP_j^0).
\]

Hence the result.

N.B. (1) These theorems are merely direct projective formulations of Carnot's theorem and the correlative. The case \( n = 1 \) corresponds to the theorem of Menelaus and its dual respectively.

(2) Also it should be remarked that the expression \( \rho_i^0 \rho_j^0 \rho_k^0 \) is independent of the choice of the point \( E \). For, if \( E' \) be any other point, we have

\[
\rho_i^0 = A_k(A_i A_j, E'P_j^0)
\]

\[
= A_k(A_i A_j, E'E') \cdot A_k(A_i A_j, E'E).
\]

But as will be shown later in Art. 4, \( \prod A_i(A_i A_j, E'E) = 1 \), so that the factors \( A_i(A_i A_j, E'E) \) give no effect to \( \rho_i^0 \cdot \rho_j^0 \cdot \rho_k^0 \). The same holds for the right side.

3. Theorem III. If a surface of \( n \)-th order meet the sides \( A_1 A_2 A_3 A_4 \) of a gauche quadrilateral \( A_1 A_2 A_3 A_4(2) \) planes drawn from the edges \( \alpha_i \alpha_j \) of a gauche quadrilateral \( \alpha_i \alpha_j \alpha_3 \alpha_4(3) \)

(1) This result is due to Laisant. Nouv. Ann. (3) 9, (1890), p. 7.

(2) The quadrilateral \( A_i A_j A_k A_l \) may be considered as a tetrahedron, in which the two edges \( A_l A_k, A_i A_i \) are counted twice but in reverse orders.
in the points $P^{(\mathcal{A})}_s$, $(s=1, 2, \ldots n)$, then
\[
\prod_{s=1}^{n} \rho^{(\mathcal{A})}_s \rho^{(\mathcal{A})}_s = 1,
\]
where
\[
\rho^{(\mathcal{A})}_s \equiv A_s \mathcal{A}_s(E, \mathcal{P}^{(\mathcal{A})}_s),
\]
$E$ being an arbitrary point not coincident with any of $A_1, A_2, A_3, A_4$.

Proof. If the coordinates of $P^{(\mathcal{A})}_s$ referred to $\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4$ be $i^{(\mathcal{A})}_1 : i^{(\mathcal{A})}_2 : i^{(\mathcal{A})}_3 : i^{(\mathcal{A})}_4$, then, as before,
\[
\prod_{s=1}^{n} \frac{i^{(\mathcal{A})}_s i^{(\mathcal{A})}_s i^{(\mathcal{A})}_s i^{(\mathcal{A})}_s}{i^{(\mathcal{A})}_1 i^{(\mathcal{A})}_2 i^{(\mathcal{A})}_3 i^{(\mathcal{A})}_4} = (-1)^n \frac{a_{1} a_{2} a_{3} a_{4}}{a_{1} a_{2} a_{3} a_{4}} = 1,
\]
where
\[
a_x i_{x_1} + a_y i_{y_1} + a_z i_{z_1} + \cdots = 0
\]
is the equation to the surface of $n$-th order. Since we might have taken $E$ for the unit point of coordinates, we have
\[
\lambda^{(\mathcal{A})}_s / \lambda^{(\mathcal{A})}_s = A_s \mathcal{A}_s(E, \mathcal{P}^{(\mathcal{A})}_s).
\]
Hence the result.

N.B. As for $E$, $e$, the similar remark might have been made as in the last Art.

4. Theorem IV. If the tangents drawn from the vertices $A_k$ of a triangle $\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3$ to a curve of $n$-th class meet the opposite sides $\mathcal{A}_4 \mathcal{A}_1$ in the points $Q^{(\mathcal{A})}_s$, $(s=1, 2, \ldots n)$, then
\[
\prod_{s=1}^{n} \rho^{(\mathcal{A})}_s \rho^{(\mathcal{A})}_s = 1,
\]
where
\[
\rho^{(\mathcal{A})}_s \equiv A_s \mathcal{A}_s(E, \mathcal{Q}^{(\mathcal{A})}_s),
\]
$E$ being an arbitrary point in the plane $\mathcal{A}_4 \mathcal{A}_2 \mathcal{A}_3$ not coincident with any of $A_1, A_2, A_3$.

These are results obtained by combining I with II', II respectively.

N.B. (1) The case $n=1$ of IV is the projective formulation of Ceva's theorem.
(2) As for $E, e$, we might have made the similar remark as in the last Art.

5. Theorem V. If the tangent planes drawn from the edges $A_iA_j$ of a quadrilateral $A_1A_2A_3A_4$ to a surface of $n$-th class meet the opposite edges $A_kA_l$ in the points $Q_{ij}^s$, $(s=1, 2, \ldots, n)$, then

$$
\prod_{s=1}^{n} \rho_{ij}^s \rho_{kl}^s \rho_{li}^s \rho_{kJ}^s = 1,
$$

where

$$
\rho_{ij}^s \equiv A_kA_l(A_iA_j, EQ_{ij}^s),
$$

$E$ being an arbitrary point, not coincident with any of $A_1, A_2, A_3, A_4$.

These are results obtained by combining the theorems III', III with III''.

N.B. (1) The case $n=1$ of V corresponds to the Ceva's theorem in space.

(2) As for $E, e$, we might have made similar remarks as in the last Art.

6. Theorem VI. If two curves of $n$-th order meet the sides $A_iA_j$ of a triangle $A_1A_2A_3$ in the points $P_{ij}^s, Q_{ij}^s$, $(s=1, 2, \ldots, n)$ respectively, then

$$
\prod_{s=1}^{n} \sigma_{ij}^s \sigma_{kl}^s \sigma_{li}^s \sigma_{kJ}^s = 1,
$$

where

$$
\sigma_{ij}^s \equiv (A_iA_j, P_{ij}^sQ_{ij}^s), (s=1, 2, \ldots, n).
$$

Proof. By the theorem II, we have

$$
\rho_{ij}^s \equiv A_k(A_iA_j, EP_{ij}^s),
$$

$$
\prod_{s=1}^{n} \rho_{ij}^s \rho_{kl}^s \rho_{li}^s \rho_{kJ}^s = (-1)^n;
$$

$$
\rho_{ij}^s \equiv A_k(A_iA_j, EQ_{ij}^s),
$$

$$
\prod_{s=1}^{n} \rho_{ij}^s \rho_{kl}^s \rho_{li}^s \rho_{kJ}^s = (-1)^n.
$$
Now
\[ s_{q_k}^o \equiv (A_i A_j \ P_i Q_j^o, \ A_k (A_i A_j, P_i Q_j^o), \ A_k (A_i A_j, E Q_j^o) \ A_k (A_i A_j, P_i E), \ A_k (A_i A_j, E Q_j^o) / A_k (A_i A_j, E P_i^o)). \]

\[ \therefore \ II \ s_{1^o}^o \ s_{2^o}^o \ s_{3^o}^o = (-1)^n = 1. \]

**Cor. 1.** If two curves of n-th order meet the sides \( A_i A_j \) of a triangle \( A_i A_j A_k \) in the points \( B_q, Q_j, (s=1, 2, \ldots, n) \) respectively, then

\[ \prod_{s=1}^{n} \frac{\sin A_i \overrightarrow{P_q^o}/k \sin A_j \overrightarrow{Q_j^o}/k}{\sin A_i \overrightarrow{P_j^o}/k \sin A_j \overrightarrow{Q_k^o}/k} = 1. \]

**Cor. 2.** If a curve of n-th order meets the sides \( A_i A_j \) of a triangle \( A_i A_j A_k \) in the points \( P_i^o, \ (s=1, 2, \ldots, n) \) respectively, then

\[ \prod_{s=1}^{n} \frac{\sin A_i \overrightarrow{P_i^o} /k}{\sin A_i \overrightarrow{P_j^o} /k} = 1. \]

**Proof.** If we take for one of the two curves any line counted n-times, we have

\[ \prod_{s=1}^{n} \frac{\sin A_i \overrightarrow{Q_j^o} /k}{\sin A_i \overrightarrow{Q_k^o} /k} = 1. \]

Hence the result.

