Systematic Development of Number-System, VI,

by

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In my paper "Axiomatic Investigation of Number-System"(1), I have developed the number-system step by step, from the natural number up to the ordinary quaternion, by the method of "pairs of numbers" using one and the same principle. By every step, a higher system is always obtained from the former one, by the two definitions and the two postulates concerning comparisons and operations of numbers. Here we may go two steps further in this direction, namely from the ordinary quaternion up to the biquaternion (the higher quaternion), and from the biquaternion up to the alternate-octonary number. By biquaternion we mean the quaternion \( a_0 + a_1j_1 + a_2j_2 + a_3j_3 \), whose coefficients \( a_0, a_1, a_2, a_3 \) are complex numbers, while the ordinary quaternion has its coefficients \( a_0, a_1, a_2, a_3 \) as real number. And as to the alternate-octonary number, we shall explain its meaning atertimes, since it is a new number-system containing biquaternion and alternate numbers as its subclass.

(G) Biquaternion.

Introduction of New Numbers.

Consider the system of ordinary quaternions already defined, and take any two numbers of them, and denote them by \( A, B \). From this pair of numbers, here we try to construct a new thing which is denoted by \( (A, B) \). In order that we may treat it as a number, we lay down the following definitions and postulates.

Definition (A). The two numbers \((A, B)\) and \((A', B')\) are said to be equal to each other when and only when \( A = A' \) and \( B = B' \).

Definition (B). The number \((A, B)\) is said to be greater than or less than the number \((A', B')\) according as \( A \) is greater than or less than \( A' \), and when \( A \) is equal to \( A' \), according as \( B \) is greater than or less than \( B' \).

Postulate (I). \((A, B) \oplus (A', B') = (A + A', B + B')\).

Postulate (II). \((A', B') = (AA' - BB', AB' + BA')\).

The assemblage of all numbers thus constructed with these definitions and postulates is called the system of biquaternions.

Here we shall study what kind of numbers is represented by the above definitions and postulates.

**Theorem 1.** The new system of numbers is a more extended one than the system of the ordinary quaternions and contains it as its subclass.

**Proof.** When \(B = 0\), the new number \((A, B)\) is equivalent to the ordinary quaternion \(A\). For, if two numbers \((A, B)B=0\) and \((A', B')B'=0\) are equal to each other, then, by Def. (A), \(A\) is equal to \(A'\), and conversely. If \((A, B)B=0\) is greater than or less than \((A', B')B'=0\), then, by Def. (B), \(A\) is greater than or less than \(A'\), and conversely. Further, if we take \(A\) and \(A'\) instead of \((A, B)B=0\) and \((A', B')B'=0\), then Postulates (I) and (II) are also satisfied. Moreover, to any ordinary quaternion \(A\), there is always a corresponding one \((A, B)B=0\) in our system of numbers, which is equivalent to it. Therefore, the class of all ordinary quaternions may be represented by the class of new numbers \(\{(A, B)B=0\}\); and we may put \((A, B)B=0 = A\).

When \(B \neq 0\), the class of such new numbers represents a quite new class of numbers. For, any element of the class \((A, B)B=0\) is equal to none of the class \((A, B)B=0\) by Def. (A), and so it cannot be equivalent to any one of ordinary quaternions. Thus the system of ordinary quaternions may be considered as a subclass of our system.

**Fundamental Properties and New Units.**

A new unit. If we put \(A = A' = 0\) and \(B = B' = 1\) in Postulate (II), then we have

\[(0, 1)^2 = (-1, 0) = -1.\]

Therefore if we put \((0, 1) = i\), then we have \(i^2 = -1\); in this respect, this unit \(i\) resembles very much to the quaternion-units \(j_1, j_2, j_3\), but it is different from them. For, as it was already shown, the quaternion-units \(j_1, j_2, j_3\) satisfy the relation \(j_r j_s = -j_s j_r\) \((r \neq s; r, s; 1, 2, 3)\), while the new unit \(i\) satisfies the relation \(i j_r = j_r i\) \((r = 1, 2, 3)\), as shown below.

By Postulate (II), we have

\[i j_1 = (0, 1) \times (j_1, 0) = (0, j_1),\]
\[j_1 i = (j_1, 0) \times (0, 1) = (0, j_1).\]
Similarly we have

\[ ij_1 = j_1 i, \quad ij_2 = j_2 i, \quad ij_3 = j_3 i. \]

**Theorem 2.** The commutative and associative laws hold good in the addition of this system.

**Theorem 3.** The associative law holds good in the multiplication of this system, but not the commutative law in general.

**Theorem 4.** The distributive law holds good always in this system of numbers.

These three theorems follow at once from Postulates (I), (II) by mere calculation.

