Axiomatic Investigation on Number-Systems, V

by

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Chapter IV.

(A) Singular Systems of Numbers.

Putting aside the idea of the ordinary system of numbers and admitting the perfect freedom of thinking, we may mentally construct many singular systems of numbers having very interesting properties. Though these systems have properties very different from those of the ordinary one, yet, by means of these systems, we may have certain clear insight of logical connection of fundamental mathematical laws and essential properties of certain fundamental elements (say, zero-element, unit-element) of a number-system. Therefore, here we shall try to mention some of them. But before entering it, we have to give the definitions of some particular elements accurately, since they occur often in our discussion.

Definition. If the result of operating the operations \(+\) and \(\times\) to any elements of a system of things is also an element of the system, then the system is called a number-system (in a wider sense).

Zero-element. The definition of zero-element is given by Benjamin Peirce and E. V. Huntington as follows.

Def. I. Any element \(Z\) satisfying the relation \((Z+Z)=Z\) is called a zero-element of the system. It is denoted by the symbol \(0\).

The more usual definition is as follows.

Def. II. Any element \(Z\) satisfying the relation \((A+Z)=A\) or \((Z+A)=A\), where \(A\) is any element of the system, is called a zero-element of the system. On account of this additive property, the zero-element is often called "the modulus of addition."

Fundamental property (a). The fundamental multiplicative property of a zero-element is as follows.

If \(Z\) is a zero-element, then the relations \((A\times Z)=Z\) and \((Z\times A)\), where \(A\) denotes any element of the system, always hold good.

If, in the number-system considered, all fundamental laws of

(1) Continuation of the paper, this Journal, Vols. 22 and 24.
operations hold good, then, from these fundamental laws and Def. I, we may deduce the above mentioned properties of zero-element. But, if a number-system does not satisfy some of fundamental laws, it may contain more than one zero-elements which are not equal to each other; or it may contain some elements satisfying Def. I, but not Def. II, or not fundamental property (a). Thus we may have several kinds of zero-elements.

Unit-element.

Def. I. Any element $U$, different from zero-element, which satisfies the relation $(U \times U) \equiv U$ is called a unit-element of the system. It is denoted by the symbol $\equiv$.

This definition is given by B. Peirce. The more usual definition is as follows.

Def. II. Any element $U$ satisfying the relation $(A \times U) \equiv A$, or $(U \times A) \equiv A$, where $A$ is any element of the system, is called a unit-element of the system. On account of this multiplicative property, the unit-element is often called "the modulus of multiplication."

If all fundamental laws hold good in the number-system considered, then from these laws and Def. I, we may deduce the property given in Def. II, and also may deduce that, if there are many unit-elements in the system, then they are all equal. But, in other cases, this is not necessarily so, and there may be an element satisfying Def. I, but not Def. II. Thus, in these cases, we may have several kinds of unit-elements.

Remark. In axiomatic treatment of number-system, the two propositions "an element $Z$ is a zero-element" and "an element $Z$ is equal to a zero-element" are not the same. To prove that an element $Z$ is a zero-element, it is necessary and sufficient that $Z$ satisfies the relation $(Z \equiv Z) \equiv Z$; but to prove that $Z$ is equal to a zero-element $R$, we have to show that $Z$ and $R$ satisfy the given condition of equality at that case. As we have already seen in the example given in Chapter II, there are several zero-elements which are equal to one another while there are also many zero-elements which are not equal to one another. Moreover, we may have an element $A$ which is not a zero-element, but is equal to a zero-element, by defining addition and multiplication suitably. For example, in a number-system consisting of all couples $(a, b)$, where $a$ and $b$ denote any ones of natural numbers and zero, if we define that $(a, b)$ and $(a', b')$ are equal when and only when $a$ and $a'$ are zeros or odd integers, and moreover if we define that $(a, b) \equiv (a', b')$
denotes an element \((a + a', b + b')\), then there is only one zero-element \((0, 0)\). But any element \((a, b)\) whose first constituent \(a\) is an odd integer is equal to the zero-element \((0, 0)\) by the definition of equality, while the element itself is not a zero-element. The same may be said of a unit-element.

To have a clear idea regarding to the relation of definitions of zero-element and unit-element, we give the two following theorems.

**Theorem.** In any system of numbers satisfying our fourteen postulates given in Chapter II, any element \(Z\) satisfying the relation \((Z \oplus Z) \ominus Z\) always satisfies the relation \((Z \otimes Z) \ominus Z\); but its converse is not true.

**Proof.** By the distributive law, we have the relation
\[
\{Z \otimes (Z \oplus Z)\} \ominus \{(Z \otimes Z) \oplus (Z \otimes Z)\},
\]
and accordingly we have the relation
\[
(Z \otimes Z) \ominus \{(Z \otimes Z) \oplus (Z \otimes Z)\}
\]
by the hypothesis and Postulates M6, E2, E1, M1. Therefore \(Z \otimes Z\) is a zero-element, and moreover \(Z\) is a zero-element by the hypothesis. But, in the system satisfying the fourteen postulates, any zero-elements are equal to one another, so that we have the relation \((Z \otimes Z) \ominus Z\).

That the converse does not hold good may be seen at once from the usual system of natural numbers.

Therefore, in any system satisfying the fourteen postulates, if we define a unit-element \(U\) as the element satisfying the relation \((U \otimes U) \oplus U\), then it is necessary to add that \(U\) does not satisfy the relation \((U \oplus U) \ominus U\), in order to distinguish the unit-element from the zero-element. But if we define the unit-element \(U\) as the element satisfying the relation \((A \otimes U) \ominus A\), is it required to add the same condition in this case also? This answer is given in the following theorem.

**Theorem.** If, in any system satisfying the fourteen postulates, an element \(U\) satisfies the relation \((A \otimes U) \ominus A\) and \((U \oplus U) \ominus U\) at the same time, then all elements of the system are all zero-elements and are equal to one another.

**Proof.** By the distributive law and the hypothesis, we have the relations
\[
\{A \otimes (U \oplus U)\} \ominus \{(A \otimes U) \ominus (A \otimes U)\}
\]
\[
\ominus A \oplus A.
\]
Similarly, we have
\[
\{U \otimes (A \oplus A)\} \ominus A \oplus A.
\]
\[
\therefore (1) \{A \otimes (U \oplus U)\} \ominus \{U \otimes (A \oplus A)\}
\]
(by Postulates E1, E2, M1, A1)
But by the hypothesis, we have the relation $U + U = U$, and accordingly by Postulate M6, we have the relation

$$(2) \quad A \lor (U + U) = (A \lor U).$$

From (1) and (2), we have the relations

$$(A \lor U) \lor (U \lor (A \lor U))$$

(by Postulates E1, E2, M1, A1),

$$\lor ( (U \lor A) \lor (U \lor A) )$$

(by distributive law),

$$\lor (A \lor U) \lor (A \lor U)$$

(by commutative law).

And at last from the hypothesis $(A \lor U) \lor A$, we have the relation

$$A \lor (A \lor A).$$

Therefore any element $A$ is a zero-element. Further, that all zero-elements are equal to one another may be proved in the usual manner.

Therefore, in any system satisfying the fourteen postulates, if we define a unit-element $U$ as the element satisfying the relation $(A \lor U) \lor U$, then it is not required to add the further condition in general, except the very particular case in which all elements of the system are zeros and are equal to one another. But though this latter case is very particular, yet we may construct such system of numbers, if we wish to have it.

For the necessity of making a distinction in later discussion, we define the several kinds of zero-elements and unit-elements as follows, though their terminology are perhaps not relevant.

*Def.* Any element which satisfies at least one relation $(Z \lor Z) = Z^{(1)}$ is called an ordinary zero-element or simply a zero-element.

*Def.* Any element which satisfies at least three relations $(Z \lor Z) = Z$, $(Z \lor A) = Z$, $(A \lor Z) = Z$ is called a perfect zero-element. There is a zero-element which satisfies the latter two relations imperfectly; namely, there is a zero-element $Z$ satisfying the relation $(zero-element Z) \lor A = (zero-element Z)$, the former zero-element being different from the latter zero-element. Such zero-element may be called the imperfect zero-element.

*Def.* Any element which satisfies all five relations $(Z \lor Z) = Z$, $(Z \lor A) = Z$, $(A \lor Z) = Z$, $(Z \lor A) = A$, $(A \lor Z) = A$ is called a true zero-element.

*Def.* Any element which satisfies at least two relations $(Z \lor Z) = Z$ and $Z \lor Z \lor Z$ is called an ordinary unit-element or simply a unit-element.

*Def.* Any element which satisfies all four relations $(Z \lor Z) = Z$, $(A \lor Z) = A$, $(Z \lor A) = A$, $(Z \lor Z) \lor Z$ is called a true unit-element.

Of course, there is an element which is an ordinary zero-element,

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(1) For the sake of simplicity, henceforth we shall use the symbol "=" instead of the symbol $\oplus$, since there is no fear of confusion in this section.
but neither perfect nor true zero-element. The same may be said of a unit-element.

Opposite and reciprocal elements.

Def. If \( A \) and \( B \) satisfy the relation \( (A \oplus B) = 0 \), then each of them is called an opposite element of the other.

Def. If \( A \) and \( B \) satisfy the relation \( (A \otimes B) = 1 \), then each of them is called a reciprocal element of the other.

Def. If \( A \) is an opposite and reciprocal element of itself at the same time, it is called the self-opposite-reciprocal-element of the system.

Remark. In the ordinary number-system, there cannot be a self-opposite-reciprocal-element; but, in our systems constructed hereafter, such element may occur.

**System (A) consisting of only three elements, all of which are zero- or unit-elements. (A simplest singular system of numbers).**

Consider a set of three elements \( A, B, C \), and define their laws of addition and multiplication by the following tables

\[
\begin{array}{c|ccc}
\oplus & A & B & C \\
\hline
A & A & B & C \\
B & B & B & C \\
C & C & C & B \\
\end{array}
\quad
\begin{array}{c|ccc}
\otimes & A & B & C \\
\hline
B & A & A & B \\
C & A & B & C \\
\end{array}
\]

For example, \( A \oplus A, A \oplus B, A \oplus C, B \oplus B \) denote \( A, B, C, B \) respectively. Moreover, in this system, two elements are said to be equal when and only when they are identical. We shall see what fundamental laws of mathematics are satisfied by this system.