**Theorem VII.** If the points in which two curves of n-th order meet the sides \( A_i A_j \) of a triangle \( A_i A_j A_k \) be joined to the opposite points, then

\[ \prod_{s=1}^{n} \frac{\sin A_i \overrightarrow{P_i^o} /k}{\sin A_i \overrightarrow{P_j^o} /k} = (-1)^n. \]

**Proof.** If we take for one of the two curves any point counted n-times, we have

\[ \prod_{s=1}^{n} \frac{\sin A_i \overrightarrow{Q_j^o} /k}{\sin A_i \overrightarrow{Q_k^o} /k} = (-1)^n. \]

Hence the result.

**Theorem IV'.** If the tangents drawn from the vertices \( a_i a_j \) of a triangle \( a_i a_j a_k \) to two curves of n-th class meet the opposite sides \( a_s \) respectively, then

\[ \prod_{s=1}^{n} \frac{\sin (a_i \overrightarrow{P_s^o})}{\sin (a_i \overrightarrow{P_j^o})} = (-1)^n. \]

**Proof.** If we take for one of the two curves any point counted n-times, we have

\[ \prod_{s=1}^{n} \frac{\sin (a_i \overrightarrow{Q_j^o} /k)}{\sin (a_i \overrightarrow{Q_k^o} /k)} = (-1)^n. \]
vertices $A_k$ by the lines $p^{(s)}_{i,j}, q^{(s)}_{i,j}$ ($s=1,2, \ldots, n$) respectively, then

$$\Pi_{s=1}^{n} \sigma_{i,j}^{(s)} \sigma_{i,j}^{(s)} = 1,$$

where

$$\sigma_{i,j}^{(s)} \equiv A_i A_j ((A_i A_k) (A_k A_j), p^{(s)}_{i,j} q^{(s)}_{i,j}),$$

$$(s=1,2, \ldots, n).$$

Cor. 1. If the points in which two curves of $n$-th order meet the sides $A_i A_j$ of a triangle $A_i A_j A_k$ be joined to the opposite vertices $A_k$ by the lines $p^{(s)}_{i,j}, q^{(s)}_{i,j}$ ($s=1,2, \ldots, n$) respectively, then

$$\Pi_{s=1}^{n} \sigma_{i,j}^{(s)} \sigma_{i,j}^{(s)} \infty_{i,j}^{(s)} = 1,$$

where

$$\infty_{i,j}^{(s)} \equiv \alpha_{i,j} (\alpha_i \alpha_j (\alpha_i \alpha_k), P^{(s)}_{i,j} Q^{(s)}_{i,j}),$$

$$(s=1,2, \ldots, n).$$

Cor. 2. If the points in which a curve of $n$-th order meet the sides $A_i A_j$ of a triangle $A_i A_j A_k$ be joined to the opposite vertices $A_k$ by the lines $p^{(s)}_{i,j}, (s=1,2, \ldots, n)$ respectively, then

$$\Pi_{s=1}^{n} \sin (\alpha_i p^{(s)}_{i,j}) \sin (\alpha_j q^{(s)}_{i,j}) = 1.$$

Cor. 2'. If the tangents drawn from the vertices $a_i a_j$ of a triangle $a_i a_j a_k$ to two curves of $n$-th class meet the opposite sides $a_k$ in the points $P^{(s)}_{i,j}, Q^{(s)}_{i,j}$ ($s=1,2, \ldots, n$) respectively, then

$$\Pi_{s=1}^{n} \sin A_i P^{(s)}_{i,j}/k \sin A_j Q^{(s)}_{i,j}/k = 1.$$

8. Theorem VIII. If two surfaces of $n$-th order meet the edges $A_i A_j$ of a gauche quadrilateral $A_i A_j A_k A_l$ in the points $P^{(s)}_{i,j}, Q^{(s)}_{i,j}$ ($s=1,2, \ldots, n$) respectively, then

$$\Pi_{s=1}^{n} \sigma_{i,j}^{(s)} \sigma_{i,j}^{(s)} = 1,$$

where

$$\sigma_{i,j}^{(s)} \equiv (A_i A_j, P^{(s)}_{i,j} Q^{(s)}_{i,j}).$$

Theorem VIII'. If the tangent planes drawn through the edges $a_i a_j$ of a gauche quadrilateral $a_i a_j a_k a_l$ to two surfaces of $n$-th class be $\pi^{(s)}_{i,j}, \tau^{(s)}_{i,j}$ ($s=1,2, \ldots, n$) respectively, then

$$\Pi_{s=1}^{n} \sigma_{i,j}^{(s)} \sigma_{i,j}^{(s)} = 1,$$

where

$$\sigma_{i,j}^{(s)} \equiv (a_i a_j, \pi^{(s)}_{i,j} \tau^{(s)}_{i,j}).$$

These theorems may be proved quite as VI and VI'.
Cor. 1°. If two surfaces of $n$-th order meet the edges $A_iA_j$ of a quadrilateral $A_1A_2A_3A_4$ in the points $P_i^s, Q_i^s$ $(s=1, 2, \ldots, n)$ respectively, then
\[
\prod_{s=1}^{n-1} \prod_{j=1}^{s-1} \frac{\sin \overrightarrow{A_iP_i^s}/k}{\sin \overrightarrow{A_iQ_i^s}/k} = 1.
\]

Cor. 2°. If a surface of $n$-th order meet the edges $A_iA_1$ of a quadrilateral $A_1A_2A_3A_4$ in the points $P_i^s, (s=1, 2, \ldots, n)$, then
\[
\prod_{s=1}^{n-1} \prod_{j=1}^{s-1} \frac{\sin \overrightarrow{A_iP_i^s}/k}{\sin \overrightarrow{A_iQ_i^s}/k} = 1.
\]

Proof. If we take for one of the two surfaces any plane counted $n$-times, we have
\[
\prod_{s=1}^{n} \prod_{j=1}^{s-1} \frac{\sin \overrightarrow{A_iP_i^s}/k}{\sin \overrightarrow{A_iQ_i^s}/k} = 1.
\]

Hence the result.

The following propositions are projective consequences of the last article.