**Theorem 5.** By means of the new unit \((0, 1) = i\), any new number \((A, B)\) may be denoted in the form \(A + Bi\).

**Proof.** By Postulate (I), we have the relation

\[ (A, B) = (A, 0) + (0, B), \]

and by Postulate (II), we have the relation

\[ (0, B) = (B, 0) \cdot (0, 1). \]

Therefore, from (1) and (2), we have

\[ (A, B) = (A, 0) + (B, 0)(0, 1) = A + Bi. \]

**Fundamental theorem (A).** Any number \((A, B)\) of the system may be expressed as a linear homogeneous function of the quaternion units \(1, j_1, j_2, j_3\) with complex coefficients.

**Lemma I.**

**Proof.**

\[ (b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3)i = (b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3)(0, 0)(0, 1) \]

But

\[ b_0 i + b_1 i j_1 + b_2 i j_2 + b_3 i j_3 = (b_0, 0)(0, 1) + b_1(0, 1)(j_1, 0) + b_2(0, 1)(j_2, 0) + b_3(0, 1)(j_3, 0) \]

\[ = (b_0, b_1)(0, j_1) + (b_2, b_3)(0, j_2) \]

\[ = (b_0, b_1)(0, j_1) + (b_2, b_3)(0, j_2) \]

\[ = b_0(j_1, 0) + b_2(j_2, 0) + b_3(j_3, 0) \]

\[ = (b_0, 0)(0, j_1) + (b_2, 0)(0, j_2) + (b_3, 0)(0, j_3) \]

\[ = (b_0, 0)(0, j_1) + (b_2, 0)(0, j_2) + (b_3, 0)(0, j_3) \]

\[ = (0, b_0) + (0, b_1 j_1) + (0, b_2 j_2) + (0, b_3 j_3). \]
Lemma II.  \[(a_r + b_r j_r)j_r = a_r j_r + b_r i j_r.\]

Proof.  \[(a_r + b_r i j_r)j_r = \{(a_r, 0) + (b_r, 0)(0, 1)\} (j_r, 0)\]

\[= (a_r, 0) + (0, b_r) (j_r, 0)\]

\[= (a_r, b_r)(j_r, 0) - (a_r j_r, b_r j_r).\]

On the other hand, we have

\[a_r j_r + b_r i j_r = (a_r j_r, 0) + (b_r, 0)(0, 1)(j_r, 0)\]

\[= (a_r j_r, 0) + (0, b_r) (j_r, 0)\]

\[= (a_r j_r, 0) + (0, b_r j_r) = (a_r j_r, b_r j_r).\]

\[\therefore (a_r + b_r i j_r)j_r = a_r j_r + b_r i j_r.\]

Proof of the fundamental theorem.

Put \[A = a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 = a_0 + \sum_{r=1}^{3} a_r j_r,\quad B = b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3\]

\[= b_0 + \sum_{r=1}^{3} b_r j_r,\] then we have

\[(A, B) = A + Bi = (a_0 + \sum_{r=1}^{3} a_r j_r) + (b_0 + \sum_{r=1}^{3} b_r j_r) i\]

\[= (a_0 + \sum_{r=1}^{3} a_r j_r) + (b_0 i + \sum_{r=1}^{3} b_r i j_r)\]  (Lemma 1)

\[= (a_0 + b_0 i) + \sum_{r=1}^{3} (a_r j_r + b_r i j_r)\]  (Theorem 2)

\[= (a_0 + b_0 i) + \sum_{r=1}^{3} (a_r + b_r i) j_r\]  (Theorem 2)

where \[a_r = a_r + b_r i = \text{complex number } (r = 0, 1, 2, 3)\].

Therefore \[(A, B)\] may be expressed as a linear function of quaternion units \(1, j_1, j_2, j_3\) with complex coefficients.

Fundamental theorem (B). The four quaternion units \(1, j_1, j_2, j_3\) are linearly independent with respect to the complex coefficients.

Proof. To show the validity of this theorem, we have to prove that \(a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 = a_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3\) (\(\alpha_0, \alpha_1, \alpha_2, \alpha_3\): complex numbers) is equal to zero when and only when the coefficients \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) are all zeros.

First suppose that \(\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3 = 0\), then we have

\[\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3 = (a_0 + b_0 i) + (a_1 + b_1 i) j_1 + (a_2 + b_2 i) j_2 + (a_3 + b_3 i) j_3\]

\[= (a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3) - (b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3) = 0.\]

Now from the definitions of zero and equality of the biquaternion
(A, B), it follows that

\[(A, B) + (A, B) = (A, B)\]  \quad (def. of zero),

or

\[(2A, 2B) = (A, B)\]  \quad (postulate of addition),

or

\[2A = A, \ 2B = B\]  \quad (def. of equality),

or

\[A = 0, \ B = 0.\]  \quad (def. of zero).

\[a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 = 0, \ b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3 = 0.\]

But, since 1, j_1, j_2, j_3 are linearly independent with respect to real coefficients, from the above, it follows that the real numbers \(a_0, a_1, a_2, a_3; b_0, b_1, b_2, b_3\) are all zero; and accordingly all the complex numbers \(a_0, a_1, a_2, a_3\) are also zero. Conversely, \(a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3\) is clearly zero when all the coefficients are zero. Thus the linear independence of the units 1, j_1, j_2, j_3 is proved.