1. The system satisfies Postulate E1. (If \( M = N \), then \( N = M \)).
2. The system satisfies Postulate E2. (If \( L = M \) and \( M = N \), then \( L = N \)).
3. The system satisfies Postulate A1. (\( L \oplus M \) is an element of the system).

That the above three are true follows at once from our definitions of equality and operation \( \oplus \).

4. The system satisfies Postulate A2. \( \{(L \oplus M) \oplus N\} = \{L \oplus (M \oplus N)\} \).

Proof. When \( L \) is \( A \), \( \{(L \oplus M) \oplus N\} \) denotes \( M \oplus N \), and \( \{L \oplus (M \oplus N)\} \) also denotes \( M \oplus N \); namely they denote the same element, and so by the definition of equality the relation \( \{(L \oplus M) \oplus N\} = \{L \oplus (M \oplus N)\} \) holds good. Similarly, when \( M \) is \( A \) or \( N \) is \( A \),
\{(L \oplus M) \oplus N\} and \{L \oplus (M \oplus N)\} denote the same element and so they are equal to each other. Therefore we have only to examine the cases, in which none of L, M, N is A. These cases are all included in the following.

1. \((B \oplus C) \oplus C = C \oplus C = B\) \(B \oplus (C \oplus C) = B \oplus B = B\),
2. \((B \oplus C) \oplus B = C \oplus B = C\)
3. \((B \oplus B) \oplus C = B \oplus C = C\)
4. \((B \oplus B) \oplus B = B \oplus B = B\)
5. \((C \oplus C) \oplus C = B \oplus C = C\)
6. \((C \oplus C) \oplus B = B \oplus B = B\)
7. \((C \oplus B) \oplus C = C \oplus C = B\)
8. \((C \oplus B) \oplus B = C \oplus B = C\)

Now, in each of these cases, \{(L \oplus M) \oplus N\} and \{L \oplus (M \oplus N)\} denote the same element as shown in the above, so that they are equal to each other.

5. The system satisfies Postulate A3. (Commutative law for addition).

This follows at once from the law of addition.

6. The system satisfies Postulate A4. (If \((L \oplus X) = (L \oplus Y)\), then \(X = Y\)).

Here we have to add the condition "if X is not a zero-element."

Now, in our system, A and B are zero-elements since the relations \((A \oplus A) = A\) and \((B \oplus B) = B\) hold good. Therefore X must be the element C. In this case, if L is A, then we have

\[L \oplus X = A \oplus C = C = A \oplus Y.\]

But in order that the relation \(A \oplus Y = C\) may hold, it is necessary that Y is C as may be seen easily from the addition-table. Therefore we have the relation \(X = Y\). When L is B or C, the similar reasoning gives the same result.

Remark. The converse "if \(X = Y\), then \((L \oplus X) = (L \oplus Y)\)" is always true without any restriction.

7. The system satisfies Postulate A6. \(\text{If } L = M\) and \(N = P\), then \((L \oplus N) = (M \oplus P)\).

8. The system satisfies Postulates M1, M3, M4, M6. The above two propositions follows at once from the definition of equality and the operations \(\oplus\) and \(\otimes\).


When A is a factor of \((L \otimes M) \otimes N\) and \(L \otimes (M \otimes N)\), both products
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reduce to the same element \( A \), so that they are equal to each other. Therefore we have only to examine the cases in which \( A \) does not occur. They may be proved as follows.

1. \( (B \times C) \times C = B \times C = B = B \times (C \times C) \),
2. \( (B \times C) \times B = B \times B = A = B \times (C \times B) \),
3. \( (B \times B) \times C = A \times C = A = B \times (B \times B) \),
4. \( (B \times B) \times B = A \times B = A = B \times (B \times B) \),
5. \( (C \times B) \times B = B \times B = A = C \times A = C \times (B \times B) \),
6. \( (C \times B) \times C = B \times C = B = C \times B = C \times (B \times C) \),
7. \( (C \times C) \times B = C \times B = B = C \times B = C \times (C \times B) \),
8. \( (C \times C) \times C = C \times C = C = C \times C = C \times (C \times C) \).
9. The system satisfies Postulate M5. (Distributive law).

When one of \( L, M, N \) is \( A \), it is at once seen that the law is true. Other cases are included in the following

1. \( B \times (B + B) = (B \times B) + (B \times B) \),
2. \( B \times (B + C) = (B \times B) + (B \times C) \),
3. \( C \times (C + C) = (C \times C) + (C \times C) \),
4. \( C \times (C + B) = (C \times C) + (C \times B) \),

since all other cases may be reduced to the above by using commutative law for addition. And that the above four relations are all true may be proved by reducing the both sides of equality to a single element by means of addition-and multiplication-tables.

11. Lastly the system does not satisfy Postulate A5, but it satisfies the converse of it.

For example, since the relation \( (C \odot C) = B = (B \odot B) \) holds good, we have the relation \( 2C = 2B \), but \( C \) is not equal to \( B \). Thus Postulate A5 does not hold good in this system. But the converse is always true, since, when \( L \) is equal to \( M \), \( L \) is identical with \( M \), so that \( \mu L \) is equal to \( \mu M \).

Thus our system of numbers satisfies all fundamental postulates except only one. But our system consists of only three elements and all these are fundamental elements of mathematics. Namely, \( A \) and \( B \) are zero-elements while \( C \) is a unit-element.

Two kinds of zero-elements. Though both \( A \) and \( B \) are zero-elements, yet they differ very strikingly in their important properties. Namely, the element \( A \) has all five characteristic properties of zero-element; but the element \( B \) has only one of them. Therefore the element \( A \) is a true zero-element and the element \( B \) an ordinary zero-element. Moreover, since the relation \( A \oplus B = B = \emptyset \) holds good, \( A \)
and \( B \) are opposite elements.

Unit-element \( C \). Unit-element \( C \) has three characteristic properties of the unit-element, so that it is a true unit-element. Moreover, since the relations \( C \oplus C = B = \emptyset \) and \( C \otimes C = C = \{1\} \) hold good, \( C \) is an opposite-reciprocal element of itself. Therefore \( C \) is a unit-element as well as a self-opposite-reciprocal element. Thus our number-system has the following remarkable properties.

1. It consists of only three elements, two of which are zero-elements while the remaining one is the unit-element.
2. It satisfies 13 fundamental postulates (which lacks only one in a set of postulates defining the ordinary number-system).
3. The zero-elements of this system belong to different kinds and are opposite to each other; and the unit element of this system is also a self-opposite-reciprocal element of the system.

It is wonderful that, in number-systems satisfying all so-called fundamental laws of arithmetic (associative, commutative, distributive laws), there exists a miniature system of numbers, such that it consists of only three elements, all of which are zero and unit.

**System (B) consisting of nine elements, all of which are either zero or unit or self-opposite-reciprocal elements of the system.**

By using the numbers of System (\( A \)), we may construct very interesting system of numbers as follows.

Let \( a, b \) denote the numbers of System (\( A \)), and with these numbers, construct a number-pair \( (a, b) \), and consider a class of all these pairs; and define their equality and operations in the following way.

1. Two elements \( (a, b) \) and \( (a', b') \) are said to be equal when and only when \( a = a' \) and \( b = b' \).
2. \( (a, b) \oplus (a', b') \) denotes the element \( (a \oplus a', b \oplus b') \).
3. \( (a, b) \otimes (a', b') \) denotes the element \( (a \otimes a', b \otimes b') \).

Then this newly constructed system of numbers satisfies the following 11 fundamental postulates without any condition, and the remaining 3 fundamental postulates under certain conditions, and moreover satisfies the converse of these last three without any condition.

1. The system satisfies Postulates \( E1, E2; A1, A2, A3, A6; M1, M2, M3, M5, M6 \). The proof of this proposition may be easily effected by means of our definitions of equality, addition, multiplication of this system, and the properties of System (\( A \)).
2. The system satisfies Postulate A4 under the condition that 
\((a, b)\) is a true zero-element of the system. The converse of this 
postulate is always true without any condition.

3. The system satisfies Postulate M4 under the condition that 
\((a, b)\) is a true unit-element of the system. The converse of this 
postulate is always true without any condition.

4. The system satisfies Postulate A5 under the condition that \(\mu\) 
is an odd integer. The converse of this postulate is always true 
without any condition.

Now our system of numbers consists of nine elements, and we 
see that four of them are zero-elements and three of them are unit-
elements while the remaining two are self-opposite-reciprocal elements.
For, we have the following relation.

\[
\begin{align*}
(A, A) \oplus (A, A) &= (A, A), \\
(A, B) \oplus (A, B) &= (A, B), \\
(B, A) \oplus (B, A) &= (B, A), \\
(B, B) \oplus (B, B) &= (B, B), \\
(A, C) \otimes (A, C) &= (A, C), \\
(C, A) \otimes (C, A) &= (C, A), \\
(C, C) \otimes (C, C) &= (C, C), \\
(B, C) \oplus (B, C) &= (B, B) = \odot, \\
(B, C) \otimes (B, C) &= (A, C) = \odot, \\
(C, B) \oplus (C, B) &= (B, B) = \odot, \\
(C, B) \otimes (C, B) &= (C, A) = \odot.
\end{align*}
\]

Therefore the four elements \((A, A), (A, B), (B, A)\) and \((B, B)\) 
are zero-elements and are denoted by \(\odot_1, \odot_2, \odot_3, \odot_4\), respectively. 
Further the three elements \((A, C), (C, A)\) and \((C, C)\) are unit-elements 
and are denoted by \(\odot_1, \odot_2, \odot_3\) respectively. Lastly the two elements 
\((B, C)\) and \((C, B)\) are self-opposite-reciprocal elements and are denoted 
by \(\alpha_1\) and \(\alpha_2\) respectively. Here we shall investigate the remarkable 
properties of these elements.

