Complete Systems of Invariants and Covariants for Triads of Ruled Surfaces\(^{(1)}\),

by

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An analytic basis for the projective differential geometry of triads of ruled surfaces whose generators are in one-to-one correspondence has been established in a recent paper by the author\(^{(2)}\). This paper, after determining the defining system of differential equations and establishing the nature of permissible transformations, derived certain invariants and covariants which were then given geometric interpretation in a number of theorems.

It is the purpose of the present investigation to develop with reasonable completeness the theory of these invariants and covariants, arriving at a system complete in the sense of a set in terms of which, and their derivatives, all invariants and covariants are expressible. A brief recapitulation will be necessary.

For defining system of differential equations we make use of a set of six ordinary first order linear and homogeneous differential equations in six dependent variables, to which are adjoined two linear and homogeneous relations between these variables:

\[
\begin{align*}
y' &= a_{11} y + a_{12} z + a_{13} \alpha + a_{14} \beta, \\
z' &= a_{21} y + a_{22} z + a_{23} \alpha + a_{24} \beta, \\
\alpha' &= a_{31} \alpha + a_{32} \beta + a_{33} \gamma + a_{34} \xi, \\
\beta' &= a_{41} \alpha + a_{42} \beta + a_{43} \gamma + a_{44} \xi, \\
\gamma' &= a_{51} y + a_{52} z + a_{53} \gamma + a_{54} \xi, \\
\xi' &= a_{61} y + a_{62} z + a_{63} \gamma + a_{64} \xi,
\end{align*}
\]

\(T\)

\[
\begin{align*}
c_{11} y + c_{12} z + c_{13} \alpha + c_{14} \beta + c_{15} \gamma + c_{16} \xi &= 0, \\
c_{21} y + c_{22} z + c_{23} \alpha + c_{24} \beta + c_{25} \gamma + c_{26} \xi &= 0\(^{(3)}\).
\end{align*}
\]

\(Q\)

A set of twenty-four linearly independent solutions of \(T\)

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\(^{(3)}\) In the first paper these two equations were solved for \(y\) and \(z\). The form used here has the advantage of symmetry.
INVARIANTS AND COVARIANTS OF RULED SURFACES.

\( y_i, z_i; \quad \alpha_i, \beta_i; \quad \gamma_i, \xi_i; \quad (i=1, 2, 3, 4), \)
satisfying the conditions

\[
(D) \quad |y_2z_3\alpha_4\beta_4| \neq 0, \quad |\alpha_1\beta_2\gamma_3\xi_4| \neq 0, \quad |\gamma_1\xi_2y_3z_4| \neq 0,
\]
always exists, and conversely, any twenty-four linearly independent functions of \( x \), satisfying conditions \((D)\), determine a system \((T)\).

The eight functions \( y_i(X), z_i(X), (i=1, 2, 3, 4) \), define two space curves \( C_y, C_z \), whose points are in one-to-one correspondence and which thus determine a ruled surface \( R_y \). Similarly for \( \alpha_i(X), \beta_i(X) \) and \( \gamma_i(X), \xi_i(X) \). Corresponding lines \( l_{yz}, l_{\alpha\beta}, l_{\gamma\xi} \) of these three ruled surfaces, that is, lines given by the same value of \( x \), are by virtue of \((D)\) non-intersecting. It follows that

\[
(C) \quad C_1 \equiv c_{11}c_{22} - c_{12}c_{21} \neq 0, \quad C_2 \equiv c_{13}c_{24} - c_{14}c_{23} \neq 0, \quad C_3 \equiv c_{15}c_{23} - c_{16}c_{25} \neq 0.
\]

As a further restriction on the coefficients we have

\[
(A) \quad A_1 \equiv a_{13}a_{24} - a_{14}a_{23} \neq 0, \quad A_2 \equiv a_{35}a_{46} - a_{36}a_{45} \neq 0, \quad A_3 \equiv a_{51}a_{62} - a_{52}a_{61} \neq 0,
\]
this being equivalent to insisting that the surfaces \( R_{yz}, R_{\alpha\beta}, R_{\gamma\xi} \) are not developables.

Permissible transformations are such as leave the geometric configuration unchanged. They are

\[
X = \xi(X),
\]

\[
(1) \quad \begin{align*}
y & = c\bar{y} + d\bar{z}, \quad \alpha = g\bar{\alpha} + h\bar{\beta}, \quad \gamma = s\bar{\gamma} + t\bar{\xi}, \\
z & = e\bar{y} + f\bar{z}, \quad \beta = j\bar{\alpha} + k\bar{\beta}, \quad \xi = v\bar{\gamma} + w\bar{\xi}, \\
D_1 & = cf - de \neq 0, \quad D_2 = gj - hj \neq 0, \quad D_3 = sv - tv \neq 0,
\end{align*}
\]

where the coefficients \( c, d, \ldots, u, v \), like the \( \alpha, \beta \), \( c, \lambda \), are functions of \( X \).

The infinitesimal transformations of the dependent variables are

\[
\begin{align*}
y & = (1 + \phi_1 \delta t)\bar{y} + \psi_1 \delta t\bar{z}, \quad \alpha = (1 + \phi_2 \delta t)\bar{\alpha} + \psi_2 \delta t\bar{\beta}, \quad \gamma = (1 + \phi_3 \delta t)\bar{\gamma} + \psi_3 \delta t\bar{\xi}, \\
z & = x_1 \delta t\bar{y} + (1 + \omega_1 \delta t)\bar{z}, \quad \beta = x_2 \delta t\bar{\alpha} + (1 + \omega_2 \delta t)\bar{\beta}, \quad \xi = x_3 \delta t\bar{\gamma} + (1 + \omega_3 \delta t)\bar{\xi}, \\
D_1 & = 1 + (\phi_1 + \omega_1) \delta t, \quad D_2 = 1 + (\phi_2 + \omega_2) \delta t, \quad D_3 = 1 + (\phi_3 + \omega_3) \delta t,
\end{align*}
\]

where \( \phi_i, \psi_i, x_i, \omega_i \) are arbitrary functions of \( x \) and \( t \) is independent of \( x \).