**Fundamental theorem (C).** The sum and product of any two numbers (A, B) and (A', B') of our new system are equal to those obtained by applying the ordinary addition and multiplication of algebra to the corresponding numbers \((a_0 + b_0 i) + (a_1 + b_1 i)j_1 + (a_2 + b_2 i)j_2 + (a_3 + b_3 i)j_3\) and \((a_0' + b_0'i) + (a_1' + b_1'i)j_1 + (a_2' + b_2'i)j_2 + (a_3' + b_3'i)j_3\).

**Proof.** (i) Sum.

Taking the formula of addition

\[(A, B) + (A', B') = (A + A', B + B')\]

given by Postulate (I), if we put

\[
\begin{align*}
A &= a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 = a_0 + \sum_{r=1}^{3} a_r j_r, \\
B &= b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3 = b_0 + \sum_{r=1}^{3} b_r j_r, \\
A' &= a_0' + a_1' j_1 + a_2' j_2 + a_3' j_3 = a_0' + \sum_{r=1}^{3} a_r' j_r, \\
B' &= b_0' + b_1' j_1 + b_2' j_2 + b_3' j_3 = b_0' + \sum_{r=1}^{3} b_r' j_r,
\end{align*}
\]

for \(A, B, A', B'\), then we have the sum of two numbers \((A, B)\) and \((A', B')\) as follows.

\[\text{Postulate (I)}\]

\[\text{(Theorem 5)}\]
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\[ \{(a_0 + a') + (b_0 + b')i\} + \sum_{r=1}^{3} \{(a_r + a'_r) + (b_r + b'_r)i\} j_r. \]

(Theorem 2, Lemmas 1, 2)

But this result is equal to that obtained by applying the ordinary addition to the two numbers

\[(A, B) = (a_0 + b_0i) + \sum_{r=1}^{3} (a_r + b_r)i j_r,\]

\[(A', B') = (a'_0 + b'_0i) + \sum_{r=1}^{3} (a'_r + b'_r)i j_r.\]

Therefore, in this case, the theorem is true.

(ii) Product.

In the formula of multiplication

\[(A, B) \times (A', B') = (A A' - B B', A B' + B A')\]

given by Postulate (II), if we put the relation (a), then we have

\[(A, B) \times (A', B') = (P_0 + P_1 j_1 + P_2 j_2 + P_3 j_3, Q_0 + Q_1 j_1 + Q_2 j_2 + Q_3 j_3),\]

where \(P_s'\) and \(Q_s'\) are real numbers.

Applying Theorems 2, 3, 4 and Lemmas 1, 2 to the above result, we have

\[(A, B) \times (A', B') = (P_0 + Q_0 i) + (P_1 + Q_1 i) j_1 + (P_2 + Q_2 i) j_2 + (P_3 + Q_3 i) j_3.\]

But this result is the same as that obtained by applying the ordinary multiplication to the two numbers

\[(A, B) = (a_0 + b_0i) + \sum_{r=1}^{3} (a_r + b_r i) j_r,\]

\[(A', B') = (a'_0 + b'_0i) + \sum_{r=1}^{3} (a'_r + b'_r i) j_r,\]

and then by applying the formulae \(j_1^2 = -1, j_2 j_3 = \pm j_1\). Therefore, in this case also, the theorem is true.

Remark. It will be noticed that, in the case of biquaternion, the fundamental law

"when the product vanishes, at least one of its factors must vanish" does not necessarily hold, while it is always true in the case of ordinary quaternion. Thus, in our system, if we take two numbers \((j_r, 1), (j_r, -1)\), for example, and make their product, then we shall see that their product vanishes while neither of its factors is zero.

\[(j_r, 1)(j_r, -1) = \{(j_r)^2 - (-1 \times 1), -j_r + j_r\} = (0, 0) = 0.\]

(H) Alternate-Octonary Number.

In higher complex numbers, the most famous and interesting
ones are quaternions and alternate numbers, both having the wide practical applications and many geometrical interpretations. Having introduced the ordinary quaternion and biquaternion into our number-system by our systematic method, I tried to introduce alternate number-system also by the same principle. But, as the quaternions and alternate numbers are different in their essential properties and the quaternion cannot be a special case of alternate numbers, it is impossible to introduce alternate numbers by our principle, namely to introduce it so as to contain quaternion as its subclass. Therefore, I tried to introduce a new number-system containing both quaternions and alternate numbers as its special cases, by using the method of "pairs of numbers already defined" (1); just as we had introduced the ordinary complex numbers, containing the real number-system and the imaginary number-system as its special cases, by the pairs of real numbers. This new system I called the alternate-octonary system, since it contains 8 independent units and also an alternate number-system.