Theorem IX. If the points in which two surfaces of $n$-th order meet the edges $A_iA_j$ of a quadrilateral $A_1A_2A_3A_4$ be joined to the opposite edges $A_kA_l$ by the planes $\pi_i^s, \tau_i^s$ $(s=1, 2, \ldots, n)$ respectively, then
\[
\prod_{s=1}^{n} \sigma_i^s = \prod_{s=1}^{n} \sigma_i^s = 1,
\]
where
\[
\sigma_i^s = A_kA_l(A_iA_j\pi_i^s\tau_i^s).
\]

Theorem IX'. If the tangent planes drawn through the edges $\alpha_i\alpha_j$ of a quadrilateral $\alpha_1\alpha_2\alpha_3\alpha_4$ to two surfaces of $n$-th class be $\pi_i^s, \tau_i^s$ $(s=1, 2, \ldots, n)$ respectively, then
\[
\prod_{s=1}^{n} \frac{\prod_{i=1}^{n} \sin \overrightarrow{A_i\pi_i^s}}{\prod_{i=1}^{n} \sin \overrightarrow{A_i\tau_i^s}} = 1.
\]

Cor. 1°. If the tangent planes drawn through the edges $\alpha_i\alpha_j$ of a quadrilateral $\alpha_1\alpha_2\alpha_3\alpha_4$ to two surfaces of $n$-th class be $\pi_i^s, \tau_i^s$ $(s=1, 2, \ldots, n)$ respectively, then
\[
\prod_{s=1}^{n} \frac{\prod_{i=1}^{n} \sin \overrightarrow{A_i\pi_i^s}}{\prod_{i=1}^{n} \sin \overrightarrow{A_i\tau_i^s}} = 1.
\]

Proof. If we take for one of the two surfaces any plane counted $n$-times, we have
\[
\prod_{s=1}^{n} \frac{\prod_{i=1}^{n} \sin \overrightarrow{A_i\pi_i^s}}{\prod_{i=1}^{n} \sin \overrightarrow{A_i\tau_i^s}} = 1.
\]

Hence the result.
Cor. 1°. If the points in which two surfaces of n-th order meet the edges \(A_iA_j\) of a quadrilateral \(A_1A_2A_3A_4\) be joined to the opposite edges \(A_kA_l\) by the planes \(\pi_i^\alpha\), \(\pi_j^\beta\) \((s=1, 2, \ldots n)\) respectively, then

\[
\prod_{s=1}^{n} \prod_{i,j} \frac{\sin (\alpha_i \pi_i^\alpha \pi_j^\beta)}{\sin (\alpha_j \pi_j^\beta \pi_i^\alpha)} = 1.
\]

Cor. 2°. If the points in which a surface of n-th order meet the edges \(A_iA_j\) of a quadrilateral \(A_1A_2A_3A_4\) be joined to the opposite edges \(A_kA_l\) by the planes \(\pi_i^\alpha\), \(\pi_j^\beta\) \((s=1, 2, \ldots n)\) respectively, then

\[
\prod_{s=1}^{n} \prod_{i,j} \frac{\sin (\alpha_i \pi_i^\alpha \pi_j^\beta)}{\sin (\alpha_j \pi_j^\beta \pi_i^\alpha)} = 1.
\]

10. \("\)Theorem X. Let a curve of n-th order meet the sides \(A_iA_j\) of a triangle \(A_1A_2A_3\) in the points \(P_{ij}^\alpha\) \((s=1, 2, \ldots n)\), and let the tangents drawn from the vertices \(A_k\) to a curve of n-th class meet the opposite side \(A_iA_j\) in the points \(Q_{ij}^\beta\) \((s=1, 2, \ldots n)\), then

\[
\prod_{s=1}^{n} \rho_{s1}^{\alpha} \rho_{s2}^{\beta} \rho_{s3}^{\gamma} = (-1)^n,
\]

where

\[
\rho_{ij}^\alpha \equiv (A_iA_j, P_{ij}^\alpha Q_{ij}^\beta).
\]

Proof. Take an arbitrary point \(E\), then we have

\[
\rho_{ij}^\alpha \equiv (A_iA_j, P_{ij}^\alpha Q_{ij}^\beta),
\]

\[
\equiv A_k(A_iA_j, EQ_{ij}^\beta) \cdot A_k(A_iA_j, P_{ij}^\alpha E).
\]
\[ \prod_{s=1}^{n} \prod_{i,j=1}^{n} \rho_{ij}^{(s)} = \prod_{s=1}^{n} \left[ \prod_{i=1}^{n} A_{i} (A_{i} A_{j}, EQ_{ij}^{(s)}) / \prod_{i=1}^{n} A_{j} (A_{i} A_{j}, EP_{ij}^{(s)}) \right], \]

\[ = (-1)^{n} \] by II and IV.

N.B. When \( n = 1 \), we obtain the anharmonic pole and polar relation with respect to the triangle. Cfr. Art. 1.

Cor. 1°. Let a curve of \( n \)-th order meet the sides \( A_{i} A_{j} \) of a triangle \( A_{1} A_{2} A_{3} \) in the points \( P_{ij}^{(s)} \) \((s = 1, 2, \ldots, n)\), and let the tangents drawn from the vertices \( A_{k} \) to a curve of \( n \)-th class meet the opposite side \( A_{i} A_{j} \) in the points \( Q_{ij}^{(s)} \) \((s = 1, 2, \ldots, n)\), then

\[ \prod_{s=1}^{n} \prod_{i,j=1}^{n} \sin \frac{A_{i}P_{ij}^{(s)} / k}{\sin A_{j}Q_{ij}^{(s)} / k} = (-1)^{n}. \]

Cor. 2°. Let a curve of \( n \)-th order meet the sides \( A_{i} A_{j} \) of a triangle \( A_{1} A_{2} A_{3} \) in the points \( P_{ij}^{(s)} \) \((s = 1, 2, \ldots, n)\), then

\[ \prod_{s=1}^{n} \prod_{i,j=1}^{n} \sin \frac{A_{i}P_{ij}^{(s)} / k}{\sin A_{j}Q_{ij}^{(s)} / k} = 1. \]

N.B. These are Carnot's theorems. Cfr. Art. 6, Cor. 2°.

11. Theorem XI. Let a surface of \( n \)-th order meet the sides \( A_{i} A_{j} \) of a quadrilateral \( A_{1} A_{2} A_{3} A_{4} \) in the points \( P_{ij}^{(s)} \) \((s = 1, 2, \ldots, n)\), and let the tangent planes drawn from the sides \( A_{i} A_{j} \) to a surface of \( n \)-th class meet the opposite sides \( A_{i} A_{j} \) in the points \( Q_{ij}^{(s)} \) \((s = 1, 2, \ldots, n)\), then

\[ \prod_{s=1}^{n} \prod_{i,j=1}^{n} \rho_{ij}^{(s)} = 1, \]

\[ \prod_{s=1}^{n} \prod_{i,j=1}^{n} \rho_{ij}^{(s)} \rho_{ij}^{(s)} = 1, \]