Remembering that the point whose coordinates are \((y'_1, y'_2, y'_3, y'_4)\) is on the tangent to the curve \( C_y \) at \( P_t \) and that the point given by \( py_i + qz_i, (i=1, 2, 3, 4) \) is on the line \( l_{yz} \), we may give immediate geometric interpretation to system \((T), (Q)\). The first equation asserts that the tangent plane to \( R_{yz} \) at \( P_t \) cuts the line \( l_{\alpha\beta} \) in the point \( a_{13}\alpha + a_{14}\beta \), the second, that the tangent plane to \( R_{yz} \) at \( P_t \) cuts \( l_{\alpha\beta} \) in
the point $a_{23}\alpha + a_{24}\beta$. Similarly for equations three to six. The first of \((Q)\) tells us that one of the straight line intersectors of the lines \(l_y, l_{\alpha \beta}, l_{\gamma \zeta}\) passes through the three points $c_{11}y + c_{12}z$ on \(l_y\), $c_{13}\alpha + c_{14}\beta$ on \(l_{\alpha \beta}\) and $c_{15}\gamma + c_{16}\zeta$ on \(l_{\gamma \zeta}\). Similarly for the second equation of \((Q)\).

If the two equations \((Q)\) adjoined to system \((T)\) are to be satisfied by the solutions of \((T)\) then the $c_{\alpha \gamma}$ are not altogether independent of the $a_{ij}$. Differentiating equations \((Q)\) and making use of \((T)\) we obtain two identities in $y, z, \alpha, \beta, \gamma, \zeta$, from which there result the desired relations. They are

\[
\begin{align*}
\begin{cases}
c_{11}' + c_{11}a_{11} + c_{12}a_{21} + c_{13}a_{31} + c_{16}a_{61} = 0, \\
c_{21}' + c_{21}a_{11} + c_{22}a_{21} + c_{23}a_{31} + c_{26}a_{61} = 0, \\
c_{12}' + c_{11}a_{12} + c_{12}a_{22} + c_{13}a_{32} + c_{16}a_{62} = 0, \\
c_{22}' + c_{21}a_{12} + c_{22}a_{22} + c_{23}a_{32} + c_{26}a_{62} = 0, \\
c_{13}' + c_{11}a_{13} + c_{12}a_{23} + c_{13}a_{33} + c_{16}a_{63} = 0, \\
c_{23}' + c_{21}a_{13} + c_{22}a_{23} + c_{23}a_{33} + c_{26}a_{63} = 0, \\
c_{14}' + c_{11}a_{14} + c_{12}a_{24} + c_{13}a_{34} + c_{16}a_{64} = 0, \\
c_{24}' + c_{21}a_{14} + c_{22}a_{24} + c_{23}a_{34} + c_{26}a_{64} = 0, \\
c_{15}' + c_{11}a_{15} + c_{12}a_{25} + c_{13}a_{35} + c_{16}a_{65} = 0, \\
c_{25}' + c_{21}a_{15} + c_{22}a_{25} + c_{23}a_{35} + c_{26}a_{65} = 0, \\
c_{16}' + c_{11}a_{16} + c_{12}a_{26} + c_{13}a_{36} + c_{16}a_{66} = 0, \\
c_{26}' + c_{21}a_{16} + c_{22}a_{26} + c_{23}a_{36} + c_{26}a_{66} = 0.
\end{cases}
\]

\[
(3)
\]

I: A Complete System of Invariants and Covariants.

If equations \((Q)\) be transformed by means of \((1)\) we find for the new coefficients,

\[
\begin{align*}
\begin{cases}
\bar{c}_{11} = c_{11} + ec_{12}, & \bar{c}_{12} = d_{11} + fc_{12}, & \bar{c}_{21} = ec_{21} + c_{22}, & \bar{c}_{22} = dc_{21} + fc_{22}, \\
\bar{c}_{13} = gc_{13} + jc_{14}, & \bar{c}_{14} = hc_{13} + kc_{14}, & \bar{c}_{23} = gc_{23} + jc_{24}, & \bar{c}_{24} = hc_{23} + kc_{24}, \\
\bar{c}_{15} = sc_{15} + uc_{16}, & \bar{c}_{16} = tc_{15} + vc_{16}, & \bar{c}_{25} = sc_{25} + uc_{26}, & \bar{c}_{26} = tc_{25} + vc_{26}.
\end{cases}
\]

From \((4)\) and \((1)\) we find

\[
\bar{c}_{11}\bar{y} + \bar{c}_{12}\bar{z} = c_{11}y + c_{12}z, \quad \bar{c}_{13}\bar{\alpha} + \bar{c}_{14}\bar{\beta} = c_{13}\alpha + c_{14}\beta, \quad \ldots, \quad \bar{c}_{26}\bar{\gamma} + \bar{c}_{26}\bar{\zeta} = c_{26}\gamma + c_{26}\zeta.
\]

We have thus six independent absolute semiconvariants

\[
\begin{align*}
\begin{cases}
\bar{y} = c_{11}y + c_{12}z, & \bar{\alpha} = c_{13}\alpha + c_{14}\beta, & \bar{\gamma} = c_{26}\gamma + c_{26}\zeta, \\
\bar{z} = c_{26}y + c_{26}z, & \bar{\beta} = c_{26}\alpha + c_{26}\beta, & \bar{\zeta} = c_{26}\gamma + c_{26}\zeta.
\end{cases}
\]

The curves $C_{\bar{y}}, C_{\bar{\alpha}}, C_{\bar{\gamma}}$; on the respective three surfaces $R_{y\alpha}, R_{z\beta}, R_{\gamma\zeta}$, are so related that triples of corresponding points, one point from each curve, are collinear. The same is true for the three curves $C_{\bar{z}}, C_{\bar{\beta}}, C_{\bar{\zeta}}$. We propose to replace the original directrix curves
\[ \dot{y}' = -\theta_1 \dot{y} + \eta_1 \ddot{z} + \theta_2 \ddot{z} - \eta_2 \ddot{\beta}, \]
\[ \dot{z}' = -P_1 \dot{y} + \sigma_1 \dot{z} + P_2 \ddot{x} - \sigma_1 \ddot{\beta}, \]
\[ \dot{\beta}' = -P_2 \dot{z} + \sigma_2 \dot{\beta} + \theta_3 \dot{\gamma} - \eta_3 \ddot{\gamma}, \]
\[ \dot{\gamma}' = -\theta_3 \dot{\gamma} + \eta_3 \ddot{\gamma} + \theta_1 \dot{y} - \eta_1 \ddot{z}, \]
\[ \zeta' = -P_3 \dot{\gamma} + \sigma_3 \ddot{\gamma} + \Pi_3 \ddot{\gamma} - \sigma_3 \ddot{\zeta}, \]
\[ \zeta' = 0, \quad \ddot{z} + \ddot{\beta} + \ddot{\gamma} = 0, \]