(a) Alternate-Octonary Number-System of the First Kind.

Introduction of New Numbers.

Consider the system of biquaternions already defined, and take any two numbers of them, and denote them by \( A, B \). From this pair of numbers, we construct a new thing which we denote by \( (A, B) \). In order that we may treat it as a number, we lay down the following definitions and postulates.

Definition (A). The two numbers \( (A, B) \) and \( (A', B') \) are said to be equal to each other when and only when \( A = A' \) and \( B = B' \).

Definition (B). The number \( (A, B) \) is said to be greater than or less than the number \( (A', B') \), according as \( A \) is greater than or less than \( A' \), and when \( A \) is equal to \( A' \), according as \( B \) is greater than or less than \( B' \).

Postulate (I). \((A, B) + (A', B') = (A + A', B + B')\).

Postulate (II). \((A, B) \times (A', B') = (A'A', AB' + BA' + BB')\).

Here by \( AB' \), is meant the following number.

\[
B = \beta_0 + \beta_1j_1 + \beta_2j_2 + \beta_3j_3,
\]

\[
B' = \beta'_0 + \beta'_1j'_1 + \beta'_2j'_2 + \beta'_3j'_3.
\]

(a) \( BB' = \sum \beta_r \beta'_r j_r j_s \quad (r, s = 1, 2, 3, \ r \neq s)\)

(1) Here they are pairs of biquaternions.
Now we shall study what kind of numbers is represented by the above definitions and postulates.

**Theorem 1.** The new system of numbers is a more extended one than the system of biquaternion and contains it as its subclass.

**Proof.** When $B=0$, the new number $(\mathfrak{A}, B)$ is equivalent to the biquaternion $\mathfrak{A}$. For, if two numbers $(\mathfrak{A}, B)_{\mathfrak{B}=0}$ and $(\mathfrak{A}', B')_{\mathfrak{B}=0}$ are equal to each other, then, by Def. (A), $\mathfrak{A}$ is equal to $\mathfrak{A}'$, and conversely. If $(\mathfrak{A}, B)_{\mathfrak{B}=0}$ is greater than or less than $(\mathfrak{A}', B')_{\mathfrak{B}=0}$, then, by Def. (B), $\mathfrak{A}$ is greater than or less than $\mathfrak{A}'$, and conversely. Further, if we take $\mathfrak{A}$ and $\mathfrak{A}'$ instead of $(\mathfrak{A}, B)_{\mathfrak{B}=0}$ and $(\mathfrak{A}', B')_{\mathfrak{B}=0}$, then Postulates (I) and (II) are also satisfied. Moreover, to any biquaternion $\mathfrak{A}$, there is always a corresponding one $(\mathfrak{A}, B)_{\mathfrak{B}=0}$ in our system of numbers, which is equivalent to it. Therefore, the class of all biquaternions may be represented by the class of new numbers $\{\mathfrak{A}, B\}_{\mathfrak{B}=0}$, and we may put $(\mathfrak{A}, B)_{\mathfrak{B}=0}=\mathfrak{A}$.

**Theorem 2.** The new system of numbers contains a system of alternate numbers as its subclass.

**Proof.** When $A=0$, the new number $(\mathfrak{A}, B)$ has the fundamental properties of alternate numbers.

(i) **In the first place**, the number $(0, B)$ satisfies the relation $(0, B)^2=0$.

For, by Postulate (II), we have the relation $(0, B) \times (0, B) = (0, BB)$; but by (a) $BB$ is zero.

\[ \therefore \quad (0, B)^2 = (0, 0). \]

And, by Postulate (I), $(0, 0) + (0, 0) = (0, 0)$, and so $(0, 0)$ is zero.

(ii) **In the second place**, the numbers $(0, B)$ and $(0, B')$ satisfy the relation...
(c) \((0, \mathbb{B}) \times (0, \mathbb{B}') = -(0, \mathbb{B}')(0, \mathbb{B})\).