Theorem XI'. If the tangent planes drawn from the sides \( \alpha_{i} \alpha_{j} \) of a quadrilateral \( \alpha_{1} \alpha_{2} \alpha_{3} \alpha_{4} \) to a surface of \( n \)-th class be \( \tau_{ij}^{(s)} \) \((s = 1, 2, \ldots, n)\) and let the joining planes of the points of intersection of a surface of \( n \)-th order with the edges \( \alpha_{i} \alpha_{j} \), to the sides \( \alpha_{i} \alpha_{j} \), respectively be \( \tau_{ij}^{(s)} \) \((s = 1, 2, \ldots, n)\), then

\[ \prod_{s=1}^{n} \prod_{i,j=1}^{n} \rho_{ij}^{(s)} \rho_{ij}^{(s)} \rho_{ij}^{(s)} = 1, \]

\[ \prod_{s=1}^{n} \prod_{i,j=1}^{n} \rho_{ij}^{(s)} \rho_{ij}^{(s)} \rho_{ij}^{(s)} = 1, \]
where

\[ \rho_i^j \equiv (A_i A_j, P_i^j Q_j^i). \]

Proof. Take an arbitrary point E, then we have

\[ \rho_i^j \equiv (A_i A_j, P_i^j Q_j^i), \]

\[ \equiv A_i A_j \cdot (A_i A_j, P_i^j Q_j^i), \]

\[ \equiv A_i A_j (A_i A_j, E_i^j Q_j^i) A_i A_j (A_i A_j, P_i^j E). \]

\[ \therefore \prod_{s=1}^{n} \prod_{t=1}^{n} \rho_i^j = \prod_{s=1}^{n} \left[ \prod_{t=1}^{n} A_i A_j (A_i A_j, E_i^j Q_j^i) \right] \prod_{s=1}^{n} \left[ \prod_{t=1}^{n} A_i A_j (A_i A_j, P_i^j E) \right] \]

\[ = 1 \quad \text{by III and V.} \]

N.B. When \( n = 1 \), we obtain the anharmonic pole and polar relation with respect to the quadrilateral. Cfr. Art. 7.

Cor. 1. Let a surface of \( n \)-th order meet the sides \( A_1 A_2 A_3 A_4 \) of a quadrilateral \( A_1 A_2 A_3 A_4 \) in the points \( P_i^j (s = 1, 2, \ldots, n) \), and let the tangent planes drawn from the sides \( A_i A_j \) to a surface of \( n \)-th class meet the opposite sides \( A_i A_j \) in the points \( Q_i^j (s = 1, 2, \ldots, n) \), then

\[ \prod_{s=1}^{n} \prod_{t=1}^{n} \sin \frac{A_i P_i^j/k}{A_j Q_i^j/k} = 1. \]

Cor. 2. Let a surface of \( n \)-th order meet the sides \( A_1 A_2 A_3 A_4 \) of a quadrilateral \( A_1 A_2 A_3 A_4 \) in the points \( P_i^j (s = 1, 2, \ldots, n) \), then

\[ \prod_{s=1}^{n} \prod_{t=1}^{n} \sin \frac{A_i P_i^j/k}{A_j Q_i^j/k} = 1. \]

12. Theorem XII. If two plane curves of \( n \)-th order meet the sides \( A_i A_j \) of a \( p \)-sided polygon in

Cor. 1. If the tangent planes drawn from the sides \( \alpha \alpha_i \) of a quadrilateral \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \) to a surface of \( n \)-th class be \( \pi_i^j (s = 1, 2, \ldots, n) \), and let the joining planes of the points of intersection of a surface of \( n \)-th order with the edges \( \alpha \alpha_i \alpha_j \) respectively be \( \tau_i^j (s = 1, 2, \ldots, n) \), then

\[ \prod_{s=1}^{n} \prod_{t=1}^{n} \sin (\alpha_i \pi_i^j) \sin (\alpha_j \tau_i^j) = 1. \]

Cor. 2. Let the tangent planes drawn from the sides \( \alpha \alpha_i \) of a quadrilateral \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \) to a surface of \( n \)-th class be \( \pi_i^j (s = 1, 2, \ldots, n) \), then

\[ \prod_{s=1}^{n} \prod_{t=1}^{n} \sin (\alpha_i \pi_i^j) = 1. \]

Theorem XII'. If the tangents drawn from the vertices \( \alpha, \alpha_i \) of a \( p \)-sided polygon to two plane
Proof I. Take any point $O$ and consider the triangles $OA_iA_j$, $OA_iA_n$, $\ldots$ $OA_iA_{2n}$. Then by II, we have

$$\pi(OA_i) \cdot \pi(A_iA_j) \cdot \pi(A_jO) = (-1)^n$$

for the first curve, where

$$\pi(A_iA_j) \equiv \prod_{s=1}^{s=n} \sigma_{ij}^{(r)}$$

and

$$\sigma_{ij}^{(r)} \equiv \sigma_{ij}(O, A_i, E_{ij} P_{ij}^{(r)}),$$

with easily comprehensive notations.

$$\pi(OA_i) \cdot \pi(A_iA_j) \cdot \pi(A_jO) = (-1)^n,$$

$$\pi(OA_i) \cdot \pi(A_iA_j) \cdot \pi(A_iO) = (-1)^n,$$

$$\pi(OA_i) \cdot \pi(A_iA_j) \cdot \pi(A_jO) = (-1)^n.$$

Let the product of the left side be denoted by $\mathfrak{P}$, so that

$$\mathfrak{P} = (-1)^{pn}.$$

For the second curve, we obtain similar equalities, thus

$$\frac{\pi(A_iA_j)}{\pi'(A_iA_j)} = \frac{\prod_{s=1}^{s=n} O(A_iA_j, E_{ij} P_{ij}^{(r)})}{\prod_{s=1}^{s=n} O(A_iA_j, E_{ij} Q_{ij}^{(r)})},$$

$$= \prod_{s=1}^{s=n} O(A_iA_j, Q_{ij}^{(r)} P_{ij}^{(r)}),$$

$$= \prod_{s=1}^{s=n} \rho_{ij}^{(r)}$$
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\[
\frac{\pi(OA)}{\pi'(OA)} = \prod_{s=1}^{n} (OA_s, E_{ij} P_{ij}^{(s)}) = \prod_{s=1}^{n} (OA_s, E_{ij} Q_{ij}^{(s)}),
\]

Similarly

\[
\frac{\pi(A_i O)}{\pi'(A_i O)} = \prod_{s=1}^{n} (A_i O, Q_{ij}^{(s)} P_{ij}^{(s)}),
\]

so that

\[
\frac{\pi(OA) \cdot \pi(A_i O)}{\pi'(OA) \cdot \pi'(A_i O)} = 1.
\]

\[
\prod_{s=1}^{n} = \frac{\pi(A_i A_j) \cdot \pi(A_j A_k) \cdots \pi(A_p A_q)}{\pi'(A_i A_j) \cdot \pi'(A_j A_k) \cdots \pi'(A_p A_q)},
\]

\[
= \prod_{s=1}^{n} \rho_{ij}^{(s)} \rho_{jk}^{(s)} \cdots \rho_{pq}^{(s)} = \frac{(-1)^p}{(-1)^p} = 1.
\]