**Principal properties of the four zero-elements.**

The first zero-element. With respect to the first zero-element, 
we have the following relations

\[
\begin{align*}
2\odot_1 &= \odot_1, \\
\odot_1^2 &= \odot_1, \\
\odot_1 + \alpha &= \alpha + \odot_1, \\
\odot_1 \times \alpha &= \odot_1 = \alpha \times \odot_1. \\
\end{align*}
\]

(\(\alpha\) is any element of the system).
Therefore this zero-element has all five fundamental properties of zero-element and so it is a true zero-element of the system. Moreover it has the fundamental property of unit-element, namely \( \varnothing_1 \times \varnothing_1 = \varnothing_1 \).

Other three zero-elements. With respect to these zero-elements, we have the following relations.

\[
\begin{align*}
2\varnothing_2 &= \varnothing_2, \\
\varnothing_2^2 &= \varnothing_1, \\
\varnothing_2 + \varnothing &= \varnothing, \\
\varnothing_2 \times \varnothing &= \varnothing_1 \text{ or } \varnothing_2. \\
\end{align*}
\]

Moreover, we have the following remarkable relations

\[
\begin{align*}
2\varnothing_3 &= \varnothing_3, \\
\varnothing_3^2 &= \varnothing_1, \\
\varnothing_3 + \varnothing &= \varnothing, \\
\varnothing_3 \times \varnothing &= \varnothing_1 \text{ or } \varnothing_3. \\
\end{align*}
\]

Moreover, we have the following remarkable relations

\[
\begin{align*}
2\varnothing_4 &= \varnothing_4, \\
\varnothing_4^2 &= \varnothing_1, \\
\varnothing_4 + \varnothing &= \varnothing, \\
\varnothing_4 \times \varnothing &= \varnothing_1 \text{ or } \varnothing_4. \\
\end{align*}
\]

Thus we see that only the zero-element \( \varnothing_1 \) satisfies all five relations \( (Z + Z) = Z, (Z + \varnothing) = (\varnothing + Z) = \varnothing, (Z \times \varnothing) = (\varnothing \times Z) = Z \) perfectly. Other three zero-elements satisfy only one relation \( (Z + Z) = Z \) perfectly; and the second relation \( (Z + \varnothing) = Z \) imperfectly. As to the third relation \( (Z \times \varnothing) = Z \), they satisfy it in a certain sense, but not in a rigorous sense. Namely, in the relation \( (Z \times \varnothing) = Z \) which they satisfy, both \( Z \)'s are zero-elements, yet they are not the same zero-element, but belong to different kinds of zero-elements. Therefore these four zero-elements present various difference among them as to the last two relations.

But they have the following interesting properties in common.

1. Each of them satisfies the relation \( (Z + Z) = Z \).
2. The \( n \)th power of each of them is equal to the true zero-element.
3. The product of any two of them is equal to the true zero-element.
4. The sum of any two of them is equal to a zero-element different from the true zero-element.
5. The product of the second, third, fourth zero-elements is equal to
6. The sum of the first, second, third zero-elements is equal to the fourth zero-element.

Principal properties of the three unit-elements.

With respect to the three unit-elements, we have the following relations.

Thus we have the theorem.

1. Each of the three unit-elements has 3 as a period with respect to addition.

2. Only the unit-element \( \Pi_3 \) satisfies three fundamental properties of unit-element and so it is the true unit-element.

3. The sum of the first and the second unit-elements is equal to the third unit-element.

4. The sum of all three unit-elements is equal to a zero-element.

5. The product of all three unit-elements is equal to the true zero-element.

Principal properties of self-opposite-reciprocal-elements.

With respect to two self-opposite-reciprocal-elements, we have the following relations.

Thus we have the theorem.

1. Each of two self-opposite-reciprocal-elements has 3 as a period
with respect to addition.

2. The \(n\)th power of each of self-opposite-reciprocal-elements is equal to a unit-element.

3. The product of the both self-opposite-reciprocal-elements is equal to a zero-element.

4. The sum of the both self-opposite-reciprocal-elements is equal to a unit-element.

5. The double of each of the self-opposite-reciprocal-elements is equal to a zero-element.

Here we recapitulate the principal properties and the wonderful singularities of our system.

1. Our number-system consists of only nine elements.

2. Our number-system contains only three kinds of numbers; zero-element, unit-element and self-opposite-reciprocal-element.

3. Our number-system satisfies 11 fundamental postulates and the converse of the remaining three postulates.

4. All of unit-elements and self-opposite-reciprocal-elements have 3 as a period with respect to addition.

5. Principal singularities of this system are as follows.
   
   (a). A multiple of a certain number which is not a zero-element is equal to a zero-element. For example, \(2\alpha_1 = \mathbf{0}_4\).

   (b). The square of a certain number which is not a unit-element is equal to a unit-element. For example, \(\alpha_2^2 = \mathbf{1}_2\).

   (c). The sum of a zero-element and a unit-element produces a number which is neither zero nor unit. For example, \(\mathbf{0}_3 + \mathbf{1}_4 = \alpha_1\).

   (d). The product of two numbers, neither of which is zero, is equal to zero. For example, \(\mathbf{1}_2 \times \alpha_1 = \mathbf{0}_3\).

   (e). The sum of two numbers, neither of which is zero, is equal to zero. For example, \(\mathbf{1}_1 + \alpha_1 = \mathbf{0}_4\).

   (f). The product of a unit-element and a number which is not a unit-element is equal to a unit-element. For example, \(\mathbf{1}_1 \times \alpha_1 = \mathbf{1}_1\).

   (g). The sum of a unit-element and a number which is not a zero-element is equal to a unit-element. For example, \(\mathbf{1}_2 + \alpha_1 = \mathbf{1}_3\).

   (h). The sum of three different unit-elements and also the product of them are both equal to zero.
System (C) containing two different unit-elements, and three different reciprocal-elements of one and the same element.

In order to construct the required system of numbers, first we shall explain some properties of a system of three numbers 0, 1, 2, obeying the following laws of operations.

1. $a + b = a + b \pmod{3}$,
2. $a \times b = a \times b \pmod{3}$,

or in tables,

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Moreover, two elements of this system are said to be equal when and only when they are identical.

Under these conventions, this system of numbers satisfies postulates $E_1, E_2; A_1, A_2, A_3, A_4, A_6; M_1, M_2, M_3, M_4, M_5, M_6$, but not $A_5$. Thus this number-system satisfies all thirteen fundamental postulates except only one. Moreover, in this system, 0 is the true zero-element, and 1 is the true unit-element, and 2 is the self-reciprocal element.

By using this system of numbers, we shall construct the following system of numbers, the number of whose elements is denumerably infinite.

Let $a$ denote the positive integers (1, 2, 3, ....) and $b$ the above numbers 0, 1, 2, and with these numbers, construct a number-pair $(a, b)$ and add to this aggregate of pairs a pair $(0, 0)$. Note that this system does not contain the elements $(0, 1)$ and $(0, 2)$. In order that we may treat these pairs of numbers as a number-system, we define their equality and operations as follows.

1. The two elements $(a, b)$ and $(a', b')$ are said to be equal when and only when $a = a'$ and $b = b'$.
2. $(a, b) \oplus (a', b')$ denotes the element $(a + a', b + b')$.
3. $(a, b) \otimes (a', b')$ denotes the element $(a \times a', b \times b')$.

(+ and $\otimes$ denote the ordinary addition and multiplication respectively).

Under these conventions, this number-system satisfies Postulates $E_1, E_2; A_1, A_2, A_3, A_4, A_6; M_1, M_2, M_3, M_4, M_6$; and that this is true may be seen easily. Moreover, this system satisfies Postulate
A5'. For, in our system, there is one and only one zero-element 
(0, 0); and as was remarked before, our system contains not the 
pairs (0, 1) and (0, 2). Therefore, when \( A \equiv (a, b) \) is not zero, \( a \) is 
different from zero, and accordingly \( \mu A \equiv \mu (a, b) \equiv (\mu a, b \oplus b \oplus \ldots \oplus b) \) 
is not equal to a zero-element.

But this system does not satisfy Postulate A5. For, take \((a, 1)\) 
and \((a, 2)\) as \( X \) and \( Y \) respectively and 3 as \( \mu \), then we have the 
relations

\[
3(a, 1) \equiv (3a, 1 \oplus 1 \oplus 1) \equiv (3a, 0),
\]
\[
3(a, 2) \equiv (3a, 2 \oplus 2 \oplus 2) \equiv (3a, 0),
\]
\[
\mu X \equiv \mu Y.
\]

But
\[
X(a, 1) \oplus Y(a, 2).
\]

Lastly, as to Postulate M5, before to determine whether the 
system satisfies it or not, we have to investigate some properties of a 
zero-element.

Discussion on zero-element.

Consider a number-system \( \{ A \} \), and suppose that this system con-
tains a subgroup \( \{ A_1 \} \) which forms a group with respect to addition 
and multiplication, namely suppose that the element which is obtained 
by operating the addition or multiplication to any two elements of 
the system \( \{ A_1 \} \) also belongs to that system, and moreover all elements 
of \( \{ A_1 \} \) belongs to \( \{ A \} \) while some of \( \{ A \} \) do not belong to \( \{ A_1 \} \).

If any element \( A_1 \) of the group \( \{ A_1 \} \) has the following two properties:

(i) \( \{ A_1 \} \) is also an element of \( \{ A_1 \} \),
(ii) \( \{ A_1 \} \otimes X \) and \( X \otimes A_1 \) are also elements of \( \{ A_1 \} \), where \( X \) is 
any element of the system \( \{ A \} \),

then we can deduce the following two propositions.

(I). When \( \{ A_1 \} \) consists of only one element, the element is a zero-
delement (perfect). For, since \( \{ A_1 \} \) contains only one element, it must 
be the element \( A_1 \) itself, therefore follows the relations

\[
A_1 \oplus A_1 = A_1,
\]
\[
A_1 \otimes X = A_1,
\]
\[
X \otimes A_1 = A_1.
\]

Therefore \( A_1 \) is a zero-element (perfect).