For, by Postulate (II), we have the relation
\[(0, \mathbb{B}) \times (0, \mathbb{B}') = (0, \mathbb{B}'),\]
and by (a), we have the relation \(\mathbb{B}B' = -\mathbb{B}'\).
\[
\therefore (0, \mathbb{B}' \mathbb{B}) = (0, -\mathbb{B}'\mathbb{B}).
\]

But by Postulate (I), we have the relation
\[(0, \mathbb{B}'\mathbb{B}) + (0, -\mathbb{B}'\mathbb{B}) = (0, \mathbb{B}' - \mathbb{B}') = (0, 0) = 0.
\]
\[
\therefore (0, -\mathbb{B}'\mathbb{B}) = -(0, \mathbb{B}'\mathbb{B}).
\]
\[
\therefore (0, \mathbb{B}') \times (0, \mathbb{B}) = -(0, \mathbb{B}') \times (0, \mathbb{B}).
\]

(iii) In the third place, take the four particular numbers \((0, 1), (0, j_1), (0, j_2), (0, j_3)\) from the class of the numbers \((0, \mathbb{B})\), and denote them by \(k_0, k_1, k_2, k_3\) respectively. Then, from the above discussion, we have the relations
\[
(\mathbb{B} = k_0, k_1, k_2, k_3)\text{ are independent with respect to the complex coefficients.}
\]

Now we may prove that any number \((0, \mathbb{B}) = (0, \alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)\) may be represented by \(\alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3\).

For, from Postulates (I) and (II), we have the relations
\[(0, \alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3) = (0, \alpha_0) + (0, \alpha_1 j_1) + (0, \alpha_2 j_2) + (0, \alpha_3 j_3)
= (\alpha_0, 0)(0, 1) + (\alpha_1, 0)(0, j_1) + (\alpha_2, 0)(0, j_2)
+ (\alpha_3, 0)(0, j_3)
= \alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3.
\]

(iv) In the fourth place, \(k_0, k_1, k_2, k_3\) are linearly independent with respect to the complex coefficients \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\). For, \(\alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3\) is equal to the number \((0, \alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)\) by (iii), and the number \((0, \alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)\) cannot be zero unless the biquaternion \(\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3\) is zero (by definitions of zero and equality), but again the biquaternion \(\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3\) cannot be zero unless the complex numbers \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) are all zero (by fundamental theorem (B) of biquaternion). Hence \(\alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3\) cannot be equal to zero unless \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) are all zero. Therefore \(k_0, k_1, k_2, k_3\) are linearly independent with respect to complex coefficients.

From the results obtained in (iii) and (iv), we may take the four numbers \(k_0, k_1, k_2, k_3\) as units of the new numbers.
(v) As multiplication formulae of alternate units, we have the following relations.

(a) \( k_r^2 = 0, \)
(b) \( k_r k_s = -k_s k_r, \)
(c) \( k_0 k_r = k_r, \)
(d) \( k_r k_s = k_t, \)

(in (d), \( r, s \) denote two of the numbers 1, 2, 3, taken cyclically; and \( r, s, t \) denote the numbers different from one another).

For we have

\[
\begin{align*}
k_0 k_1 &= (0, 1)(0, j_1) = (0, j_1) = k_1, \\
k_0 k_2 &= (0, 1)(0, j_2) = (0, j_2) = k_2, \\
k_0 k_3 &= (0, 1)(0, j_3) = (0, j_3) = k_3, \\
k_1 k_2 &= (0, j_1)(0, j_2) = (0, j_1 j_2) = (0, j_3) = k_3, \\
k_2 k_3 &= (0, j_2)(0, j_3) = (0, j_2 j_3) = (0, j_1) = k_1, \\
k_3 k_1 &= (0, j_3)(0, j_1) = (0, j_3 j_1) = (0, j_2) = k_2.
\end{align*}
\]

(vi) From the above relations, we may prove the following relation

\[ (0, \sum \alpha j_r j_s) = \sum \alpha k_r k_s \quad r \neq s. \]

For, since \( j_r j_s = \pm j_t \), by the property of quaternion-units, we have the relation

\[ \sum \alpha j_r j_s = \sum \pm \alpha j_t. \]

\[ \therefore \quad (0, \sum \alpha j_r j_s) = (0, \sum \pm \alpha j_t) = \sum \pm \alpha k_t. \quad \text{(by (iii)).} \]

But \( k_r k_t = (0, j_r)(0, j_s) = (0, j_r j_s) = (0, \pm j_t) = \pm k_t. \)

\[ \therefore \quad \sum \alpha k_r k_s = \sum \pm \alpha k_t. \]

\[ \therefore \quad (0, \sum \alpha j_r j_s) = \sum \alpha k_r k_s. \]

(vii) The sum and product of any two numbers \((0, b)\) and \((0, b')\) of our system are equal to those obtained by applying the ordinary addition and multiplication of algebra to the corresponding numbers \(\beta_0 k_0 + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3\) and \(\beta_0' k_0 + \beta_1' k_1 + \beta_2' k_2 + \beta_3' k_3\).

\[ \therefore \quad (a) \quad (0, b) + (0, b') = (0, b + b') \]

\[ = \{0, (\beta_0 + \beta_0') + (\beta_1 + \beta_1') j_1 + (\beta_2 + \beta_2') j_2 \\
+ (\beta_3 + \beta_3') j_3 \} \]

\[ = (\beta_0 + \beta_0') k_0 + (\beta_1 + \beta_1') k_1 + (\beta_2 + \beta_2') k_2 \\
+ (\beta_3 + \beta_3') k_3. \]

But the result obtained by applying the ordinary addition to the
numbers \( \beta_0 k_0 + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 \) and \( \beta_0' k_0 + \beta_1' k_1 + \beta_2' k_2 + \beta_3' k_3 \) is clearly \( (\beta_0 + \beta_0') k_0 + (\beta_1 + \beta_1') k_1 + (\beta_2 + \beta_2') k_2 + (\beta_3 + \beta_3') k_3 \). Q. E. D.