**Proof II.** Let the coordinates of \( A_s, P_{ij}^{(s)} \) be \((x^{(s)})\) and \((\lambda^s x^{(s)} - \mu^s x^{(s)})\) respectively. In order that the point \((\lambda x^{(s)} - \mu x^{(s)})\) may lie on the curve of \( n \)-th order

\[ f((x)) = 0, \]

we must have

\[ f((x)) \equiv \lambda^n f((x^{(s)})) + \cdots + \mu^n f((x^{(s)})) = 0. \]

If \( \lambda^s / \mu^s \), \((s=1, 2, \ldots, n)\) be the \( n \) roots of this equation, we have

\[
\prod_{s=1}^{n} \frac{\lambda^s}{\mu^s} = (-1)^n \frac{f((x^{(s)}))}{f((x^{(s)}))}.
\]

\[
\prod_{t=1}^{n} \prod_{i=1}^{n} \frac{\lambda^s}{\mu^s} = (-1)^n \frac{f((x^{(s)})) f((x^{(s)})) f((x^{(s)})) \cdots f((x^{(s)}))}{f((x^{(s)})) f((x^{(s)})) f((x^{(s)})) \cdots f((x^{(s)}))} = (-1)^n.
\]

Similarly, for the second curve, we have

\[
\prod_{t=1}^{n} \prod_{j=1}^{n} \frac{\lambda^s}{\mu^s} = (-1)^n.
\]
Now
\[ \frac{\lambda_{ij}^{(p)}}{\rho_{ij}^{(p)}} : \frac{\lambda_{ji}^{(p)}}{\rho_{ji}^{(p)}} = (A_i A_j, Q_i^{(p)}, P_j^{(p)}) = \rho_{ij}^{(p)}. \]

\[ \therefore \prod_{s=1}^{n} \rho_{i(s)}^{(p)} \rho_{j(s)}^{(p)} = \frac{(-1)^{2n}}{(-1)^{2n}} = 1. \]

N.B. These theorems are to be considered each as a generalization of II and II' respectively.

Cor. 1°. If two curves of n-th order meet the sides A_i A_j of a p-sided polygon A_1 A_2 \ldots A_p in \( P_i^{(p)}, Q_j^{(p)} (s=1, 2, \ldots, n) \) respectively, then

\[ \prod_{s=1}^{n} \left[ \frac{\sin \overrightarrow{A_i Q_i^{(p)}} / k \sin \overrightarrow{A_j P_j^{(p)}} / k}{\sin \overrightarrow{A_i Q_i^{(p)}} / k \sin \overrightarrow{A_j P_j^{(p)}} / k} \right] = 1. \]

Cor. 2°. If a curve of n-th order meet the sides A_i A_j of a p-sided polygon A_1 A_2 \ldots A_p in \( P_i^{(p)} (s=1, 2, \ldots, n) \), then

\[ \prod_{s=1}^{n} \left[ \frac{\sin \overrightarrow{A_i P_i^{(p)}} / k \sin \overrightarrow{A_j P_j^{(p)}} / k}{\sin \overrightarrow{A_i P_i^{(p)}} / k \sin \overrightarrow{A_j P_j^{(p)}} / k} \right] = \frac{(-1)^{2n}}{(-1)^{2n}} = 1. \]

Cor. 1°. If the tangents drawn from the vertices a_1 a_2 of a p-sided polygon a_1 a_2 \ldots a_p to two plane curves of n-th class be \( p_{i(s)}^{(p)}, q_j^{(p)} (s=1, 2, \ldots, n) \) respectively, then

\[ \prod_{s=1}^{n} \left[ \frac{\sin (\overrightarrow{a_1 q_i^{(p)}}) \sin (\overrightarrow{a_2 p_j^{(p)}})}{\sin (\overrightarrow{a_1 q_i^{(p)}}) \sin (\overrightarrow{a_2 p_j^{(p)}})} \right]. \]

Cor. 2°. If the tangents drawn from the vertices a_1 a_2 of a p-sided polygon a_1 a_2 \ldots a_p to a curve of n-th class be \( p_{i(s)}^{(p)}, q_j^{(p)} (s=1, 2, \ldots, n) \), then

\[ \prod_{s=1}^{n} \left[ \frac{\sin (\overrightarrow{a_1 q_i^{(p)}}) \sin (\overrightarrow{a_2 p_j^{(p)}})}{\sin (\overrightarrow{a_1 q_i^{(p)}}) \sin (\overrightarrow{a_2 p_j^{(p)}})} \right]. \]

provided it be true when \( n=1(1) \).

N.B. The left side is Carnot's theorem (1).

13. Theorem XIII. If two surfaces of n-th order meet the sides A_i A_j of a p-sided polygon

Theorem XIII'. If the tangent planes drawn from the sides a_1 a_2 of a p-sided polygon in space

(1) Here I followed Salmon's convention concerning the signs. See Salmon, Higher Plane Curves, (1879), p. 106.
$A, A_1, \ldots, A_n$ in space in $P_{ij}^{(x)}, Q_{ij}^{(y)}$ 
$(s=1, 2, \ldots, n)$ respectively, then

$$\prod_{s=1}^{n} \rho_{ij}^{(x)} \rho_{ij}^{(y)} \ldots \rho_{ij}^{(n)} = 1,$$

where

$$\rho_{ij}^{(x)} \equiv (A_i A_j, P_{ij}^{(x)} Q_{ij}^{(y)}).$$

These theorems may be proved quite as XII and XII'.

N.B. Here we might have stated corollaries similar to Cor. 1' and Cor. 2' in the last article.

14. Theorem XIV (1). If 
the joining lines of two given 
points $K, E$ with each vertex of a 
$(2m+1)$-sided polygon $A_i A_1 \ldots \ldots A_{m+1}$ meet the opposite sides $A_i A_j$ in the points $P_{ij}, Q_{ij}$ respectively, then

$$\rho_{i2} \rho_{i3} \ldots \rho_{2m+1} = 1,$$

where

$$\rho_{ij} \equiv (A_i A_j, P_{ij}^{(x)} Q_{ij}^{(y)}).$$

Proof. Let the coordinates of $A_i$ be $(x^{(i)})$, and let the coordinates of $P_{ij}$ be $(\lambda x^{(k)} - \mu x^{(l)})$, then we have

$$\begin{vmatrix}
  x_1^{(i)} & x_2^{(i)} & x_3^{(i)} \\
  x_1^{(k)} & x_2^{(k)} & x_3^{(k)} \\
  \lambda x_1^{(m+1)} - \mu x_2^{(n+2)} & \lambda x_2^{(m+1)} - \mu x_3^{(n+2)} & \lambda x_3^{(m+1)} - \mu x_3^{(n+2)}
\end{vmatrix} = 0,$$

where $(x^{(k)})$ are the coordinates of $K$, i.e.