where

\[ C_i \theta = a_{3i} c_{13} c_{21} - a_{2i} c_{12} c_{21} + a_{1} c_{10} c_{21} - a_{0} c_{10} c_{21}, \]
\[ C_1 \eta = a_{31} c_{13} c_{12} - a_{32} c_{13} c_{11} + a_{21} c_{10} c_{11} - a_{22} c_{10} c_{11}, \]
\[ C_2 \eta = a_{13} c_{11} c_{21} - a_{14} c_{11} c_{21} + a_{23} c_{12} c_{21} - a_{24} c_{12} c_{21}, \]
\[ C_3 \eta = a_{33} c_{11} c_{21} - a_{32} c_{12} c_{21} + a_{31} c_{10} c_{21} - a_{30} c_{10} c_{21}, \]
\[ C_4 \eta = a_{14} c_{11} c_{12} - a_{24} c_{12} c_{11} + a_{34} c_{12} c_{11} - a_{40} c_{12} c_{11}, \]
\[ C_5 \eta = a_{14} c_{11} c_{13} - a_{24} c_{12} c_{13} + a_{34} c_{12} c_{13} - a_{40} c_{12} c_{13}, \]
\[ C_6 \eta = a_{33} c_{11} c_{13} - a_{32} c_{12} c_{13} + a_{31} c_{10} c_{13} - a_{30} c_{10} c_{13}, \]

Now \( \dot{y}' \), \ldots, \( \dot{\zeta}' \) are derivatives of absolute semicovariants and are therefore themselves absolute semicovariants. Because of the functional independence of \( \dot{y}, \ldots, \dot{\zeta} \) it follows from (I) that \( \theta_1, \eta_1, \Pi_1, \sigma_1, \sigma_j \), \( j = 1, 2, 3 \), are absolute seminvariants. It can be shown that they are independent.\(^{(1)}\)

The twelve seminvariants given by equations (6) contain twenty-four of the thirty-six coefficients of system \((T), (Q)\). The problem of finding the total number of independent absolute seminvariants containing the \( a_{ij}, c_{xh} \), next presents itself. The method of attack is well known.

We first apply the infinitesimal transformation (2) to system \((T), (Q)\), finding thereby the infinitesimal changes in the coefficients.\(^{(2)}\)

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\(^{(1)}\) Their Jacobian with respect to the \( a_{ij} \) is equal to unity.

\(^{(2)}\) It is not necessary to supply these here. The \( \delta_{c_{xh}} \) are expressions of the type \( \delta_{c_{xh}} = (\phi_{x} c_{x1} + \chi_{x} c_{x1}) \delta t \). For the \( \delta a_{ij} \) see "first paper."
Any function $U(a_{ij}, c_{\lambda})$, invariant under transformation of the dependent variables must satisfy the equation

$$\partial U = \sum \frac{\partial U}{\partial a_{ij}} \partial a_{ij} + \sum \frac{\partial U}{\partial c_{\lambda}} \partial c_{\lambda} = 0,$$

and conversely.

The expressions for $\partial a_{ij}, \partial c_{\lambda}$ contain the arbitrary functions $\phi_j, \psi_j, \chi_j, \omega_j, (j=1, 2, 3)$, together with their first derivatives. When these expressions are substituted in (7) there results an identity from which, by equating to zero the coefficients of $\phi_j, \psi_j, \chi_j, \omega_j, \phi_j', \psi_j', \chi_j', \omega_j'$, there is obtained a system of partial differential equations whose solutions are the invariant functions sought.

In the case under consideration there are thus obtained twenty-four such partial differential equations. Twelve of these are of the type

$$\frac{\partial U}{\partial a_{ij}} = 0,$$

where $(i, j=1, 2), (i, j=3, 4), (i, j=5, 6)$. It follows that twelve of the $a_{ij}$ do not appear in the absolute seminvariants of the type sought. They are indeed those twelve $a_{ij}$ which are absent from the twelve invariants already found in (6).

By making use of equations (8) the remaining twelve equations are simplified. Four of them are

$$\left\{ \begin{array}{l}
-a_{13} \frac{\partial U}{\partial a_{13}} - a_{14} \frac{\partial U}{\partial a_{14}} + a_{51} \frac{\partial U}{\partial a_{51}} + a_{61} \frac{\partial U}{\partial a_{61}} + c_{11} \frac{\partial U}{\partial c_{11}} + c_{21} \frac{\partial U}{\partial c_{21}} = 0, \\
-a_{12} \frac{\partial U}{\partial a_{12}} - a_{24} \frac{\partial U}{\partial a_{24}} + a_{52} \frac{\partial U}{\partial a_{52}} + a_{62} \frac{\partial U}{\partial a_{62}} + c_{12} \frac{\partial U}{\partial c_{12}} + c_{22} \frac{\partial U}{\partial c_{22}} = 0, \\
-a_{23} \frac{\partial U}{\partial a_{23}} - a_{34} \frac{\partial U}{\partial a_{34}} + a_{53} \frac{\partial U}{\partial a_{53}} + a_{63} \frac{\partial U}{\partial a_{63}} + c_{13} \frac{\partial U}{\partial c_{13}} + c_{23} \frac{\partial U}{\partial c_{23}} = 0, \\
-a_{32} \frac{\partial U}{\partial a_{32}} - a_{24} \frac{\partial U}{\partial a_{24}} + a_{52} \frac{\partial U}{\partial a_{52}} + a_{62} \frac{\partial U}{\partial a_{62}} + c_{12} \frac{\partial U}{\partial c_{12}} + c_{22} \frac{\partial U}{\partial c_{22}} = 0.
\end{array} \right.$$  

The eight omitted can be obtained from (9) by permuting the $a_{ij}$ and $c_{\lambda}$ which, for the $c$'s replaces $c_{\lambda}$ with $c_{\lambda+2}$ and for the $a$'s replaces $a_{ij}$ with $a_{i+2,j+2}$, it being understood that wherever $i+2$ or $j+2$ exceeds 6, we use its residue, mod. 6.

These twelve equations are independent and form a complete system. They involve twenty-four independent variables, twelve $a$'s and twelve $c$'s. There are therefore $24-12=12$ independent solutions. All other solutions are expressible as functions of the twelve. For
these twelve independent solutions we can take the $\theta, \eta, P_j, \sigma_j, (j=1, 2, 3)$, given by (6).