(II). If there is a zero-element (perfect) in the given system of 
numbers, then the subgroup \( \{ A_1 \} \) always contains it. For, denote the 
zero-element of the system by \( A_1 \), and take any element \( A_1 \) of \( \{ A_1 \} \), 
then by the property of \( \{ A_1 \} \), \( A_1 \otimes X \) is an element of \( \{ A_1 \} \), so that 
\( (A_1 \otimes A_1) \) must be an element of \( \{ A_1 \} \). But, by the property of zero-
element, \( A_i \otimes A_p \) is \( A_p \) itself. Therefore the zero-element \( A_p \) belongs to the subgroup \( \{ A_1 \} \).

By the above two propositions, when a given system of numbers contains a zero-element (perfect), the subgroup \( \{ A_1 \} \) may be called a generalized zero-group; and all elements, except the perfect zero-element contained in it, are called pseudo-zero-elements.

In the system \((B)\), the four elements \((0, 0, 0, 0)\) form a generalized zero-group, since the element which is obtained by operating addition and multiplication to any two elements of the system and to themselves is also an element of the system; and moreover, the product of any one of them and any element of System \((B)\) is also one of them. In this case, the element \((0, 0)\) is a true zero-element while the elements \((0, 0, 0)\) are pseudo-zero-elements.

In the system \((C)\), the subgroup \{(a, 0)\}(a=0, 1, 2, \ldots\) forms a generalized zero-group, and \((0, 0)\) is a true zero-element while all others are pseudo-ones.

Next we shall consider the zero-element of a number-system whose elements consists of two or more constituents. Denote any one element of the system by \((a, b, c, \ldots)\). If its constituents \(a, b, c, \ldots\) are connected by the relations \(b=\Psi_1(a), c=\Psi_2(a), \ldots\), where \(\Psi\) is a one valued function of \(a\), then the second, third, \ldots constituents are completely determined by the first constituent \(a\) and so the aggregate \\{(a, b, c, \ldots)\} may be put in a one-to-one relation with the aggregate \\{a\}. Thus, in such a case, \{(a, b, c, \ldots)\} may be treated as \\{a\}, and need not to be treated separately. By this reason, we give the following definition with respect to these number-systems.

Def. A number-system whose elements consist of two or more independent constituents is called a compound number-system; and in contrast with this, any number-system whose elements consist of only one constituent or whose elements may be put in a one-to-one relation with the elements of the above is called a simple number-system.

Def. Two constituents \(a, b\) to a number \((a, b)\) are said to be independent of each other with respect to addition and multiplication when, in the combinations
\[
(a, b) \oplus (a', b') = (a_1, b_1),
(a, b) \otimes (a', b') = (a_2, b_2),
\]
\(a_1\) and \(a_2\) are composed of \(a\) and \(a'\) only, and \(b_1\) and \(b_2\) of \(b\) and \(b'\) only.
Though the constituents of an element of a compound number-system are independent in themselves, yet it may happen that they are not independent of addition and multiplication. For example, though, in the ordinary complex number \((a, b)\), \(a\) and \(b\) are independent of each other, yet they are not so with respect to the multiplication, since we have the following multiplication-formula

\[(a, b) \times (a', b') = (aa' - bb', ab' + ba').\]

Therefore we have to distinguish two kinds of compound number-systems.

If constituents of elements of a compound number-system are independent of each other with respect to addition and multiplication, then the number-system is called the system of the first kind, and in all other cases, it is called the system of the second kind.

Now consider a compound number-system of the first kind and suppose that the first constituent of the sum of the two elements \((a, b)\) and \((a', b')\) is composed of \(a\) and \(a'\) operated by the operation \(\oplus\) in any manner whatever and also the second is in a similar manner, so that \((a, b) \oplus (a', b')\) denotes a number of the form \(\{\Psi(a, a', \oplus), \Phi(b, b', \oplus)\}\). Similarly, suppose that the result of multiplying two elements \((a, b)\) and \((a', b')\) denotes a number of the form \(\{\Psi_1(a, a', \otimes), \Phi_1(b, b', \otimes)\}\) taking the operation \(\otimes\) instead of the operation \(\oplus\). Then in such a system of numbers, if both constituents \(a, b\) be perfect zero-elements, the number \((a, b)\) is itself a perfect zero-element. For, \((a, b) \oplus (a, b)\) denotes a number \(\{\Psi(a, a, \oplus), \Phi(b, b, \oplus)\}\). But since \(a \oplus a\) and \(b \oplus b\) denote \(a\) and \(b\) respectively (\(a\) and \(b\) being zero-elements), so also \(\Psi(a, a, \oplus)\) and \(\Phi(b, b, \oplus)\) must denote \(a\) and \(b\) respectively. Therefore \((a, b) \oplus (a, b)\) denotes a number \((a, b)\) itself. Next, since \(a\) and \(b\) are perfect zero-elements, \(a \otimes a'\) and \(b \otimes b'\) denote \(a\) and \(b\) respectively whatever \(a'\) and \(b'\) may be, so that \(\Psi_1(a, a', \otimes)\) and \(\Phi_1(b, b', \otimes)\) also denote \(a\) and \(b\) respectively, whatever \(\Psi_1\) and \(\Phi_1\) may be. Therefore \((a, b) \otimes (a', b')\) denote a number \((a, b)\) itself for any number \((a', b')\) of the system. Accordingly \((a, b)\) is a perfect zero-element. From this consideration, we derive the following definition.

Def. If one of constituents of an element of a compound number-system, whose constituents are independent of each other with respect to addition and multiplication, be a perfect zero-element, then the element is called the semi-zero-element of the system.

In the following two theorems, it is supposed that the addition (multiplication) of any two elements of the system is defined by the
addition (multiplication) of their corresponding constituents and the equality of them is defined by the identity of their corresponding constituents.

Theorem. In any compound number-system containing a semi-zero-element, the fundamental postulate ‘if \( A \neq 0 \) and \( AX = AY \), then \( X = Y \)” cannot hold good always.

Proof. Any compound number-system contains two or more different elements having the same first constituent. For, if there is only one element having \( p \) as its first constituent, whatever \( p \) may be, then it follows that all elements of the system are entirely determined by the first constituent only and so the system may be considered as a simple number-system, contrary to the supposition that our system is compound. Therefore the system must contain two or more different elements having the same first constituent. Denote these two elements by \((p, q_1)\) and \((p, q_2)\) respectively.

Now take a semi-zero-element \( A = (m, n) \) and suppose that \( n \) is a perfect zero-element; and further take \((p, q_1)\) and \((p, q_2)\) as \( X \) and \( Y \). Then we have the relations

\[
A \otimes X = (m, n) \otimes (p, q_1) = \{ \Phi(m, p, \otimes), n \}, \\
A \otimes Y = (m, n) \otimes (p, q_2) = \{ \Phi(m, p, \otimes), n \},
\]

Therefore by the definition of equality, we have the relation

\[
(A \otimes X) = (A \otimes Y).
\]

But \( X \) and \( Y \) are not equal since \( q_1 \) and \( q_2 \) are not identical with each other. Therefore we see that, from the relation \((A \otimes X) = (A \otimes Y)\), the relation \( X \equiv Y \) does not necessarily follow; namely Postulate A5 does not hold good in such a number-system.

Theorem. In any compound number-system \( \{(a, b)\} \), the aggregate of semi-zero-elements may be divided into two groups, such that one of them has the first constituents of all their elements as zero-elements while the other has the second constituents as zero-elements. Each of these groups forms a generalized zero-group, provided that the element \((0, 0)\) is a perfect zero-element of the system. In this case, all semi-zero-elements are pseudo-zero-elements.

Proof. Firstly, this aggregate forms a group with respect to addition and multiplication, since the result of operating addition or multiplication to any two elements \((a, b_1)\) and \((a, b_2)\) of the aggregate is \( \{ \Psi(o, a, \oplus), \Phi(b_1, b_2, \oplus) \} \), or \( \{ \Psi(o, a, \otimes), \Phi(b_1, b_2, \otimes) \} \); and by the property of a perfect zero-element and of our number-system, \( \Psi(o, a, \oplus) \) and \( \Psi(o, a, \otimes) \) are both \( a \), and \( \Phi(b_1, b_2, \oplus) \) and \( \Phi(b_1, b_2, \otimes) \) are both
b', so that each of (o, b)⊕(o, b) and (o, b)⊗(o, b) denotes the element of the form (o, b). Secondly, any element (o, b) of the aggregate has the following two properties:

1. (o, b)⊕(o, b) denotes an element of the aggregate,
2. (o, b)⊗(o, b) and (o, b)⊗(o, b) denote elements of the aggregate, since Φ(o, a, b) denotes o always, whatever a may be. Thirdly, this aggregate contains the zero-element (o, o). Therefore this aggregate forms a generalized zero-group.

Now it is natural that a compound number-system usually treated contains a semi-zero-element as it is natural that a simple number-system usually treated contains a zero-element; and in most cases, Postulate M5 does not hold good for semi-zero-element in a compound number-system as it does not hold for zero-element in a simple number-system. Therefore when we treat simple and compound number-systems as a whole, it seems to me that it will be better to rewrite Postulate M5 as follows,

"If A is neither zero-element nor semi-zero-element and also if AX = AY, then X = Y".

Now let us return to the discussion of our number-system. In our system, any element of the form (a, o) is a semi-zero-element as well as pseudo-zero-element, and (o, o) is the true zero-elements. Besides these elements, there is neither zero-element nor semi-zero-element. Therefore, under our rewritten Postulate M5, we may prove that our number-system satisfies Postulate M5. For, since A≡(a, b) is neither zero-element nor semi-zero-element, both a and b are not zero. Therefore from the relation

\[(A \otimes X) = (A \otimes Y),\]
\[(a, b) \otimes (m, n) = (a, b) \otimes (m', n'),\]
\[(a \times m, b \otimes n) = (a \times m', b \otimes n'),\]
\[a \times m = a \times m', b \otimes n = b \otimes n',\]

the relations m=m', and n=n' and accordingly the relation (m, n)=(m', n') follows at once by our conventions of addition, multiplication and equality.