$$\frac{\lambda^{m+1, m+2}}{\mu^{m+1, m+2}} = \begin{vmatrix}
  x_1^{(i)} & x_2^{(i)} & x_3^{(i)} \\
  x_1^{(k)} & x_2^{(k)} & x_3^{(k)} \\
  x_1^{(m+1)} & x_2^{(m+2)} & x_3^{(m+2)}
\end{vmatrix}.$$
Similarly,

\[ \frac{\lambda_{m+2, m+3}}{\mu_{m+2, m+3}} = \left| \begin{array}{ccc} \alpha_1^{(0)} & \alpha_2^{(k)} & \alpha_3^{(m+1)} \\ \alpha_1^{(m+2)} & \alpha_2^{(0)} & \alpha_3^{(m+1)} \\ \alpha_1^{(m+2)} & \alpha_2^{(0)} & \alpha_3^{(m+1)} \end{array} \right|, \]

\[ \frac{\lambda_{m+3, m+4}}{\mu_{m+3, m+4}} = \left| \begin{array}{ccc} \alpha_1^{(0)} & \alpha_2^{(k)} & \alpha_3^{(m+2)} \\ \alpha_1^{(m+3)} & \alpha_2^{(k)} & \alpha_3^{(m+2)} \\ \alpha_1^{(m+3)} & \alpha_2^{(k)} & \alpha_3^{(m+2)} \end{array} \right|. \]

\[ \begin{array}{c}
\vdots \\
\vdots \\
\end{array} \]

\[ \frac{\lambda_{1, 2}}{\mu_{1, 2}} = \left| \begin{array}{ccc} \alpha_1^{(m+2)} & \alpha_2^{(0)} & \alpha_3^{(m)} \\ \alpha_1^{(m+2)} & \alpha_2^{(0)} & \alpha_3^{(m)} \\ \alpha_1^{(m+2)} & \alpha_2^{(0)} & \alpha_3^{(m)} \end{array} \right|. \]

\[ \begin{array}{c}
\vdots \\
\vdots \\
\end{array} \]

\[ \frac{\lambda_{m, m+1}}{\mu_{m, m+1}} = \left| \begin{array}{ccc} \alpha_1^{(2m+1)} & \alpha_2^{(k)} & \alpha_3^{(m+1)} \\ \alpha_1^{(2m+1)} & \alpha_2^{(k)} & \alpha_3^{(m+1)} \\ \alpha_1^{(2m+1)} & \alpha_2^{(k)} & \alpha_3^{(m+1)} \end{array} \right|. \]

Now

\[ \frac{\left| \begin{array}{ccc} \alpha_1^{(k)} & \alpha_2^{(0)} & \alpha_3^{(m+2)} \\ \alpha_1^{(m+2)} & \alpha_2^{(0)} & \alpha_3^{(m+2)} \\ \alpha_1^{(m+2)} & \alpha_2^{(0)} & \alpha_3^{(m+2)} \end{array} \right|}{\left| \begin{array}{ccc} \alpha_1^{(0)} & \alpha_2^{(k)} & \alpha_3^{(m)} \\ \alpha_1^{(m+2)} & \alpha_2^{(k)} & \alpha_3^{(m)} \\ \alpha_1^{(m+2)} & \alpha_2^{(k)} & \alpha_3^{(m)} \end{array} \right|} = -1, \text{ etc.} \]

\[ \therefore \frac{\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_{2m+1}}{\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_{2m+1}} = (-1)^{2m+1} = -1. \]

Similarly, if the coordinates of \( Q_{ij} \) be \( (\lambda'_{a^{(0)}} - \mu'_{a^{(0)}}) \), we have

\[ \frac{\lambda'_{12} \cdot \lambda'_{23} \cdot \ldots \cdot \lambda'_{2m+1}}{\mu'_{12} \cdot \mu'_{23} \cdot \ldots \cdot \mu'_{2m+1}} = -1. \]

Now

\[ \rho_{ij} = (A_i A_j, Q_{ij} P_{ij}) = \frac{\lambda'_{ij}}{\mu'_{ij}}, \]

\[ \therefore \rho_{12} \rho_{23} \ldots \rho_{2m+1} = (-1)/(-1) = 1. \]

\textit{N. B.} If the number of sides be \( 2m \) we draw a straight line through one of the vertices, \( A_i \) say, and take it for the \((2m+1)\)-th side. Then we obtain \( XIV \), and correlatively for \( XIV' \).

\textbf{15. Theorem XV.} If the joining planes of two given straight lines \( k, c \) with each vertex of a \((2m+1)\)-sided gauche polygon \( A_i A_j \ldots A_{2m+1} \) meet the opposite side \( A_i A_j \) in the points \( P_{ij}, Q_{ij} \) respectively, then

\[ \rho_{12} \rho_{23} \ldots \rho_{2m+1} = 1, \]

where

\[ \rho_{ij} = (A_i A_j, Q_{ij} P_{ij}). \]

\textbf{Theorem XV'.} If the intersection-points of two given lines \( k, c \) with each face of a \((2m+1)\)-faced polygon \( \alpha_i \alpha_j \ldots \alpha_{2m+1} \) in space be connected with the opposite side \( \alpha_i \alpha_j \) by the planes \( \pi_{ij}, \tau_{ij} \) respectively, then

\[ \rho_{12} \rho_{23} \ldots \rho_{2m+1} = 1, \]

where

\[ \rho_{ij} = (\alpha_i \alpha_j, \tau_{ij} \pi_{ij}). \]
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Proof. Project the whole figure upon an adequate plane. Then to the image so obtained the theorem XIV is applicable, and projecting the figure back to the original figure, we obtain the result, since double ratios are not destroyed at all thereby.

N.B. (1) If the number of sides be $2m$ we consider an arbitrary line through one of the vertices as the $(2m+1)$-th side of null length, and the theorem remains still true. And correlative for XV'.

(2) If in XIV, XV, we write

$$\frac{\lambda_{ij}}{\mu_{ij}} = \frac{\sin A_i P_{ij}}{\sin A_i P_{ij}}; \quad \frac{\lambda'_{ij}}{\mu'_{ij}} = \frac{\sin A_i Q_{ij}}{\sin A_i Q_{ij}};$$

we obtain the corresponding metric theorems, and correlative for XIV' and XV'.

16. Theorem XVI. If the joining lines of a given point $K$ to each vertex of a $(2m+1)$-sided polygon $A_1A_2A_3,...,A_{2m+1}$ meet the opposite side $A_iA_j$ in the point $P_{ij}$ and if an arbitrary straight line $e$ cuts the sides $A_iA_j$ in the points $Q_{ij}$ then

$$\rho_{12}\rho_{23}...\rho_{lm+1,l}=1,$$

where

$$\rho_{ij} \equiv (A_iA_j Q_{ij}P_{ij}).$$

Theorem XVII. If the intersection-points of a given straight line $k$ with each side of a $(2m+1)$-sided polygon $a_1a_2...a_{2m+1}$ be connected with the opposite vertex $a_i$ by the straight line $p_{ij}$ and if an arbitrary point $E$ be connected with the vertex $a_{ij}$ by the straight line $e_{ij}$ then

$$\rho_{12}\rho_{23}...\rho_{mn+1,m}=1,$$

where

$$\rho_{ij} \equiv (a_i a_j q_{ij}p_{ij}).$$

Proof. For $K$, as in theorem XIV, we have

$$\frac{\lambda_{12} \lambda_{23} \cdots \lambda_{2m+1,l}}{\mu_{12} \mu_{23} \cdots \mu_{l,m+1}} = -1.$$

For $E$, as in the second proof for theorem XII, we have

$$\frac{\lambda'_{12} \lambda'_{23} \cdots \lambda'_{2m+1,l}}{\mu'_{12} \mu'_{23} \cdots \mu'_{2m+1,l}} = (-1)^{2m+1} = -1.$$

Now

$$\frac{\lambda'_{ij}}{\mu'_{ij}} : \frac{\lambda_{ij}}{\mu_{ij}} \equiv (A_iA_j Q_{ij}P_{ij}) \equiv \rho_{ij},$$

$$\therefore \rho_{12}\rho_{23}...\rho_{2m+1,1}=1.$$
**N. B.** If the number of sides be $2m$, we take an arbitrary straight line passing through any one of the vertices and take it for the $(2m + 1)$-th side, then the theorem remains also true. And corollarily.