We have found that the first derivatives of the $c_{\alpha\lambda}$ are expressible in terms of $c_{\alpha\lambda}$ and $a_{ij}$. It follows that higher derivatives of $c_{\alpha\lambda}$ are expressible in terms of $c_{\alpha\lambda}$ and of the $a_{ij}$ and their derivatives. It is not necessary therefore to consider seminvariants containing derivatives of $c_{\alpha\lambda}$.

In searching for seminvariants containing $a_{ij}, a_{ij}', c_{\alpha\lambda}$, we must make use of the expressions for

$$\delta a_{ij}' = \frac{d}{dX} (\delta a_{ij}),$$

in addition to those for $\delta a_{ij}, \delta c_{\alpha\lambda}$.

Forming from these sixty expressions a system of partial differential equations by equating to zero the coefficients of $\phi_i, \Phi_j, \chi_i, \omega_j; \phi_i', \Phi_j', \chi_i', \omega_j'; \phi_i'', \Phi_j'', \chi_i'', \omega_j''$, we obtain thirty-six equations in sixty variables. Twelve of these equations are of the type

$$\frac{\partial U}{\partial a_{ij}'} = 0,$$

so that twelve of the sixty variables do not enter into these new invariants. The missing variables are $a_{ij}', (i,j=1, 2), (i,j=3, 4), (i,j=5, 6)$.

We are thus able to reduce the number of equations in our system to twenty-four and the number of independent variables to forty-eight. The equations are independent and form a complete system as before. There must be therefore $48-24=24$ independent solutions, that is, twenty-four absolute seminvariants containing $a_{ij}, a_{ij}', c_{\alpha\lambda}$. The twelve already obtained can be counted as twelve of the twenty-four. Since the derivatives of these twelve are also seminvariants and independent of their primitives and of each other, our list of twenty-four is complete. All absolute seminvariants containing $a_{ij}, a_{ij}', c_{\alpha\lambda}$ only are thus expressible in terms of the original twelve and their first derivatives.

Repeating this process to find absolute seminvariants containing $a_{ij}, a_{ij}', a_{ij}''$, $c_{\alpha\lambda}$, we find that $a_{ij}'', (i,j=1, 2), (i,j=3, 4), (i,j=5, 6)$, do not occur in these invariants, that the number of partial differential equations in the system is seventy-two, the number of independent variables thirty-six and hence that the number of independent solutions is thirty-six. Twenty-four are already at hand and the second derivatives of the original twelve provide the remaining twelve. So on indefinitely.
We have thus shown that $\theta_j, \eta_j, \Pi_j, \sigma_j$, $(j=1, 2, 3)$, form a complete system of absolute seminvariants in the sense that all others can be expressed in terms of these twelve and of their derivatives.

First derivatives of $y, \ldots, \zeta$, are expressible in terms of $y, \ldots, \zeta$; $a_\alpha, c_\alpha$, by means of $(T)$. By making repeated use of $(T)$ and (3), derivatives of any order $n$, of $y, \ldots, \zeta$, can be expressed in terms of $y, \ldots, \zeta$; $a_\alpha, a_\alpha'$, $\ldots, a_\alpha^{n-1}, c_\alpha$. It follows that all absolute semicovariants can be expressed as functions of the dependent variables only together with the $c_\alpha$ and the $a_\alpha$ and their derivatives.

The infinitesimal changes in the dependent variables are

$$\delta y = -(f_1y + f_2z)\delta t, \quad \delta x = -(f_1x + f_2y)\delta t, \quad \delta y = -(f_3y + f_4z)\delta t,$$

$$\delta z = -(f_5y + f_6z)\delta t, \quad \delta \beta = -(f_7x + f_8y)\delta t, \quad \delta \xi = -(f_{10}y + f_{11}z)\delta t.$$

Since these expressions involve only the twelve arbitrary functions $f_\alpha, f_\beta, f_\xi, \omega_j, (j=1, 2, 3)$, the system of partial differential equations whose solutions are absolute semicovariants involving derivatives of $a_\alpha$ up to those of order $n-1$, will be formed from the system whose solutions are absolute seminvariants involving derivatives of $f_\alpha$ of the same order, by annexing to the first twelve equations of this latter system the twelve terms

$$-y \frac{\partial U}{\partial y}, -x \frac{\partial U}{\partial x}, \ldots, -\xi \frac{\partial U}{\partial \xi}; \quad -z \frac{\partial U}{\partial y}, -\beta \frac{\partial U}{\partial x}, \ldots, -\gamma \frac{\partial U}{\partial \xi},$$

one term to each equation.

The augmented system will contain the same number of equations as the one from which it is formed but it will involve six additional independent variables. The equations will be independent and the system complete. The number of independent solutions of the augmented system will thus exceed by six the number of independent solutions of the system from which it is formed. It follows that there are precisely six independent absolute semicovariants. For these six we can take those already found in (5). All absolute semicovariants can be expressed in terms of these six and seminvariants. We have thus a complete system of absolute seminvariants and semicovariants.

A transformation of the independent variable $X = \xi(X)$ replaces $a_\alpha$ with $\bar{a}_\alpha$ where

$$\xi' \bar{a}_\alpha = a_\alpha,$$

and leaves $c_\alpha$ as well as $y, z, \ldots, \zeta$, unchanged. It follows that the six semicovariants $\bar{y}, \bar{z}, \ldots, \bar{\xi}$, are absolute covariants and the
twelve seminvariants \( \theta_j, \eta_j, \Pi_j, \sigma_j \) (\( j = 1, 2, 3 \)), are relative invariants. The ratios of any eleven of these latter to the remaining one will be absolute invariants, since all twelve are of the same weight. Moreover any eleven invariants so obtained will be independent.

The infinitesimal transformation of the independent variable is

\[
(11) \quad X = X + \phi \delta t,
\]

where \( \phi(X) \) is an arbitrary function. We find

\[
(12) \quad X' = 1 + \phi' \delta t, \quad X'' = \phi'' \delta t, \ldots.
\]

By (10),

\[
\bar{\theta}_j = \frac{1}{\xi'} \delta t,
\]

and this, in view of (12), can be written

\[
\bar{\theta}_j = \theta_j (1 - \phi' \delta t),
\]

so that

\[
(13) \quad \delta \theta_j = - \theta_j \phi' \delta t, \quad \delta \theta'_j = - (\theta_j \phi'' + \theta'_j \phi') \delta t, \ldots.
\]

Similar expressions hold for \( \eta_j, \Pi_j, \sigma_j \).

All absolute invariants must be combinations of seminvariants. Therefore in searching for absolute invariants it will be sufficient to find functions of the seminvariants which are unchanged by transformation of the independent variable.

