On Solution of Functional Equations,

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Theorem. Suppose that we have solved a given functional equation containing one unknown function which must be determined so as to depend on one independent variable as the solution, under the condition that the solution must be a continuous function and under several other additional conditions such as its differentiability. Then if the independent variable be multiplied by a certain factor the solution thus obtained is the most general solution of the given functional equation, which is to be got when we seek the solution as subjected to the only one condition of continuity without taking any of the other additional, previously assigned, conditions into considerations, provided that the given functional equation has such a form that from the two values of any solution whatever corresponding to two fixed values $a, b$, say, of the independent variable $x$ on which that solution depends, the values of that solution corresponding to an everywhere dense set of points lying within the intervals $(a, b)$ can be determined uniquely, and independently from the end-points $a, b$ themselves, and provided that corresponding to the two values $x_1, x_1'$ of the independent variable of the one solution there exist two values $x_2, x_2'$ of the independent variable of the other solution, so that

$$\varphi(x_1) = F(x_2) \quad \text{and} \quad \varphi(x_1') = F(x_2').$$

For, let the one solution under the only one condition of continuity be $\varphi(x)$, and let the other solution found by using the other several conditions such as the differentiability, beside the condition of continuity, be $F(x)$. Then we can determine by means of the given functional equation the values

$$\varphi \{ \theta(x_1' - x_1) \} \quad \text{and} \quad F' \{ \theta(x_2' - x_2) \},$$

from $\varphi(x_1)$ and $\varphi(x_1')$, and $F'(x_2)$ and $F'(x_2')$ respectively, where $\theta$ is determined so that $\theta(x_1' - x_1)$ and $\theta(x_2' - x_2)$ correspond to an everywhere dense set of points $x$ lying in the intervals $(x_1, x_1')$ and $(x_2, x_2')$ respectively. For example, $\theta$ may be taken equal to $\frac{m}{2^n}$ in which $m$ and
n are both positive integers, m being <2^n.
Since \( \varphi(x_1) = F(x_2) \) and \( \varphi(x_1') = F(x_2') \),
we must have
\[
\varphi(\theta(x_1' - x_1)) = F(\theta(x_2' - x_2)).
\]
By the theory of sets of points, this equation must hold good for any
other values of \( \theta \) (>0, and <1) than \( \frac{m}{2^n} \), because the both functions
\( \varphi(x) \) and \( F(x) \) are continuous. Therefore
\[
\varphi(x) = F(c x),
\]
where \( x \) may take any value lying within the interval \((x_1, x_1')\) and \( c \) is
equal to the constant \( \frac{x_2' - x_2}{x_1' - x_1} \).

Corollary. Particularly, if the end-points \( x_1', x_1' \) for the one solution
and the end-points \( x_2, x_2' \) for the other solution are to be taken coinci-
dent, so that \( x_1 = x_2, x_1' = x_2' \), the constant \( c \) becomes 1 and therefore
the two functions \( \varphi(x) \) and \( F(x) \) are entirely identical; and in this
case we may take off the phrase "and independently from the end-
points \( a, b \)" in the enunciation of the above theorem.

M. Andrade has applied this method of proof to the solution of
Poisson's or d'Alembert's equation
\[
f(x + y) + f(x - y) = 2f(x)f(y),
\]
in his paper "Sur l'équation fonctionnelle de Poisson", Bull. Soc.
Math. France, T. 28 (1900), p. 58, and the above generalisation has
been suggested indeed by his paper. The method can be made of use
very widely, for example for the cases in which we are proposed to de-
termine all the functions satisfying a given addition-theorem. I have
thought it important to call attention to the theorem explicitly enu-
nicated in its general form as above. In many cases, the independent
variable in the solution which was found by using several other con-
ditions than continuity, is accompanied with an arbitrary constant
multiplier, as we see in the case of Poisson's equation. If such be the
case, we need not to consider the factor \( \frac{x_2' - x_2}{x_1' - x_1} \), though \( x_1 = x_2, \) and
\( x_1' = x_2' \), and we may say the both solutions are quite identical.

The theorem may be extended to the case of functions of more
than one independent variable.