**17. Theorem XVII.** If the joining planes of a given straight line $k$ to each vertex of a $(2m + 1)$-sided polygon $A_1A_2 \cdots \cdots A_{2m+1}$ in space meet the opposite side $A_iA_j$ in the point $P_{ij}$ and if an arbitrary plane $\varepsilon$ cut the sides $A_iA_j$ in the points $Q_{ij}$ then

$$\rho_{12} \rho_{23} \cdots \cdots \rho_{2m+1,1} = 1,$$

where

$$\rho_{ij} = (A_iA_j, Q_{ij}, P_{ij}).$$

**Theorem XVII'.** If the intersection-points of a given straight line $k$ with each face of a $(2m + 1)$-faced polygon $\alpha_1\alpha_2 \cdots \cdots \alpha_{m+1}$ in space be connected with the opposite side $\alpha_i\alpha_j$ by the planes $\pi_{ij}$, and if an arbitrary point $E$ is connected with the sides $\alpha_i\alpha_j$ by the planes $\tau_{ij}$ then

$$\rho_{12} \rho_{23} \cdots \cdots \rho_{2m+1,1} = 1,$$

where

$$\rho_{ij} = (\alpha_i\alpha_j, \tau_{ij}, \pi_{ij}).$$

**N. B.** (1) If the number of sides be $2m$, we take an arbitrary straight line passing through any one of the vertices and take it for the $(2m + 1)$-th side, then the theorem remains also true. And corollarily.

(2) In XVI, XVII and in XVI', XVII', if we write

$$\rho_{ij} = \frac{\sin A_iQ_{ij}/k}{\sin A_iP_{ij}/k} : \frac{\sin A_jQ_{ij}/k}{\sin A_jP_{ij}/k}$$

and

$$\rho_{ij} = \frac{\sin (a_i q_{ij})}{\sin (a_j q_{ij})} : \frac{\sin (a_j q_{ij})}{\sin (a_j q_{ij})}$$

respectively, we obtain corresponding metric theorems.

**18.** In conclusion, we remark that our principle of infinite plurality of projective geometries\(^{(1)}\) enables us to give an infinite number of analogies of the preceding theorems and whence to deduce an infinite number of theorems. Some simple examples will be given below.—

From theorems VII and VII', we have the

Theorem. If two curves of $2n$-th order $C, C'$ pass through three given points $a_1a_3, a_3a_1, a_1a_2$ each $n$-times, and if $t_{ij}^{(s)}, t_{ij}'^{(s)} (s=1, 2, \ldots n)$ be the tangents at $a_ia_j$ to $C$ and $C'$ respectively, then

$$\prod_{i=1}^{2n} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} \rho_{3i}^{(s)} = 1,$$

where

$$\rho_{ij}^{(s)} \equiv (a_ia_j, t_{ij}^{(s)}, t_{ij}'^{(s)}).$$

Cor. 1. The condition (1) may be rewritten

$$\prod_{i=1}^{2n} \prod_{j=1}^{2n} \frac{\sin(a_ia_j t_{ij}^{(s)}) \sin(a_ia_j t_{ij}'^{(s)})}{\sin(a_ia_j t_{ij}^{(s)}) \sin(a_ia_j t_{ij}'^{(s)})} = 1.$$

Cor. 2. If two rational quartic curves have three nodes $a_1a_k$, $a_ka_3$, $a_3a_1$ in common, and if the tangents at $a_ia_k$ to them be $t_{ij}^{(s)}$, $t_{ij}^{(s)}$, $t_{ij}^{(s)}$, $t_{ij}'^{(s)}$ respectively, then

$$\rho_{ij}^{(s)} \rho_{ij}'^{(s)} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} = 1,$$

where

$$\rho_{ij}^{(s)} \equiv (a_ia_j, t_{ij}^{(s)}, t_{ij}'^{(s)}).$$

The theorems II and II' enable us to formulate the following theorems:

Theorem. If a curve of $2n$-th order $C$ passes through three given points $a_2a_3, a_3a_1, a_1a_2$ each $n$-times, and if $t_{ij}^{(s)} (s=1, 2, \ldots n)$ be the tangents at $a_ia_j$ to $C$, then

$$\prod_{i=1}^{2n} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} = (-1)^n,$$

where

$$\rho_{ij}^{(s)} \equiv a_ia_j(a_ia_j, E_{ij}^{(s)}),$$

$E$ being an arbitrary point.

Theorem. If two curves of $2n$-th class $C, C'$ touch three given lines $A_2A_3, A_3A_1, A_1A_2$ each $n$-times, and if $T_{ij}^{(s)}, T_{ij}'^{(s)} (s=1, 2, \ldots n)$ be the contact-points on $A_ia_j$ of $C$ and $C'$ respectively, then

$$\prod_{i=1}^{2n} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} \rho_{3i}^{(s)} = 1,$$

where

$$\rho_{ij}^{(s)} \equiv (A_ia_j, T_{ij}^{(s)}, T_{ij}'^{(s)}).$$

Cor. 1. The condition (1) may be rewritten

$$\prod_{i=1}^{2n} \prod_{j=1}^{2n} \frac{\sin(A_ia_j T_{ij}^{(s)}) \sin(A_ia_j T_{ij}'^{(s)})}{\sin(A_ia_j T_{ij}^{(s)}) \sin(A_ia_j T_{ij}'^{(s)})} = 1.$$

Cor. 2. If two corational quartic curves have three bitangents $A_1A_2, A_2A_3, A_3A_1$ in common and if the contact points on $A_ia_j$ of them be $T_{ij}^{(s)}, T_{ij}'^{(s)}, T_{ij}^{(s)}, T_{ij}'^{(s)}$ respectively, then

$$\rho_{ij}^{(s)} \rho_{ij}'^{(s)} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} = 1,$$

where

$$\rho_{ij}^{(s)} \equiv (A_ia_j, T_{ij}^{(s)}, T_{ij}'^{(s)}).$$

Theorem. If a curve of $2n$-th class $C$ touches three given lines $A_1A_2, A_2A_3, A_3A_1$ each $n$-times, and if $T_{ij}^{(s)} (s=1, 2, \ldots n)$ be the contact-points on $A_ia_j$ of $C$, then

$$\prod_{i=1}^{2n} \rho_{ij}^{(s)} \rho_{ij}'^{(s)} = (-1)^n,$$

where

$$\rho_{ij}^{(s)} \equiv A_ia_j(A_ia_j, cT_{ij}^{(s)}),$$

c being an arbitrary line.
Cor. 1°. The condition (2) may be rewritten
\[
\frac{\sin (\alpha, t'_1) \sin (\alpha, t'_2) \sin (\alpha, t'_3)}{\sin (\alpha, t'_1) \sin (\alpha, t'_2) \sin (\alpha, t'_3)} = 1.
\]