Proceeding as before we must have for such a function

\[
(14) \quad \delta U = \sum \frac{\partial U}{\partial \theta_j} \delta \theta_j + \sum \frac{\partial U}{\partial \eta_j} \delta \eta_j + \sum \frac{\partial U}{\partial \Pi_j} \delta \Pi_j + \sum \frac{\partial U}{\partial \sigma_j} \delta \sigma_j = 0,
\]

\( (j = 1, 2, 3) \).

Making use of the first of (13) we find a single partial differential equation

\[
\sum \theta_j \frac{\partial U}{\partial \theta_j} + \sum \eta_j \frac{\partial U}{\partial \eta_j} + \sum \Pi_j \frac{\partial U}{\partial \Pi_j} + \sum \sigma_j \frac{\partial U}{\partial \sigma_j} = 0, \quad (j = 1, 2, 3),
\]

involving twelve independent variables. The number of independent solutions of this equation is thus eleven. There are therefore eleven independent absolute invariants involving the twelve fundamental seminvariants. For these we may take the eleven mentioned above.

If we wish to determine the number of independent invariants involving the twelve fundamental seminvariants together with their first derivatives then from (14) and the first two of (13) we find
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\[
\sum \theta \frac{\partial U}{\partial \theta_j} + \sum \eta_i \frac{\partial U}{\partial \eta_i} + \sum \Pi_j \frac{\partial U}{\partial \Pi_j} + \sum \sigma_j \frac{\partial U}{\partial \sigma_j}
\]

\[
+ \sum \theta_i \frac{\partial U}{\partial \theta_i} + \sum \eta_i' \frac{\partial U}{\partial \eta_i'} + \sum \Pi_i' \frac{\partial U}{\partial \Pi_i'} + \sum \sigma_i' \frac{\partial U}{\partial \sigma_i'} = 0,
\]

\[
\sum \theta_j \frac{\partial U}{\partial \theta_j'} + \sum \eta_j \frac{\partial U}{\partial \eta_j'} + \sum \Pi_j \frac{\partial U}{\partial \Pi_j'} + \sum \sigma_j \frac{\partial U}{\partial \sigma_j'} = 0, \quad (j = 1, 2, 3)
\]

as the system of partial differential equations to be satisfied.

The number of independent solutions is twenty-two. The invariants already found account for half of this number and their first derivatives can be taken for the other half, since they are independent of each other and of their primitives.

There is no need to pursue this matter further in detail. The number of independent absolute invariants involving the fundamental twelve seminvariants together with their derivatives up to those of order \(n - 1\) inclusive, is \(11n\) and for those we can take the ratios of any eleven of the original twelve to the remaining one together with the first \(n - 1\) derivatives of these eleven ratios. We have thus a complete system of absolute invariants and covariants.

The twelve relative invariants given by (6) form a complete system from another point of view. If these twelve are thought of as given functions of \(x\) then system (I), (R) is determined to within constants of integration. Since all defining systems for our triad of ruled surfaces \(R_{v,}, R_{a,}, R_{\gamma}\), are equivalent to system (I), (R), i.e. are obtainable from (I), (R) by transformations of type (1), and since all such systems have the same invariants, then when these twelve invariants are given the triad of ruled surfaces is projectively determined. The choice of these twelve functions is not unrestricted since, by (A), we must have

\[
\eta_j \Pi_j - \theta_j \sigma_j = 0, \quad (j = 1, 2, 3).
\]

It is of interest to note that system (I) may be transformed into one whose coefficients are all absolute invariants by so choosing the new independent variable that \(x'\) shall be equal to any one of the twelve fundamental seminvariants.

We conclude this section by listing the more important invariants of the first paper together with their values in terms of the fundamental invariants of the present discussion. We have

\[
I_1 = \sigma_1 - \theta_1, \quad I_1^{(1)} = \sigma_2 - \theta_2, \quad I_1^{(2)} = \sigma_3 - \theta_3,
\]

\[
I_2 = \eta_1 \Pi_1 - \theta_1 \sigma_1, \quad I_2^{(1)} = \eta_2 \Pi_2 - \theta_2 \sigma_2, \quad I_2^{(2)} = \eta_3 \Pi_3 - \theta_3 \sigma_3,
\]
INVARANTS AND COVARIANTS OF RULED SURFACES.

\[ J_2 = \eta_1 \Pi_2 + \Pi_1 \eta_2 - \theta_1 \sigma_2 - \sigma_1 \theta_2, \]
\[ J_2^{(1)} = \eta_1 \Pi_2 + \Pi_1 \eta_2 - \theta_1 \sigma_2 - \sigma_1 \theta_2, \]
\[ J_2^{(2)} = \eta_2 \Pi_2 + \Pi_1 \eta_2 - \theta_2 \sigma_1 - \sigma_2 \theta_1, \]
\[ J_3 = \eta_1 \sigma_2 \Pi_2 + \Pi_1 \eta_2 \sigma_3 + \sigma_1 \Pi_2 \eta_3 - \theta_1 \eta_2 \theta_3 - \Pi_1 \theta_2 \eta_3 - \Pi_1 \theta_2 \eta_3 = \theta_1 \sigma_2 \theta_3 - \sigma_1 \sigma_2 \eta_3. \]

Of these \( I_2, I_2^{(1)}, I_2^{(2)} \) cannot vanish by virtue of condition (A) as applied to system \((I)\).

For the covariants \( C_1^{(0)}, C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, L_1^{(0)}, L_1^{(1)}, K_1^{(1)}, K_1^{(2)}, M_1^{(1)}, M_1^{(2)} \), of the first paper we have, except for factors involving \( C_1, C_2, C_3 \), the respective forms,