Thus our system satisfies Postulates E1, E2; A1, A2, A3, A4, A5' A6; M1, M2, M3, M4, M5, M6, and also Huntington's definition of equality, but not Postulate A5. Therefore this system also supplies an example which may be used to prove the impossibility of
deducing Postulate $A_4^{(1)}$ from Huntington's Postulates $A_1, A_2, A_3, A_4', A_5'; M_1, M_2, M_3, M_4, M_5$, when Postulate $M_3$ is taken in rewritten form.

Some properties of this number-system.

Theorem. In this system, there are two unit-elements $(1, 1)$ and $(1, 0)$. The first element is a modulus of multiplication and so is the true unit-element. The second is a semi-zero-element as well as an ordinary unit-element.

Proof. For, we have the relations

$$(1, 1) \circ (1, 1) = (1, 1),$$

$$(1, 1) \circ (a, b) = (a, b),$$

$$(1, 0) \circ (1, 0) = (1, 0).$$

Theorem. The aggregate of elements produced from the second unit-element by addition and multiplication forms a generalized zero-group when the true zero-element $(0, 0)$ is added to it. But the same aggregate produced from the true unit-element cannot form a generalized zero-group though it forms a subgroup with respect to addition and multiplication.

Proof. All the elements produced from the element $(1, 0)$ by addition and multiplication are of the form $(n, 0)$, since we have the relations

$$(1, 0)^n = (1, 0),$$

$$n(1, 0) = (n, 0) \quad n = 1, 2, 3, \ldots$$

Now it may be easily seen that the aggregate $\{(0, 0), (n, 0)\}$ forms a generalized zero-group.

Next all elements produced from $(1, 1)$ by addition and multiplication are of the form $(n, n \mod 3)$, since we have the relations

$$(1, 1)^n = (1, 1),$$

$$n(1, 1) = (n, n \mod 3).$$

Here the aggregate $\{(0, 0), (n, n \mod 3)\}$ forms a subgroup with respect to addition and multiplication. But it cannot be a generalized zero-group, since the product $(4, 0)$ of the elements $A = (2, 2)$ and $X = (2, 0)$ does not belong to the aggregate $\{(0, 0), (n, n \mod 3)\}$, $A$ being an element of this aggregate and $X$ being that of the given system.

Theorem. All elements of this system (except two elements $(0, 0)$ and $(1, 2)$) are represented by the sum of the two unit-elements; and the excepted two elements $(0, 0)$ and $(1, 2)$ are represented by the difference of them.

Proof. Denote the two unit-elements $(1, 0)$ and $(1, 1)$ by $e_1$ and $e_2$

---

(1) Our Postulates $A_5, A_5'$; $M_5$ correspond to Huntington's Postulates $A_4, A_4'$, $M_3$, respectively.
respectively, then we have the following relations
\[
\begin{align*}
(3p, 0) &= 3pe_1 \\
(3p, 1) &= (3p - 2)e_2 + 2e_1 \\
(3p, 2) &= (3p - 1)e_2 + e_1 \\
(3p + 1, 0) &= (3p + 1)e_1 \\
(3p + 1, 1) &= (3p + 1)e_2 \\
(3p + 1, 2) &= (3p - 1)e_2 + 2e_1p = 1 \\
(3p + 2, 0) &= (3p + 2)e_1 \\
(3p + 2, 1) &= (3p + 1)e_2 + e_1p = 0 \\
(3p + 2, 2) &= (3p + 2)e_2
\end{align*}
\]
Moreover \((0, 0)\) and \((1, 2)\) are obtained from the element \(X\) satisfying the following relations
\[
\begin{align*}
(1, 0) \oplus X &= (1, 0) \quad \text{or} \quad (e_1 \oplus X) = e_1, \\
(1, 0) \oplus X &= 2(1, 1) \quad \text{or} \quad (e_1 \oplus X) = 2e_2.
\end{align*}
\]
Remark. The above two units \(e_1\) and \(e_2\) are different from the units \(i, j, k\) used in complex number or quaternion. If we define the element satisfying the relations \((a \otimes a) = a\) and \((a \oplus a) = a\) as a primary unit and the element satisfying the relations \((a \otimes a) = - (\text{primary unit})\) and \((a \oplus a) = a\) as a secondary unit, then quaternion units \(i, j, k\) are secondary units while our units \(e_1\) and \(e_2\) are primary ones.

Reciprocal element.

Theorem. In this system, there are two and only two elements which have their reciprocal elements with respect to the first unit-element, and they are all self-reciprocal elements. As to the second unit-element, there are three elements having their reciprocal ones, and all these three are reciprocal elements of one and the same element.

Proof. In order that \((a, b)\) may be a reciprocal element of \((u, v)\) with respect to the first unit-element, the relation
\[
(a, b) \otimes (u, v) = (1, 1),
\]
or
\[
(a \times u, b \otimes v) = (1, 1),
\]
or
\[
a \times u = 1 \quad \text{and} \quad b \otimes v = 1
\]
must hold. Therefore \(a\) and \(u\) must be both 1, and \(v\) must be 1 or 2 according as \(b\) is 1 or 2; and when \(b = 0\), \(b \otimes v = 1\) cannot hold good. Thus we have two and only two pairs of reciprocal elements
\[
(a, b) \equiv (1, 1), \quad (u, v) \equiv (1, 1); \\
(a, b) \equiv (1, 2), \quad (u, v) \equiv (1, 2).
\]
Namely both elements are self-reciprocal elements.

Next in order that the same relation
\[
(a, b) \otimes (u, v) = (1, 0)
\]
may hold good with respect to the second unit-element, we should have the relations

\[ a \times u = 1, \quad b \circ v = 0. \]

Therefore, as before, \( a \) and \( u \) must be both 1, and \( v \) must be always 0 when \( b \) is 1 or 2. But if \( b \) is 0, then \( v \) may be 0 or 1 or 2. Thus we have the following pairs of reciprocal elements

\[
\begin{align*}
(a, b) &= (1, 0), \\
(u, v) &= (1, 1) \\
&\quad (1, 2); \\
(a, b) &= (1, 1), \\
(u, v) &= (1, 0); \\
(a, b) &= (1, 2), \\
(u, v) &= (1, 0).
\end{align*}
\]

But the latter two are included in the first. Thus we have the following interesting proposition.

"The second unit-element has three different reciprocal elements of it."

(B) On Distributive Law.

Let us recapitulate our fundamental postulates of number-system, established in Chapter II, and consider their formal properties. For convenience of comparison, here we write down their essential parts.

E1. If \( A \oplus B \), then \( B \oplus A \).

E2. If \( A \oplus B \) and \( B \oplus C \), then \( A \oplus C \).

A1. \( A \oplus B \) is an element of the system.

A2. \( \{ (A \oplus B) \oplus C \} \equiv \{ A \oplus (B \oplus C) \} \).

A3. \( (A \oplus B) \oplus (B \oplus A) \).

A4. If \( (A \oplus X) \equiv (A \oplus Y) \), then \( X \equiv Y \).

A5. If \( \mu X \equiv \mu Y \), then \( X \equiv Y \).

A6. If \( A \oplus B \) and \( C \oplus D \), then \( (A \oplus C) \oplus (B \oplus D) \).

M1. \( A \otimes B \) is an element of the system.

M2. \( \{ (A \otimes B) \otimes C \} \equiv \{ A \otimes (B \otimes C) \} \).

M3. \( (A \otimes B) \otimes (B \otimes A) \).

M4. If \( A \oplus 0 \) and \( (A \otimes X) \equiv (A \otimes Y) \), then \( X \equiv Y \).

M5. \( \{ A \otimes (B \oplus C) \} \equiv \{ (A \otimes B) \oplus (A \otimes C) \} \).

M6. If \( A \oplus B \) and \( C \oplus D \), then \( (A \otimes C) \equiv (B \otimes D) \).

Of these Postulates, all of them, except only one postulate M5, are simple and regular in form, and are symmetry with respect to the symbols \( \oplus \), \( \otimes \) and \( \equiv \). But only Postulate M5, which expresses the relation of two operations \( \oplus \) and \( \otimes \), is neither simple nor regular in form. Namely, the both sides of the equality are different in the number of the letters \( A, B, C \), and of the operations \( \oplus, \otimes \), and of the
brackets. Nor it is symmetry with respect to the operations $\oplus$ and $\otimes$, namely, if we interchange these operations, the resulting relation differs from that of the former. In this respect, this postulate is different strikingly from other postulates, as Euclid's postulate of parallels in geometry is different from other ones. Thus if a man will attempt to construct an ideal set of fundamental laws of numbers, free from all restraints of the ordinary numbers and their laws, he will try to change the distributive law (Postulate M5) though he will use all others unchanged. Now let us try to do this, namely let us try to construct a new system of numbers, whose fundamental laws are as simple, regular and symmetrical as possible. We shall preserve all other postulates except distributive law and replace the latter by more simple or more symmetrical ones containing two operations $\oplus$ and $\otimes$. Then what properties will have the system thus constructed? We shall see that, though these new systems are simple and regular in the forms of fundamental laws, yet they are very singular ones and perhaps would not be of any use. Thus, by such investigation, we are led to get the deep insight of the ordinary distributive law and to know the true meaning and true importance of it though its form is irregular. Before to enter into these studies, we must consider the definition and properties of zero-element and unit-element in the number-systems lacking the distributive law.

Zero-and unit-elements in a system lacking the distributive law.

Theorem (a). The propositions "if $(Z\oplus Z)\equiv Z$, then $(A\oplus Z)\equiv A$" and its converse "if $(A\oplus Z)\equiv A$, then $(Z\oplus Z)\equiv Z$" can be deduced without the distributive law.

Proof. By Postulate A3, we have the relation $(A\oplus B)\equiv (B \otimes A)$ for any elements $A$, $B$ of the system; therefore if we put $A$ instead of $B$, we have the relation $(A\oplus A)\equiv (A\oplus A)$. From this last relation, we have the relation $A\equiv A$ by Postulate A4.

Next, by Postulate A6, from the relation $A\equiv A$ and $(Z\oplus Z)\equiv Z$, we have the relation

$$A\equiv (Z\oplus Z)\equiv (A\oplus Z),$$

or

$$A\equiv (Z\oplus Z)\equiv (A\oplus Z) \quad \text{(by Prostulates A2, E2)},$$

or

$$A\equiv (Z\oplus Z)\equiv A \quad \text{(by Postulate A4)}.$$  

Q. E. D.

Similarly we may prove the converse theorem without the distributive law.