Cor. 2°. If the nodal tangents of a rational quartic be \( t'_i \) (\( i, j = 1, 2, 3; s = 1, 2 \)), then
\[
\frac{\sin (\alpha, t'_1) \sin (\alpha, t'_2) \sin (\alpha, t'_3)}{\sin (\alpha, t'_1) \sin (\alpha, t'_2) \sin (\alpha, t'_3)} = 1.
\]

Cor. 3°. When \( n = 1 \), the curve is a conic and the condition (2) becomes
\[
\frac{\sin (\alpha, t'_1)}{\sin (\alpha, t'_1)} = 1.
\]

N.B. It is well known that \((t'_i a_k), (t'_j a_i), (t'_k a_j)\) are collinear and therefore Cor. 3° is nothing but a projective consequence of Menelaus' theorem. And correlatively.

The theorems VIII and VIII' enables us to formulate the following theorems:

**Theorem.** If two surfaces of 3n-th order go through the sides \( \alpha, \alpha, \alpha \) of a gauche quadrilateral \( \alpha, \alpha, \alpha, \alpha \) each n-times and have tangent planes \( \pi_{ij}^{(s)}, \pi'_{ij}^{(s)} \) (\( s = 1, 2, \ldots, n \)) respectively therealong, then
\[
(3) \quad \prod_{s=1}^{n} \sigma_{ij}^{(s)} \sigma_{ij}^{(s)} = 1,
\]
where
\[
\sigma_{ij}^{(s)} \equiv (\alpha, \alpha, \pi_{ij}^{(s)}, \pi'_{ij}^{(s)}).
\]

**Theorem.** If two surfaces of 3n-th class touch the sides \( \alpha, \alpha, \alpha \) of a gauche quadrilateral \( \alpha, \alpha, \alpha, \alpha \) each n-times and have contact points \( P_{ij}^{(s)}, P'_{ij}^{(s)} \) (\( s = 1, 2, \ldots, n \)) respectively thereon, then
\[
(3) \quad \prod_{s=1}^{n} \sigma_{ij}^{(s)} \sigma_{ij}^{(s)} = 1,
\]
where
\[
\sigma_{ij}^{(s)} \equiv (\alpha, \alpha, P_{ij}^{(s)}, P'_{ij}^{(s)}).
\]
Cor. 1°. The condition (3) may be rewritten

\[ \prod_{i=1}^{n} \prod_{j=1}^{s} \frac{\sin \alpha_i \pi_i \phi_j}{\sin \alpha_j \pi_j \phi_i} = 1. \]

Cor. 2°. If two rational cubic surfaces go through the sides \( \alpha, \alpha_j \) of a gauche quadrilateral \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and have tangent planes \( \pi_{ij}, \pi'_ij \) respectively therealong, then

\[ \sigma_1, \sigma_2, \sigma_3, \sigma_4 = 1, \]

where

\[ \sigma_{ij} = (\alpha_i, \alpha_j, \pi_{ij}, \pi'_ij). \]

The theorems III and III' enable us to formulate the following theorems:

Theorem. If a surface of 3n-th order go through the sides \( \alpha, \alpha_j \) of a gauche quadrilateral \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) each \( n \)-times and have tangent planes \( \pi_{ij} \) \( (s=1, 2, \ldots, n) \) therealong, then

\[ \prod_{i=1}^{n} \prod_{j=1}^{s} \sin \alpha_i \pi_i \phi_j = 1, \]

where

\[ \phi_{ij} = \alpha_i \phi_j (\alpha_i, \phi_j, \pi_{ij}). \]

Cor. 1°. The condition (4) may be rewritten

\[ \prod_{i=1}^{n} \prod_{j=1}^{s} \frac{\sin \alpha_i \pi_i \phi_j}{\sin \alpha_j \pi_j \phi_i} = 1. \]

Cor. 2°. If four of the six lines connecting the conic nodes of a rational cubic surface form the

Theorem. If a surface of 3n-th class touch the sides \( \alpha, \alpha_j \) of a gauche quadrilateral \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) each \( n \)-times and have contact points \( P_{ij}, P'_{ij} \) respectively thereon, then

\[ \prod_{i=1}^{n} \prod_{j=1}^{s} \sin \alpha_i \phi_j / k \sin \alpha_j \phi_i / k = 1. \]

Cor. 1°. The condition (3) may be rewritten

\[ \prod_{i=1}^{n} \prod_{j=1}^{s} \frac{\sin \alpha_i P_{ij} \phi_j / k}{\sin \alpha_j P_{ij} \phi_i / k} = 1. \]

Cor. 2°. If four of the six lines of intersection of the planes of conical contact of a corational
gauche quadrilateral $\alpha, \alpha, \alpha, \alpha,$ and the tangent planes along $\alpha, \alpha,$ to the rational cubic be $\pi_{ij},$ then
\[
\sin(\alpha, \pi_{ij}) \sin(\alpha, \pi_{2i}) \sin(\alpha, \pi_{3i}) \sin(\alpha, \pi_{4i})
\]
\[
\frac{\sin(\alpha, \pi_{1i}) \sin(\alpha, \pi_{2i}) \sin(\alpha, \pi_{3i}) \sin(\alpha, \pi_{4i})}{\sin(\gamma, \pi_{4i})}
\]
\[= 1.
\]
cubic surface form the gauche quadrilateral $A_i, A_i, A_i, A_i,$ and the contact-points on $A_i, A_i,$ of the corational cubic be $P_{ij},$ then
\[
\frac{\sin A_i P_{12}}{k} \sin \frac{A_i P_{23}}{k} \sin \frac{A_i P_{34}}{k}
\]
\[
\frac{\sin A_i P_{12}}{k} \sin \frac{A_i P_{23}}{k} \sin \frac{A_i P_{34}}{k}
\]
\[= 1.
\]
Dec. 15, 1921.

ERRATA.

I.

These Journals, vol. 17, Nos. 3, 4:
P. 256, line 17 from the top, for "2px(p+\ldots)" read "2px/(p+\ldots)";
P. 258, " 14 " 14 " bottom, for "can" read "can be";
" 11 " 11 " " for "space" read "space(1)";
" 3 " 3 " " for "Fig. 3" read "Fig. 4";
P. 268, " 12 " " " for "Fig. 4" read "Fig. 5";
" 6 " 6 " " for "Fig. 5" read "Fig. 6".

II.

These Journals, vol. 17, Nos. 3, 4:
P. 269, line 1 from the top, "(n=3)" should be added to the end.

III.

These Journals, vol. 19, Nos. 1, 2:
P. 77, line 10 from the bottom, for "are collinear. | are concurrent." read "are collinear, when and only when m+m' is | are concurrent, when and only when m+m' is even."
P. 78, line 7 from the bottom, for "always the case," read "the case, when and only when m+m' is even."