(b) \((0, B) \times (0, B') = (0, BB')\)

\[ = \{0, \sum \beta_r \beta_r' j_r j_r + \sum (\beta_0 \beta_r' - \beta_r \beta_0') j_r\} \]

\[ = \{0, \sum \beta_r \beta_r' j_r j_r + \sum (\beta_0 \beta_r' - \beta_r \beta_0') j_r\} \]

\[ = \sum \beta_r \beta_r' k_r k_r + \sum (\beta_0 \beta_r' - \beta_r \beta_0') k_r. \] (By (iii)(vi)).

But the result obtained by applying the ordinary multiplication to the numbers \( \beta_0 k_0 + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 \) and \( \beta_0' k_0 + \beta_1' k_1 + \beta_2' k_2 + \beta_3' k_3 \) and by substituting the relations

\[ k_r^2 = 0, \quad k_r k_r = - k_r k_r, \]

is

\[ \sum \beta_r \beta_r' k_r k_r + \sum (\beta_0 \beta_r' - \beta_r \beta_0') k_r. \]

Therefore both results are identical.

(viii) The number \((0, B)\) obeys the distributive law.

By Postulates (I) and (II), we have the following relations

(a) \((0, B) \times (0, B') + (0, B'') = (0, B)(0, B' + B'') = (0, BB' + BB'')\),

(b) \((0, B)(0, B') + (0, B)(0, B'') = (0, BB') + (0, BB'')\)

\[ = (0, BB' + BB''). \]

But by the convention for \( BB' \), we have

\[ BB' + BB'' = \sum \beta_r (\beta_r' + \beta_r'') j_r j_r + \sum (\beta_0 \beta_r' - \beta_r \beta_0') j_r \quad (r, s = 1, 2, 3) \]

\[ + \sum (\beta_0 \beta_0' + \beta_0' + \beta_0'' - \beta_s \beta_0') j_s \quad (s = 1, 2, 3) \]

\[ = \sum (\beta_r \beta_r' + \beta_r \beta_r'') + \sum (\beta_0 \beta_r' - \beta_r \beta_0') j_r \quad (r, s = 1, 2, 3) \]

Now, since \( \beta, \beta', \beta'' \) are complex numbers and they obey the distributive, associative and commutative laws, we have

\[ BB' + BB'' = \sum \beta_r (\beta_r' + \beta_r'') j_r j_r + \sum (\beta_0 \beta_r' - \beta_r \beta_0') j_r \quad (r, s = 1, 2, 3) \]

\[ + \sum (\beta_0 \beta_0' + \beta_0' + \beta_0'' - \beta_s \beta_0') j_s \quad (s = 1, 2, 3) \]

\[ = (0, BB' + BB''). \]

\[ \therefore (0, B)(0, B') + (0, B'') = (0, B)(0, B') + (0, B)(0, B''). \]

Similarly, the relation

\[ (0, B') + (0, B'') = (0, B')(0, B') + (0, B'')(0, B') \]

also holds good.

(ix) The number \((0, B)\) obeys the associative and commutative laws for addition.
This follows at once from Postulate (1) and the property of the biquaternion $\mathfrak{B}$.

From the above discussions, we see that the new number $(0, \mathfrak{B}) = (0, \alpha_0 + \alpha_1i_1 + \alpha_2i_2 + \alpha_3i_3)$ has all characteristic properties of alternate numbers, namely, (i) the characteristic relations between the units

$$k_r^2 = 0, \quad k_r k_s = -k_s k_r$$

also hold good in all numbers of the system

$$(0, \mathfrak{B})^2 = 0, \quad (0, \mathfrak{B})(0, \mathfrak{B'}) = -(0, \mathfrak{B'})(0, \mathfrak{B});$$

and (ii) it may be represented in the form $\alpha_0k_0 + \alpha_1k_1 + \alpha_2k_2 + \alpha_3k_3$, and may be treated in the same manner as in the ordinary addition and multiplication.