\[ \begin{align*}
\hat{y} \hat{\sigma} - \hat{z} \hat{\alpha} & \equiv \hat{\alpha} \xi \equiv \hat{\beta} \gamma \equiv \hat{\gamma} \hat{z} - \xi \hat{y}, \\
\Pi_2 \hat{y} \hat{\alpha} - \sigma_2 \hat{y} \hat{\beta} - \theta_2 \hat{z} \hat{\alpha} + \eta_2 \hat{\beta}, \\
\Pi_3 \hat{\alpha} \gamma - \sigma_3 \hat{\alpha} \zeta - \theta_3 \hat{\beta} \gamma + \eta_3 \hat{\zeta}, \\
\Pi_1 \hat{\gamma} \hat{y} - \sigma_1 \hat{\gamma} \hat{z} - \theta_1 \hat{\xi} \hat{y} + \eta_1 \hat{\xi}, \\
\Pi_1 \hat{y}^2 - (\theta_1 + \sigma_1) \hat{y} \hat{z} + \eta_1 \hat{z}^2, \\
\Pi_1 \hat{\gamma}^2 - (\theta_1 + \sigma_1) \hat{\gamma} \hat{\zeta} + \eta_1 \hat{\zeta}^2, \\
\Pi_2 \hat{y}^2 - (\theta_2 + \sigma_2) \hat{y} \hat{z} + \eta_2 \hat{z}^2, \\
\Pi_2 \hat{\gamma}^2 - (\theta_2 + \sigma_2) \hat{\gamma} \hat{\zeta} + \eta_2 \hat{\zeta}^2, \\
\Pi_3 \hat{\alpha}^2 - (\sigma_3 + \sigma_3) \hat{\alpha} \hat{\beta} + \eta_3 \hat{\beta}^2, \\
\Pi_3 \hat{\gamma}^2 - (\sigma_3 + \sigma_3) \hat{\gamma} \hat{\zeta} + \eta_3 \hat{\zeta}^2.
\end{align*} \]

II. Geometric Significance of the Invariant

System \((I), (R)\).

The rulings on the three surfaces \( R_y, R_{x\theta}, R_{y\zeta} \) correspond in sets of three, each set made up of one line from each surface. Any three curves lying one on each surface are thus put in point correspondence, those points corresponding which are cut out by sets of corresponding rulings. This correspondence will be one-to-one providing each curve cuts every ruling of the surface on which it lies once only. If all triples of corresponding points on three such curves are such that their three points are collinear, then we shall say of these curves that they constitute a set in alignment. Consider now three one-parameter families of curves, one family on each surface, the parameter being the same for all three families. The curves of three such families thus correspond in sets of three. If all sets of corresponding curves in three such families are sets in alignment then we shall say of these three families of curves that they constitute a system in alignment.

In view of the above definitions it results that the directrix curves for the defining system of differential equations \((I), (R)\) constitute two sets in alignment. We wish to find the most general transformation of the dependent variables of type (1) which will replace these two sets with two other sets in alignment.
The distinguishing feature of a defining system of differential equations for which the directrix curves constitute two sets in alignment, is the form of the adjoined equations \((R)\). So long as a linear relation exists between \(y, \alpha, \gamma\) and a second such relation between \(z, \beta, \zeta\), the directrix curves will consist of two sets in alignment. For these linear relations the necessary and sufficient conditions on the coefficients of transformation \((1)\) are found to be

\[
g : j = s : u = e : c, \quad h : k = t : v = d : f,
\]
or in different form

\[
(17) \quad g = \mu e, \quad j = \mu e, \quad s = \rho c, \quad u = \rho c, \quad h = v d, \quad k = v f, \quad t = \tau d, \quad v = \tau f,
\]
where \(\mu, \nu, \rho, \tau\) are arbitrary functions not identically zero.

If the new variables are \(\bar{y}, \bar{\alpha}, \bar{\gamma}; \bar{z}, \bar{\beta}, \bar{\zeta}\), then the new equations corresponding to \((R)\) are found to be

\[
(18) \quad \bar{y} + \mu \bar{\alpha} + \rho \bar{\gamma} = 0, \quad \bar{z} + v \bar{\beta} + \tau \bar{\zeta} = 0.
\]

A semicanonical form of system \((T)\) for which the directrix curves are intersector curves \(^1\) is characterized by the conditions

\[
(19) \quad a_0 = 0, \quad (i, j = 1, 2), \quad (i, j = 3, 4), \quad (i, j = 5, 6).
\]

If, starting with system \((I)\) we make that transformation \((1)\) for which relations \((17)\) hold, then we may further insist that one or more pairs of directrix curves shall be intersector curves, that is that \((19)\) be satisfied wholly or in part. The curves \(C_y, C_z\) on \(R_{yz}\) will be such a pair providing \((c, e)\) and \((d, f)\) are any two distinct pairs of simultaneous solutions of the linear system

\[
(20) \quad u' = -\theta_1 u + \eta_1 v, \quad v' = -\Pi_1 u + \sigma_1 v. \quad \text{\((2)\)}.
\]

It follows that in an infinite number of ways it is possible to so select the directrix curves that they shall constitute two sets in alignment while at the same time the pair on \(R_{yz}\) are intersector curves. Similarly for \(R_{\alpha \beta}\) and \(R_{\gamma \zeta}\).

The curves \(C_\alpha, C_\beta\) on \(R_{\alpha \beta}\) will be intersector curves providing \((\mu c, \mu e)\) and \((\nu d, \nu f)\) are two distinct pairs of simultaneous solutions of the system

\[
(21) \quad u' = -\theta_2 u + \eta_2 v, \quad v' = -\Pi_2 u + \sigma_2 v.
\]

\(^1\) An intersector curve on \(R_{yz}\) is one such that its tangent at any point \(P\) intersects that ruling of \(R_{\alpha \beta}\) which corresponds to the ruling of \(R_{yz}\) through \(P\). See "first paper" as also a paper, "Ruled surfaces in correspondence," by E. P. Lane, Trans. Am. Mat. Soc. vol. 25 (1923).

\(^2\) "First paper," also "Ruled surfaces in correspondence" by E. P. Lane.
If all four curves are to be intersector curves then from (20) and (21) we will have

\[(22)\]
\[c' = -\theta_1 c + \eta e, \quad c'' = -\Pi_1 c + \sigma_1 e, \quad m'c' + \mu'c = (-\theta_2 c + \eta e) \mu, \quad m'e' + \mu'e = (-\Pi_2 c + \sigma_2 e) \mu,\]
together with a second set of four equations formed from (22) by replacing \(c, e, \mu\) with \(d, f, \nu\) respectively. By eliminating \(c', e', \mu', \mu\) from (22) we find that \((c, e)\) must satisfy the condition

\[(23)\]
\[(\Pi_1 - \Pi_2) c^2 - (\theta_1 - \theta_2 + \sigma_1 - \sigma_2) ce + (\eta_1 - \eta_2) e^2 = 0.\]

In the same way there results the condition

\[(24)\]
\[(\Pi_1 - \Pi_2) d^2 - (\theta_1 - \theta_2 + \sigma_1 - \sigma_2) df + (\eta_1 - \eta_2) f^2 = 0.\]