Theorem (b). The proposition "if $(Z\oplus Z)\equiv Z$, then $(A\otimes Z)\equiv Z$" cannot be deduced without the distributive law.
Proof. Consider the class of numbers 0, 1, 2, ..., and define their equality and operations as follows.

1. \(A\) and \(B\) are said to be equal when and only when they are identical.
2. \(A \oplus B\) denotes the element \((A + B)\).
3. \(A \otimes B\) denotes the element \((A + B + 1)\).

Then this class of numbers satisfies all thirteen postulates except the distributive law, but it does not satisfy the relation \((A \otimes Z) \equiv Z\), since \(A \otimes Z\) denotes the element \((A + Z + 1)\) and it is different from \(Z\) even when \(Z\) satisfies the relation \((Z \oplus Z) \equiv Z\), namely \(Z\) denotes the element 0. Hence it follows that the relation \((A \otimes Z) \equiv Z\) cannot be deduced from the above thirteen postulates. (Of course, this class does not satisfy the distributive law).

Remark. It is well known fact that the proposition "if \((Z \oplus Z) \equiv Z\), then \((A \otimes Z) \equiv Z\)" can be deduced from the above thirteen postulates and distributive law.

Theorem (c). The distributive law cannot be deduced from the above thirteen postulates and the proposition "if \((Z \oplus Z) \equiv Z\), then \((A \otimes Z) \equiv Z\)."

Proof. Consider the class of numbers 0, 1, 2, ..., and define their equality and operations as follows.

1. \(A\) and \(B\) are said to be equal when and only when they are identical.
2. \(A \oplus B\) denotes the element \((A + B)\).
3. \(A \otimes B\) denotes 0 if at least one of \(A\), \(B\) be zero, and it denotes the elements \((A + B + 1)\), if both of \(A\), \(B\) be not zero.

Then it is clear that this class satisfies \(E1, E2; A1, A2, A3, A4, A5, A6\) since equality and addition are the ordinary ones. Therefore we have only to examine Postulates \(M\)'s.

1. It satisfies Postulate \(M1\), since \(A \otimes B\) denotes \(A + B + 1\) or 0.
2. It satisfies Postulate \(M2\), since both \((A \otimes B) \otimes C\) and \(A \otimes (B \otimes C)\) denote 0 when at least one of \(A\), \(B\), \(C\) is zero, and denote \(A + B + C + 2\) in all other cases.
3. It satisfies Postulates \(M3\) and \(M6\), as may be easily seen.
4. It satisfies Postulate \(M4\). For, if \(X\) is 0, then, by the above convention 3, \(A \otimes X\) is also 0; and accordingly \(A \otimes Y\) is also 0 by the hypothesis \((A \otimes X) \equiv (A \otimes Y)\). But, since \(A\) is not 0, it follows that \(Y\) must be 0 from the convention 3. Therefore, in this case, we have the relation \(X \equiv Y\) under the hypothesis \(A \oplus 0\) and \((A \otimes X) \equiv (A \otimes Y)\). Next if \(X\) is not 0, then \((A \otimes X) \equiv 0\) and accordingly \((A \otimes Y) \equiv 0\) and
accordingly \( Y \oplus 0 \).

Therefore

\[
A \otimes X = A + X + 1, \\
A \otimes Y = A + Y + 1.
\]

Therefore

\[
A + X + 1 \oplus A + Y + 1 = X \oplus Y.
\]

5. It satisfies the proposition "if \((Z \oplus Z) \oplus Z\), then \(A \otimes Z \oplus Z\)." For, in our system, the element which satisfies the relation \((Z \oplus Z) \oplus Z\) is the element 0 and only this. And by the convention 3, \(A \otimes 0\) denotes 0. Therefore we have the relation \((A \otimes Z) \oplus Z\).

6. It does not satisfy the distributive law. For, \(A \otimes (B \oplus C)\) denotes \(A + (B + C) + 1\) while \((A \otimes B) \oplus (A \otimes C)\) denotes \((A + B + 1) + (A + C + 1)\). Therefore \(A \otimes (B \oplus C)\) is not equal to \((A \otimes B) \oplus (A \otimes C)\) in general.

Thus we know that the distributive law cannot be deduced from the thirteen postulates and the proposition "if \((Z \oplus Z) \oplus Z\), then \((A \otimes Z) \oplus Z\)."

From Theorem (b), we see that, when we reject the ordinary distributive law, we must take the element satisfying the two relations

(i) \((Z \oplus Z) \oplus Z\),

(ii) \((A \otimes Z) \oplus Z\)  \(A: \) any element of the class

at the same time, as the zero-element of the class, since the property (ii) cannot be deduced from the property (i) and the thirteen postulates. From Theorem (c), we see that, in the class of numbers satisfying the thirteen postulates and the proposition "if \((Z \oplus Z) \oplus Z\), then \((A \otimes Z) \oplus Z\)," we may take another distributive law instead of the ordinary one, since the thirteen postulates and the above proposition do not contain the ordinary distributive law as their logical consequence.

Theorem (d). In a class of numbers satisfying all thirteen postulates except the distributive law, the two relations \((Z \oplus Z) \oplus Z\) and \((A \otimes Z) \oplus Z\) are independent of each other.

Proof. In a class of numbers having the property enunciated in the theorem, there may be an element satisfying the relation \((Z \oplus Z) \oplus Z\), but not the relation \((A \otimes Z) \oplus Z\); and conversely, an element satisfying the latter relation, but not the former one. An example of the element having the former property is the element 0 in the class of numbers given in Theorem (b). An example of the element having the latter property is given in the following class of numbers.

Take the class of numbers 0, 1, 2, 3, \ldots, and define their
equality and operations as follows.

1. A and B are said to be equal to each other when and only when they are identical.

2. \( A \oplus B \) denotes the element \( (A + B + 1) \).

3. \( A \otimes B \) denotes the element \( A \times B \).

Then it may be seen at once that this class of numbers satisfies Postulates E1, E2; A1, A2, A3, A4, A5, A6; M1, M2, M3, M4, M6. But it does not satisfy the distributive law. Now in this system, take the element zero, then it satisfies the relation \( (A \otimes 0) \oplus A = 0 \), but not the relation \( (0 \oplus 0) \otimes 0 \), since \((0 \oplus 0)\) denotes \(0 + 0 + 1 = 1\).

Theorem (e). The proposition “if U is a unit-element, then \((A \otimes U) \oplus A = A\)”¹ and its converse “if \(A \oplus(\text{zero-element})\) and \((A \otimes U) \oplus A\),”¹ then U is a unit-element” can be deduced without the distributive law.

Proof. The former part of this theorem may be proved in a similar manner as in Theorem (a), and the latter part may be proved as follows.

From the relation \((A \otimes U) \oplus A\), we have the relation

\[
\{(A \otimes U) \otimes U\} \oplus (A \otimes U), \quad \text{(by Postulate M6),}
\]

or

\[
A \otimes (U \otimes U) \oplus (A \otimes U), \quad \text{(by Postulates E2, M2),}
\]

or (1) \( (U \otimes U) \oplus U \) (by Postulate M4)

But U is not a zero-element. For, if so, we would have the relation \((A \otimes U) \oplus U\) by definition of zero-element, and combined with the relation \((A \otimes U) \oplus A\), we would have the relation

\[
A \oplus U
\]

by Postulates E1 and E2. Therefore, if U is a zero-element, from the above, A would also be equal to zero-element, contrary to the hypothesis. Therefore

(2) \( U \oplus \text{zero-element} \).

From (1) and (2), we conclude that U is a unit-element.

Cor. The latter proposition cannot be deduced without the distributive law, if, in the definition of zero-element Z, we do not give the property “\((A \otimes Z) \oplus Z\)” to it.

Proof. Consider the class of numbers 0, 1, 2, 3, ... and define their equality and operations as follows.

1. A and B are said to be equal when and only when they are identical with each other.

---

¹ In the former proposition, the relation \((A \otimes U) \oplus A\) holds good for any element A of the system. But, in the latter proposition, it is sufficient that the relation \((A \otimes U) \oplus A\) holds good for only one element A which is not zero.
2. \( A \oplus B \) denotes the element \( (A + B) \).
3. \( A \otimes B \) denotes the element \( (A + B) \) when at least one of \( A, B \) is 0, and denotes the element \( (A + B + 1) \) in all other cases.

Then it may be seen at once that the class satisfies Postulates E1, E2; A1, A2, A3, A4, A5, A6; M1, M2, M3, M6, but not the distributive law M5. Moreover, that this class satisfies M4 may be seen as follows.

By the definition of multiplication, \( A \otimes X \) denotes \( A + X \) or \( A + X + 1 \), and \( A \otimes Y \) denotes \( A + Y \) or \( A + Y + 1 \); and by the hypothesis \( A \otimes X \) is equal to \( A \otimes Y \), therefore we have the relations

\[
A + X = A + Y \quad (1),
\]

or

\[
A + X = A + Y + 1 \quad (2),
\]

or

\[
A + X + 1 = A + Y \quad (3),
\]

or

\[
A + X + 1 = A + Y + 1 \quad (4).
\]

But (2) and (3) cannot occur. For, since \( A \oplus 0 \), \( A \otimes X \) denotes \( A + X \) when and only when \( X \) is 0, and accordingly from (2) we have the relation \( Y + 1 = 0 \) or \( Y = -1 \). But our class contains not the number \(-1\), so that (2) cannot hold good in our class of numbers. The same may be said of (3). From (1) and (4), we have \( X = Y \) at once and accordingly \( X \oplus Y \). Therefore our class of numbers satisfies all thirteen postulates except the distributive law.