**Theorem 3.** The number of our system consists of the sum of biquaternion and alternate number.

**Proof.** By Postulate (1), we have

$$(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}, 0) + (0, \mathfrak{B}),$$

and by Theorem 2, we have

$$(\mathfrak{A}, 0) = \mathfrak{A} = \alpha_0 + \alpha_1i_1 + \alpha_2i_2 + \alpha_3i_3,$n

$$(0, \mathfrak{B}) = (0, \beta_0 + \beta_1i_1 + \beta_2i_2 + \beta_3i_3) = \beta_0k_0 + \beta_1k_1 + \beta_2k_2 + \beta_3k_3.$$n

Therefore we have

$$(\mathfrak{A}, \mathfrak{B}) = (\alpha_0 + \alpha_1i_1 + \alpha_2i_2 + \alpha_3i_3) + (\beta_0k_0 + \beta_1k_1 + \beta_2k_2 + \beta_3k_3).$$

**Theorem 4.** Multiplication formulae for quaternion-units and alternate-units are as follows:

<table>
<thead>
<tr>
<th>$j_r^2 = -1$</th>
<th>$k_r^2 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j_r j_s = -j_s j_r$</td>
<td>$k_r k_s = -k_s k_r$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
(j_0, j_1) &= j_0 k_1 = k_0, \\
(k_0 j_1) &= j_1 k_0 = j_1, \\
(k_0 j_2) &= j_2 k_0 = j_2, \\
(k_0 j_3) &= j_3 k_0 = j_3,
\end{align*}
\]

or

$$(k_0 j_r = j_r k_0 = k_r).$$

\[
\begin{align*}
k_0 j_r &= (0, 1)(j_r, 0) = (0, 1) = k_r, \\
(j_r k_0) &= (j_r, 0)(0, 1) = (0, j_r) = k_r.
\end{align*}
\]
The latter three formulae (ii), (iii), (iv) may be expressed in the following formulae

\[
\begin{align*}
\text{(ii)} & \\
& \begin{cases}
  k_1.1=1.k_1=k_1, \\
  k_1j_1=j_1k_1=-k_0, \\
  k_1j_2=-j_2k_1=k_3, \\
  k_1j_3=-j_3k_1=-k_2.
\end{cases}
\end{align*}
\]
\[
\cdot \cdot \cdot \\
\begin{align*}
& \begin{cases}
  k_3.t=0.j_s(t, 0)=(0.j_s)=(0.j_3)=k_3, \\
  j_2k_2=(j_2, 0)(0.j_1)=(0.j_2j_1)=(0, -j_3)=-k_3.
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\text{(iii)} & \\
& \begin{cases}
  k_2.1=1.k_2=k_2, \\
  k_2j_1=-j_1k_2=-k_3, \\
  k_2j_2=j_2k_2=-k_0, \\
  k_2j_3=-j_3k_2=k_1.
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\text{(iv)} & \\
& \begin{cases}
  k_3.1=1.k_3=k_3, \\
  k_3j_1=-j_1k_3=-k_2, \\
  k_3j_2=j_2k_3=-k_1, \\
  k_3j_3=-j_3k_3=-k_0.
\end{cases}
\end{align*}
\]

\(\varepsilon\) denotes +1 when \(r, s, t\) take the numbers 1, 2, 3 in cyclical order, and it denotes -1 when they take 1, 2, 3 counterclockwise.

**Theorem 5.** The sum and product of any two numbers \((A, B)\) and \((A', B')\) of our system are equal to those obtained by applying the ordinary addition and multiplication of algebra to the corresponding numbers \(\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r + \sum_{s=0}^{3} \beta_s k_s\) and \(\alpha_0' + \sum_{r=1}^{3} \alpha_r' j_r + \sum_{s=0}^{3} \beta_s' k_s\).

**Lemma 1.**

\[
\{0, (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r) (\beta_0' + \sum_{s=0}^{3} \beta_s' j_s)\} = (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r) \sum_{s=0}^{3} \beta_s' k_s.
\]

Here the right-hand side is to be calculated by the ordinary multiplication formula.

**Proof.** On one hand, we have

\[
(1) \quad \{0, (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r) (\beta_0' + \sum_{s=0}^{3} \beta_s' j_s)\} = \{0, (\alpha_0 \beta_0' - \alpha_1 \beta_1' - \alpha_2 \beta_2' - \alpha_3 \beta_3') + (\alpha_0 \beta_1' + \alpha_1 \beta_0' + \alpha_2 \beta_3' - \alpha_3 \beta_2') j_1 \\
+ (\alpha_0 \beta_2' - \alpha_1 \beta_3' + \alpha_2 \beta_0' + \alpha_3 \beta_1') j_2 + (\alpha_0 \beta_3' + \alpha_1 \beta_2' - \alpha_2 \beta_1' + \alpha_3 \beta_0') j_3\}
\]