Since \(c/e = d/f\), it follows that

\[(25)\]
\[c|e + d|f = (\theta_1 - \theta_2 + \sigma_1 - \sigma_2) (\Pi_1 - \Pi_2), \quad cd|ef = (\eta_1 - \eta_2) (\Pi_1 - \Pi_2).\]

Making use of (25) and (1) as modified by (17), we find

\[(26)\]
\[\left\{\begin{array}{l}
\mu \nu (\Pi_3 - \Pi_2) D_1 \bar{y} \bar{z} = [(\Pi_1 - \Pi_2) \bar{y}^2 - (\theta_1 - \theta_2 + \sigma_1 - \sigma_2) \bar{y} \bar{z} + (\eta_1 - \eta_2) \bar{z}^2] ef,
\mu \nu (\Pi_2 - \Pi_1) D_1 \bar{a} \bar{b} = [(\Pi_1 - \Pi_2) \bar{a}^2 - (\theta_1 - \theta_2 + \sigma_1 - \sigma_2) \bar{a} \bar{b} + (\eta_1 - \eta_2) \bar{b}^2] ef.
\end{array}\right.\]

It follows that in only one way is it possible to so select the directrix curves that they shall constitute two sets in alignment while at the same time the pair on \(R_{yz}\) as well as the pair on \(R_{ab}\) are intersector curves. The pair on \(R_{yz}\) are given by the factors of the first covariant of (26) and the pair on \(R_{ab}\) by the factors of the second covariant of (26). Similarly for \(R_{ab}, R_{y\zeta}\) and \(R_{y\zeta}, R_{yz}\).

Finally we may insist that all three pairs of directrix curves shall be intersector curves. This is equivalent to adding to (22) the two conditions

\[(27)\]
\[\rho c' + \rho' c = (-\theta_2 c + \eta e) \rho, \quad \rho c' + \rho' e = (-\Pi_2 c + \sigma_2 e) \rho.\]

Eliminating \(c', e', \rho', \rho\) from (27) and the first two of (22) we find

\[(28)\]
\[(\Pi_1 - \Pi_2) c^2 - (\theta_1 - \theta_2 + \sigma_1 - \sigma_2) ce + (\eta_1 - \eta_2) e^2 = 0.\]

(23) and (28) can both hold if and only if

\[(29)\]
\[\frac{\Pi_1 - \Pi_2}{\Pi_1 - \Pi_2} = \frac{\theta_1 - \theta_2 + \sigma_1 - \sigma_2}{\theta_1 - \theta_2 + \sigma_1 - \sigma_2} = \frac{\eta_1 - \eta_2}{\eta_1 - \eta_2}.\]

A second set of six conditions from (22), (27) by replacing \(c, e, \mu, \rho\) with \(d, f, \nu, \tau\), leads again to (29). We conclude then that only in case the twelve fundamental invariants satisfy conditions (29) is it possible to so choose the directrix curves that they shall constitute two sets in alignment while at the same time all six are intersector curves.
The point whose coordinates are \((y^{(n)}, z^{(n)}, \beta^{(n)}, \gamma^{(n)})\) where \(y^{(n)}\) is the \(n\)th derivative of \(y\) traces a curve, the \(n\)th tangential of the curve \(C_y\). Similarly for \(\alpha^{(n)}, \ldots, \xi^{(n)}\). The six tangentials of any given order can be taken as directrix curves of three ruled surfaces \(R_{yz}^{(n)}, R_{y\beta}^{(n)}, R_{y\gamma}^{(n)}\), whose lines will correspond in sets of three. With respect to the original surfaces \(R_{yz}, R_{y\beta}, R_{y\gamma}\), we shall speak of the set \(R_{yz}^{(n)}, R_{y\beta}^{(n)}, R_{y\gamma}^{(n)}\), as the \(n\)th derived triad.

From equations (R) we find that
\[
(30) \quad \ddot{y}^{(n)} + \ddot{z}^{(n)} + \ddot{\beta}^{(n)} + \ddot{\gamma}^{(n)} = 0, \quad \ddot{z}^{(n)} + \ddot{\beta}^{(n)} + \ddot{\gamma}^{(n)} = 0, \quad (\eta = 0, 1, 2, \ldots),
\]
so that when the original triad \(R_{yz}, R_{y\beta}, R_{y\gamma}\) of ruled surfaces is referred to the covariant curves \(C_y, C_z, \ldots, C_{\xi}\) as directrix curves, all derived triads including that of order zero have the property that their directrix curves constitute two sets in alignment. By reference to (18) it is seen that the most general transformation of type (1) which leaves the above property undisturbed is that for which \(\mu = v = \rho = \tau = 1\), that is, for which
\[
(31) \quad g = s = c, \quad j = u = c, \quad h = t = d, \quad k = v = f.
\]
The invariance of the coefficients of system (I), (R) is not preserved under the general transformation (1) nor under the special transformations (17) and (31). But if in (1) we use coefficients independent of \(x\) then the transformed system of differential equations will also have invariant coefficients. If, in addition, we assume the relation (31) then we shall have the most general system which possesses all the properties of system (I), (R).

From equations (I) it follows that the first tangentials of the points \((\ddot{y}), (\ddot{z})\) cannot lie on \(l_{u\beta}\). For this can happen only if \(\theta_1 = \eta_1 = I_1 = \sigma_1 = 0\). But this would make \(I_2 = 0\) contrary to hypothesis. Similarly the first tangentials of \((\ddot{\alpha}), (\ddot{\beta})\) cannot lie on \(l_{u\xi}\) nor can those of \((\ddot{\gamma}), (\ddot{\xi})\) lie on \(l_{yz}\). The same situation holds for the most general system possessing all the properties of system (I), (R).

On the other hand the second tangentials of \((\dddot{y}), (\dddot{z})\) can lie on \(l_{u\beta}\), those of \((\dddot{\alpha}), (\dddot{\beta})\) on \(l_{u\xi}\), and those of \((\dddot{\gamma}), (\dddot{\xi})\) on \(l_{yz}\).