Now, in this class of numbers, there is only one element 0 satisfying the relation \( Z \oplus Z \equiv Z \), and for this element 0, the second relation \( A \otimes Z \equiv Z \) does not hold good. But the relations \( A \oplus 0, \ (A \otimes Z) \equiv Z \) always hold good. Moreover this element 0 satisfies the relation \( Z \otimes Z \equiv Z \) while it does not satisfy the relation \( Z \otimes 0 \). Namely, in this class of numbers, the proposition (A) "if \( A \oplus 0 \) and \( (A \otimes U) \equiv A \), then \( (U \otimes U) \equiv U \) and \( U \oplus 0 \)" does not hold good in general. Hence we may conclude that when a zero-element does not satisfy the relation \( (A \otimes Z) \equiv Z \), the proposition (A) cannot be deduced from the set of the thirteen postulates lacking the distributive law.

It is to be noted that the element 0 of our class has very remarkable properties, namely it has three fundamental properties of unit-element \( (U \otimes U) \equiv U \), \( (A \otimes U) \equiv A \), \( (U \otimes A) \equiv A \), and the three fundamental properties of zero-element \( (Z \oplus Z) \equiv Z \), \( (A \oplus Z) \equiv A \), \( (Z \oplus A) \equiv A \) at the same time.

Conclusion. From the above discussions, we may deduce the theorem.

The necessary and sufficient condition in order that the zero-element
and the unit-element may have all of their fundamental properties in a
class of numbers satisfying all thirteen fundamental postulates, but not the
distributive law, is that their definitions are given as follows:

the element $Z$ is called a zero-element when and only when it satisfies
the relations $(Z + Z) = Z$ and $(A \times Z) = Z$;
the element $U$ is called a unit-element when and only when it satisfies
the relations $(U \times U) = U$ and $U \oplus (\text{zero-element})$.

Theorem (f). The proposition "($Z + Z$) is an ordinary or a perfect
or a true zero-element according as $Z$ is an ordinary or a perfect or a
true zero-element" can be deduced without the distributive law.

Proof. (I). When $Z$ is an ordinary zero-element, we have the
relation $(Z + Z) = Z$, and accordingly by Postulate A6, we have the
relation $\{(Z + Z) \oplus (Z + Z)\} = (Z + Z)$, which shows that $(Z + Z)$, is also
an ordinary zero-element.

(II). When $Z$ is a perfect zero-element, we have the relations
$(A \times Z) = Z$ and $(Z \times A) = Z$ besides the above relation $(Z + Z) = Z$. Now
from these relations and the relation $A \oplus A$, we have the following
relations

$\{A \times (Z + Z)\} = (A \times Z)$

$\oplus Z$

$\oplus Z + Z$

Therefore $\{A \times (Z + Z)\} = (Z + Z)$.

Similarly we have the relation

$\{(Z + Z) \times A\} = (Z + Z)$.

Therefore $(Z + Z)$ is a perfect zero-element.

(III). When $Z$ is a true zero-element, we have the relation
$(A + Z) = A$ and $(Z + A) = A$ besides the above three. Now from the
relations $(Z + A) \oplus A$ and $(Z + Z) \oplus Z$, we have the relation

$\{(Z + Z) \oplus (Z + A)\} = (Z + A)$

by Postulate A6.

Therefore $\{(Z + Z) \oplus A \oplus Z\} \oplus (A + Z)$

by Postulates E2, A2, A3.

Therefore $\{(Z + Z) \oplus A\} \oplus A$

by Postulate A4.

Similarly we have the relation

$A \oplus (Z + Z) \oplus A$.

Therefore $Z + Z$ is a true zero-element.

Cor. The same may be said of the combination $A \oplus A \oplus \ldots \oplus A$.

Remark. If "equality" be an undefined relation in the axiomatic
treatment of a number-system, the two propositions "$Z + Z$ is a zero-
element" and "$Z \oplus Z$ is equal to a zero-element" are entirely different as was already remarked. The definition of a zero-element "$(Z \oplus Z) \oplus Z$" means that $2Z$ is equal to the zero-element $Z$, and never means that $2Z$ is a zero-element. In order that $2Z$ may be a zero-element, it is necessary to prove that the relation "$2Z \oplus 2Z \oplus 2Z$" holds good.

Theorem (g). The proposition "$Z \otimes Z$ is a perfect or a true zero-element according as $Z$ is a perfect or a true zero-element" can be deduced without the distributive law. But the proposition "$Z \otimes Z$ is an ordinary zero-element when $Z$ is an ordinary zero-element" cannot be deduced without the distributive law.

Proof. (I). When $Z$ is a perfect zero-element, we have the relations $(Z \oplus Z) \oplus Z$, $(A \otimes Z) \oplus Z$, $(Z \otimes A) \oplus Z$, and since $A$ may be any element whatever, we have the relation $(Z \otimes Z) \oplus Z$ from the above. Accordingly we have the following relations

$((Z \otimes Z) \oplus (Z \otimes Z)) \ominus (Z \otimes Z)$ (by Postulate A6),

$\ominus Z$ (by the hypothesis),

$(Z \otimes Z)$ (by the above relation).

Therefore $(a)$ $((Z \otimes Z) \oplus (Z \otimes Z)) \ominus (Z \otimes Z)$ (by Postulate E2).

Further we have the relation

$((A \otimes Z) \otimes Z) \ominus (Z \otimes Z)$ (by Postulate M6),

or $(b)$ $(A \otimes (Z \otimes Z)) \ominus (Z \otimes Z)$ (by Postulate M2).

Similarly we have the relation

$(c)$ $((Z \otimes Z) \otimes A) \ominus (Z \otimes Z)$.

From the relations $(a)$, $(b)$, $(c)$, we conclude that $(Z \otimes Z)$ is a perfect zero-element.

(II). When $Z$ is a true zero-element, we have the relations $(Z \oplus A) \ominus A$ and $(A \oplus Z) \ominus A$ besides the above three. From the relations $(Z \otimes Z) \ominus Z$ and $A \ominus A$ we have the relations

$((Z \otimes Z) \oplus A) \ominus (Z \oplus A) \ominus (Z \otimes Z) \ominus A$.

Therefore

$((Z \otimes Z) \oplus A) \ominus A$.

Similarly we have the relation $(A \oplus (Z \otimes Z)) \ominus A$. Moreover, as in (I), we have the relations $((Z \otimes Z) \oplus (Z \otimes Z)) \ominus (Z \otimes Z)$, $(A \otimes (Z \otimes Z)) \ominus (Z \otimes Z)$, and $((Z \otimes Z) \otimes A) \ominus (Z \otimes Z)$ besides the above relations. Therefore $Z \otimes Z$ is a true zero-element.

(III). That the relation $((Z \otimes Z) \oplus (Z \otimes Z)) \ominus (Z \otimes Z)$ cannot be deduced from the relation $(Z \oplus Z) \ominus Z$ without distributive law may be proved as follows.

Consider a class of numbers $0, 1, 2, \ldots$ and define their equality and operations in the following manner:
1. \(A\) and \(B\) are said to be equal when and only when they are identical,

2. \(A\oplus B\) denotes the element \((A + B)\),

3. \(A\otimes B\) denotes the element \((A + B + 1)\).

Then it is at once seen that this class satisfies Postulates E1, E2; A1, A2, A3, A4, A5, A6; M1, M2, M3, M4, M6, but not the distributive law. Now, in this class, 0 is an ordinary zero-element, but neither perfect nor true zero-element, since the relation \((0\oplus 0)\oplus 0\) holds good while the relation \((A\otimes 0)\oplus 0\) does not hold. But \((0\otimes 0)\) is not an ordinary zero-element. For, since we have the relations \(0\otimes 0 = 0 + 0 + 1 = 1\), \((0\otimes 0)\oplus (0\otimes 0) = 1 + 1 = 1 + 1\), so we must have the relation \((0\otimes 0)\oplus (0\otimes 0)\oplus (0\otimes 0)\).

Namely, in this class, 0 is an ordinary zero-element, but \((0\otimes 0)\) is not so.

Now we proceed to discuss the properties of number-systems obeying the thirteen fundamental postulates and a new distributive law. As new distributive laws, first we take the following ones, since they are most simple, regular and symmetrical in form.

\[
\begin{align*}
\text{(I)} & \quad \{(A\oplus B)\otimes C\} \equiv \{(A\otimes B)\oplus C\}, \\
\text{(II)} & \quad \{A\oplus (B\otimes C)\} \equiv \{A\otimes (B\oplus C)\}, \\
\text{(III)} & \quad \{(A\otimes C)\oplus (B\otimes C)\} \equiv \{(A\oplus C)\otimes (B\oplus C)\}.
\end{align*}
\]

Henceforth we shall use the symbol "\(=\)" instead of the symbol \(\equiv\) for the sake of simplicity, but shall preserve the symbols \(\oplus\) and \(\otimes\), since results of the operations are very different from the ordinary ones.

(I). The first case in which the distributive law is given by

\[\{(A\oplus B)\otimes C\} = \{(A\otimes B)\oplus C\} .\]

Suppose that the number-system contains a zero-element. Since the above distributive law holds good for any values of \(A, B, C\), suppose that \(A\) and \(B\) are zero-elements and \(C\) any element of the system, then we have the relation "(zero element) = C". Thus we have the following striking proposition.

**Theorem.** If a system of numbers contains a zero-element, then all elements of the system are equal to the zero-element.

Next, suppose that this system contains no zero-element, but a unit-element, then taking \(C\) as that unit-element and \(A, B\) as any elements of the system, we have the relation

\[(A\oplus B) = (A\otimes B)\oplus 1 .\]

Thus we have the theorem.
Theorem. If the system contains a unit-element, but not a zero-element, then the sum of any two elements of the system differs only by unit from the product of them.

(II). The second case in which the distributive law is given by

\[ \{A \oplus (B \otimes C)\} = \{A \otimes (B \oplus C)\}. \]

First suppose that the system of numbers contains a zero-element, and take \(B\) and \(C\) as zero-elements and \(A\) as any element of the system, then we have the relation \(A\) = (zero-element). Thus, in this system also, we have the theorem.

Theorem. If the system of numbers contains a zero-element, then all elements of the system are equal to the zero-element.

Next, suppose that the system contains a unit-element, but not a zero-element, then by Postulate A1, \((1 \oplus 1)\), \((1 \oplus 1 \oplus 1)\), ... are also elements of our system. Now, in the distributive law, if we take \(B\) and \(C\) as unit-elements, we have

\[ A \oplus 1 = A \otimes (1 \oplus 1). \]

Further if we take \(C\) as unit-element always and \(B\) as \((1 \oplus 1)\), \((1 \oplus 1 \oplus 1)\), then we have

\[ A \oplus (1 \oplus 1) = A \otimes (1 \oplus 1 \oplus 1), \]
\[ A \oplus (1 \oplus 1 \oplus 1) = A \otimes (1 \oplus 1 \oplus 1 \oplus 1). \]

\[ \cdots \cdots \cdots \cdots \cdots \]

If we denote \((1 \oplus 1)\), \((1 \oplus 1 \oplus 1)\), ... by 2, 3, ... respectively, then we have the following interesting relations

\[ A \oplus 1 = A \otimes 2, \]
\[ A \oplus 2 = A \otimes 3, \]
\[ A \oplus 3 = A \otimes 4, \]
\[ \cdots \cdots \cdots \cdots \cdots \]
\[ A \oplus n = A \otimes (n \oplus 1). \]

Hence theorem.

If the system of numbers contains a unit-element, but not a zero-element, then the sum of any element \(A\) and any positive integer \(n\) is equal to the product of that element and the next greater integer \(n \oplus 1\).

(III). The third case in which the distributive law is given by

\[ \{(A \otimes C) \oplus (B \otimes C)\} = \{(A \oplus C) \otimes (B \oplus C)\}. \]

Suppose that the system of numbers contains a zero-element, and also suppose that \(A\) is the zero-element, then from the above distributive law, we have the relation
or

\[ B \otimes C = C \otimes (B \oplus C) \]
\[ C \otimes B = C \otimes (B \oplus C) \] (by commutative law).

But, in order that this relation may hold good, \( C \) must be equal to the zero-element; for, if it would not be so, we must have the relation

\[ B = B \oplus C \]

by Postulate M4. Since this relation must hold good for any value of \( B \), we should have the relation

\[ 0 = C \]

by giving the value 0 to \( B \), contrary to the hypothesis. Thus we have the theorem.

**Theorem.** If the system of numbers contains a zero-element, then all elements of the system are equal to the zero-element.

Next, suppose that the system of numbers contains a unit-element, but not a zero-element, and suppose that \( C \) is the unit-element, then we have the relation \( A \oplus B = (A \oplus 1) \otimes (B \oplus 1) \). Hence we have an interesting theorem.

**Theorem.** If the system of numbers contains a unit-element, but not a zero-element, then the sum of any two elements of the system is equal to the product of the next greater elements. \(^{(1)}\)

Further, put \( A = B \) in the distributive law, then we have the relation

\[ (A \otimes C) \oplus (A \otimes C) = (A \oplus C) \otimes (A \oplus C), \]

or

\[ 2(A \otimes C) = (A \oplus C)^2. \]

Hence we have the theorem.

**Theorem.** The square of the sum of any two elements of the system is equal to the double product of them.

Cor. 1. By repeated addition of the above formula, we have

\[ 2n(A \otimes C) = n(A \oplus C)^2. \]

Cor. 2. If we put \( A = 1 \) in the above theorem, we have the following proposition.

The double of any element of the system is equal to the square of the next greater number.

Thus far, we have treated three systems of numbers obeying a new distributive law of the simple, regular and symmetric form, and have found such striking fact, that all elements of these systems are equal to a zero-element if the system contains a zero-element. Now we shall try simply to study the systems of numbers obeying a new distributive law of a similar form as that of ordinary one. They are the following three.

\(^{(1)}\) Here the term “next greater element of A” means the element \((A \oplus 1)\).
(IV). \[ A \oplus (B \otimes C) = (A \oplus B) \otimes (A \oplus C), \]

(V). \[ A \oplus (B \otimes C) = (A \otimes B) \oplus (A \otimes C), \]

(VI). \[ A \otimes (B \oplus C) = (A \oplus B) \odot (A \oplus C). \]

The first of these three is obtained by interchanging the operations \( \oplus \) and \( \otimes \) in the both sides of the ordinary distributive law; and the second is obtained by doing the same process to the left-hand side of the ordinary law; and the third by doing the same process to the right-hand side of the ordinary one.

(IV). The fourth case in which the distributive law is given by

\[ \{ A \oplus (B \otimes C) \} = \{ (A \oplus B) \otimes (A \oplus C) \}. \]

Suppose that the system of numbers contains zero-elements, and \( B \) and \( C \) are both zeros, then we have the relation

\[ A = A \otimes A. \]

Therefore any element \( A \) of the system is a zero-element or a unit-element. If the system contains a unit-element besides the zero-element, then by putting \( A=1, \ B=1 \) and \( C=0 \), we have the relation

\[ 1 = 1 \oplus 1. \]

Since any element is equal to itself in any system satisfying 13 postulates, we have the relation

\[ 1 \oplus 1 = 1 \oplus 1 \oplus 1 \]

by Postulate A6. Similarly we have the relations

\[ 1 \oplus 1 \oplus 1 = 1 \oplus 1 \oplus 1 \oplus 1, \]

\[ \ldots \ldots \ldots \ldots \ldots \]

Therefore \[ 1 = 1 \oplus 1 = 1 \oplus 1 \oplus 1 = \ldots \ldots \ldots \ldots \]

But, on the other hand, from the property of the zero-element \( 1 \oplus 0 = 1 \) and the relation \( 1 = 1 \oplus 1 \), we have

\[ 1 \oplus 0 = 1 \oplus 1 \] (by Postulate E2),

or

\[ 0 = 1 \] (by Postulate A4).

Thus if we admit that the system contains zero-element as well as unit-element, then the zero-element must be equal to the unit-element, and in this case we have the following striking relations

\[ 0 = 1 = 1 \oplus 1 = 1 \oplus 1 \oplus 1 = \ldots \ldots \]

But if we do not admit that 0 is equal to 1, then the system can contain no unit-element, so that all elements of the system are zero-elements. (It should be noticed that the two propositions "two elements are identical" with each other and "two elements are equal to each other" are entirely different in axiomatic treatment of mathematics). Thus we have the theorem.
Theorem. If the system of numbers contains a zero-element, but not a unit-element, then all elements are zeros; if it contains a unit-element, but not a zero-element, then all elements are unit-element; if it contains both elements, then the zero-element is equal to the unit-element.

(V). The fifth case in which the distributive law is given by
\[ \{A \oplus (B \otimes C)\} = \{(A \times B) \oplus (A \times C)\}. \]

Suppose that the system contains zero-elements and \(B\) and \(C\) are both zeros, then, from the distributive law, we have the relation \(A=0\). Therefore, in this case, all elements of the system are equal to the zero-element.

Next, suppose that the system contains a unit-element, but not a zero-element, then by putting \(B\) and \(C\) as unit-elements, we have the relation
\[ A \oplus 1 = A \oplus A, \]
and accordingly we have the relation \(A=1\) by Postulate A4. Therefore, in this case, all elements are equal to the unit-element. Further, we put \(A\) as a unit-element, then we have
\[ 1 \oplus (B \otimes C) = B \oplus C, \text{ or } B \oplus C = B \otimes C \oplus 1. \]
Thus we have the theorem.

Theorem. If the system of numbers contains a zero-element, then all elements are equal to the zero-element; if the system contains a unit-element, but not a zero-element, then all elements are equal to the unit-element. Accordingly, by Postulates E1 and E2, all elements are equal to one another in both cases, so that any combinations of any elements of the system are also equal to one another by Postulates A1 and M1.

(VI). The sixth case in which the distributive law is given by
\[ \{A \otimes (B \oplus C)\} = \{(A \oplus B) \otimes (A \oplus C)\}. \]

Suppose that the system of numbers contains a zero-element and \(C\) is that zero-element, then we have the relation
\[ A \otimes B = (A \oplus B) \otimes A = A \otimes (A \oplus B). \]
From this relation, we may deduce that \(A\) must be equal to the zero-element by a similar reasoning as in the case (III). Thus, in this case also, any element of the system is equal to the zero-element.

Next suppose that the system contains a unit-element, but not a zero-element, then by putting \(A=1\) and \(B=C\), we have the relation
\[ B \oplus B = (B \oplus 1) \otimes (B \oplus 1), \]
or
\[ 2B = (B \oplus 1)^2. \]
In this relation, if we put \(B=1\), then we have the relation
\[ 1 \oplus 1 = (1 \oplus 1) \otimes (1 \oplus 1), \]
and since \((1\oplus 1)\) is not a zero-element, we may conclude that \((1\oplus 1)\)
must be a unit-element. Moreover, in this case, we may prove that
the relation \(1=1\oplus 1\) also holds good. For, by putting \(A=B=C=1\)
in the distributive law, we have the relation
\[
1 \times (1\oplus 1) = (1\oplus 1) \times (1\oplus 1),
\]
and from this relation, again we have the relation \(1=1\oplus 1\) by
Postulate M4.
Therefore
\[
1=1\oplus 1=1\oplus 1\oplus 1.
\]
By adding 1 to the both sides of \(1\oplus 1\oplus 1=1\oplus 1\oplus 1\oplus 1\), we have the relation
\[
1\oplus 1\oplus 1=1\oplus 1\oplus 1\oplus 1,
\]
and so on. Thus we have the theorem.

Theorem. If the system of numbers contains a zero-element, then all
elements are equal to the zero-element. If it contains a unit-element, but
not a zero-element, then the double of any element is equal to the square
of the next greater element, and moreover all elements produced by adding
unit-element repeatedly are equal to one another.

Conclusion. All systems of numbers having new distributive laws,
of which we have hitherto discussed, have very peculiar property that,
if the system contains a zero-element, then all elements are equal to
the zero-elements. But such a system of numbers having all its ele-
ments as zeros or all its elements equal to one another will perhaps
not be of any use in practical calculation, nor will be of much use in
theoretical purpose. Only the system of numbers having the ordinary
distributive law is free from such peculiarity. In this respect, we can
understand how the ordinary distributive law plays an important role
in a system of numbers, though its form is irregular. But the ordinary
distributive law is a relation naturally holding good in the operations
of natural numbers which is most primitive. It is wonderful to see that
this natural and primitive law has so striking predominance over all
other similar ones.