By differentiating the first and second of equations (I) once each and eliminating from these \(\dddot{\alpha}, \dddot{\beta}, \dddot{\gamma}, \dddot{\xi}\), and their derivatives by the use of equations (I), (R), we find
\[
(32) \quad \begin{cases}
\dddot{y}'' + p_{11} \dddot{y}' + p_{12} \dddot{z}' + (\theta_1 p_{11} + I_1 p_{12} + t_1) \dddot{y} - (\eta_1 p_{11} + \sigma_1 p_{12} + u_1) \dddot{z} = 0, \\
\dddot{z}'' + p_{21} \dddot{y}' + p_{22} \dddot{z}' + (\theta_1 p_{21} + I_1 p_{22} + v_1) \dddot{y} - (\eta_1 p_{21} + \sigma_1 p_{22} + w_1) \dddot{z} = 0,
\end{cases}
\]
where
(33) \[
\begin{align*}
I_2^{(1)} p_{11} &= (\theta_1 + \theta_2) I_2^{(1)} - \theta_2 x_2 x_3 + \theta_2 z_2 x_3 + \theta_3 y_2 x_3 - \eta_2 x_3 \sigma_3 - \eta_2 y_2 + \sigma_3 \theta_3', \\
I_2^{(1)} p_{12} &= -\eta_1 + \eta_2 I_2^{(1)} + \theta_2 x_2 x_3 - \theta_2 z_2 x_3 - \eta_2^2 x_3 - \eta_2 x_3 \sigma_3 - \theta_3 \sigma_3', - \eta_2 \theta_3', \\
I_2^{(1)} p_{21} &= (\eta_1 + \eta_2) I_2^{(1)} - \theta_2 x_2 x_3 + \theta_2 z_2 x_3 + \theta_3 y_2 x_3 - \eta_2 x_3 \sigma_3 - \eta_2 y_2 + \sigma_3 \theta_3', \\
I_2^{(1)} p_{22} &= -\eta_1 + \eta_2 I_2^{(1)} + \theta_2 x_2 x_3 - \theta_2 z_2 x_3 - \eta_2^2 x_3 - \eta_2 x_3 \sigma_3 - \theta_3 \sigma_3', - \eta_2 \theta_3'.
\end{align*}
\]

Equations (32) constitute a defining system of differential equations for the ruled surface $R_{\nu_2}$. Similar systems for the surfaces $R_{\alpha_2}, R_{\beta_2}, \alpha_2, \beta_2, \gamma_2$ can be obtained from (32), (33), (34) by permuting letters, the dependent variables undergoing the cyclic substitutions $(y \alpha \gamma), (z \beta \zeta)$ and all single subscripts undergoing the cyclic substitution $(123)^{(1)}$.

In view of the first two equations of (I), equations (32) can be written
\[
\begin{align*}
\frac{\partial \nu_1'}{\partial \theta_1} - \nu_2' z + \nu_2 z' &= -\nu_2' x_2 + \eta_2 x_3 - \eta_2 \sigma_3, \\
\frac{\partial \nu_2'}{\partial \theta_2} - \nu_1' y + \nu_1 y' &= -\nu_1' x_2 + \eta_2 x_3 - \eta_2 \sigma_3.
\end{align*}
\]

If now
\[
t_1 = u_1 = v_1 = w_1 = 0,
\]
then the second tangentials of $(\tilde{y}), (\tilde{z})$ will lie on $l_{\alpha_2}$. We suppose further that
\[
t_2 = u_2 = v_2 = w_2 = 0,
\]
and
\[
t_3 = u_3 = v_3 = w_3 = 0,
\]
so that $(\tilde{\alpha}''), (\tilde{\beta}'')$ lie on $l_{\gamma_2}$ and $(\tilde{\gamma}''), (\tilde{\zeta}'')$ lie on $l_{\nu_2}$.

Under conditions (37) the coefficients of $\tilde{\alpha}, \tilde{\beta}$ in (35) take the forms
\[
\begin{align*}
\theta_2 p_{11} + \eta_2 p_{12} &= \theta_2 \alpha_2 + \theta_3 \beta_2 + \theta_3 \gamma_2 - \eta_1 x_3 - \eta_2 z_3 - \eta_3 \gamma_1 = \theta, \\
\eta_2 p_{11} + \sigma_2 p_{12} &= \theta_2 x_2 + \theta_3 x_3 - \eta_1 x_3 - \eta_2 z_3 - \eta_3 x_1 = \eta, \\
\theta_2 p_{21} + \eta_2 p_{22} &= \theta_2 \alpha_2 + \theta_3 \beta_2 + \theta_3 \gamma_2 - \eta_1 x_3 - \eta_2 z_3 - \eta_3 \gamma_1 = \eta, \\
\eta_2 p_{21} + \sigma_2 p_{22} &= \theta_2 x_2 + \theta_3 x_3 - \eta_1 x_3 - \eta_2 z_3 - \eta_3 x_1 = \eta.
\end{align*}
\]

Since $\theta, \eta, x, \sigma$ are invariant under the substitution (123) we may write without further computation,
\[
\begin{align*}
\tilde{y}'' &= -\theta \tilde{\alpha} + \eta \beta, \\
\tilde{\alpha}'' &= -\theta \tilde{\gamma} + \eta \zeta, \\
\tilde{\gamma}'' &= -\theta \tilde{y} + \eta \tilde{z}, \\
\tilde{\beta}'' &= -\theta \tilde{\sigma} + \eta \zeta, \\
\tilde{\zeta}'' &= -\theta \tilde{\sigma} + \eta \zeta.
\end{align*}
\]

(1) The manner in which the invariants $I, J$ are permuted can be found by referring to equations (15).
The collinearity of \((\tilde{y}'''), (\tilde{z}'''), (\tilde{r}''')\) and of \((\tilde{y}'''), (\tilde{z}'''), (\tilde{r}''')\) are easily verified from (40) in view of equations \((R)\).

From the above results we find that the second derived triad of ruled surfaces will coincide with the original triad \(R_\mu, B_\alpha, R_\nu\) if and only if, conditions (36), (37), (38) are satisfied.

It should be noted that in this coincidence the surfaces are once permuted, \(R_\sigma^{(2)}\) coinciding with \(R_\alpha\) and so on around.

The defining system of differential equations for the second derived triad when it coincides with the original triad can be obtained by differentiating equations (40) once each and employing (40) and \((I), (R)\) to eliminate the undesirable variables and imposing conditions (36), (37), (38). Two repetitions of the complete process above will result in the coincidence of the sixth derived triad with the original, the coincidence this time being without permuting in the sense that

\[
R_\mu^{(6)} \equiv R_\nu, \quad R_\alpha^{(6)} \equiv R_\beta, \quad R_\gamma^{(6)} \equiv R_\delta.
\]

In conclusion it is worth while to point out that the advantage of system \((I), (R)\) over any other defining system in which the coefficients are not invariants and the dependent variables not covariants lies in the fact that geometric significance attaches to every whatever combination of the coefficients or of the coefficients and dependent variables which is homogeneous and isobaric, providing only no transformation coefficients other than constants or invariants enter into it.

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