\[
= \{0, \alpha_0 \beta_0' - \alpha_1 \beta_1' - \alpha_2 \beta_2' - \alpha_3 \beta_3' + (\alpha_0 \beta_1' + \alpha_1 \beta_0' + \alpha_2 \beta_3' - \alpha_3 \beta_2') j_1 \\
+ (\alpha_0 \beta_2' - \alpha_1 \beta_3' + \alpha_2 \beta_0' + \alpha_3 \beta_1') j_2 + (\alpha_0 \beta_3' + \alpha_1 \beta_2' - \alpha_2 \beta_1' + \alpha_3 \beta_0') j_3\}
\]
And on the other hand, we have

\[ (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r)(\sum_{r=0}^{3} \beta_r j_r k_r) = \alpha_0 \sum_{r=0}^{3} \beta_r j_r k_r + \alpha_1 \sum_{r=0}^{3} \beta_r j_1 k_r + \alpha_2 \sum_{r=0}^{3} \beta_r j_2 k_r + \alpha_3 \sum_{r=0}^{3} \beta_r j_3 k_r \]

From (1) and (2), the legitimacy of our lemma follows at once.

**Lemma II.** In the product of biquaternion and alternate number of our system, their units \( j \) and \( k \) may be interchanged; namely

\[ (\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)(\beta_0 k_0 + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3) = (\alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3)(\beta_0 + \beta_1 j_1 + \beta_2 j_2 + \beta_3 j_3). \]

**Proof.** On one hand, we have

\[ (\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)(\beta_0 k_0 + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3) = \alpha_0 \sum_{r=0}^{3} \beta_r k_r + \alpha_1 \sum_{r=0}^{3} \beta_r j_r k_r + \alpha_2 \sum_{r=0}^{3} \beta_r j_2 k_r + \alpha_3 \sum_{r=0}^{3} \beta_r j_3 k_r \]

And on the other hand, we have

\[ (\alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3)(\beta_0 + \beta_1 j_1 + \beta_2 j_2 + \beta_3 j_3) \]

Therefore (1) and (2) are equal to each other.
Proof of Theorem 5. In the case of the sum, it may be proved easily that this theorem is true. Therefore we shall prove it in the case of the product.

In the first place, by Postulates (I) and (II), we have

\[(\mathbb{A}, \mathbb{B})(\mathbb{A}', \mathbb{B}')=\mathbb{A}'(\mathbb{A}' + \mathbb{B}' + \mathbb{B}')+(\mathbb{A}', 0)\]

But, by Theorems 1, 2, we have

\[(1) \quad (\mathbb{A}', 0) = \mathbb{A}' = (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r)(\alpha_0' + \sum_{r=1}^{3} \alpha_r' j_r),\]

\[(2) \quad (0, \mathbb{B}'') = (\sum_{r=0}^{3} \beta_r k_r)(\sum_{s=0}^{3} \beta_s' k_s),\]

and also by Lemmas 1, 2, we have

\[(3) \quad (0, \mathbb{A}')(0, \mathbb{A}''') = 0, (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r)(\beta_0' + \sum_{s=1}^{3} \beta_s' j_s)\]

\[= (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r)(\sum_{s=0}^{3} \beta_s' k_s),\]

\[(4) \quad (0, \mathbb{B}') = 0, (\beta_0 + \sum_{r=1}^{3} \beta_r j_r)(\alpha_0' + \sum_{s=1}^{3} \alpha_s' j_s)\]

\[= (\beta_0 + \sum_{r=1}^{3} \beta_r j_r)(\sum_{s=0}^{3} \alpha_s' k_s)\]

\[= (\sum_{r=0}^{3} \beta_r k_r)(\alpha_0' + \sum_{s=1}^{3} \alpha_s' k_s).\]

\[\therefore \quad (\mathbb{A}, \mathbb{B})(\mathbb{A}', \mathbb{B}') = (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r)(\alpha_0' + \sum_{s=1}^{3} \alpha_s' j_s) + (\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r)(\sum_{s=0}^{3} \beta_s' k_s)\]

\[+ (\sum_{r=0}^{3} \beta_r k_r)(\alpha_0' + \sum_{s=1}^{3} \alpha_s' k_s) + (\sum_{r=0}^{3} \beta_r k_r)(\sum_{s=1}^{3} \beta_s' k_s).\]

But this result is the same as that obtained by applying the method of ordinary multiplication to the numbers \((\alpha_0 + \sum_{r=1}^{3} \alpha_r j_r) + (\sum_{r=0}^{3} \beta_r k_r)\) and \((\alpha_0' + \sum_{s=1}^{3} \alpha_s' j_s) + (\sum_{s=1}^{3} \beta_s' k_s)\).

**Theorem 6.** Eight units of our number-system, \(1, j_1, j_2, j_3, k_0, k_1, k_2, k_3\), are linearly independent with respect to complex coefficients.

**Proof.** In the preceding theorems, we have proved the linear independency of the four quaternion units \(1, j_1, j_2, j_3\), and also of the four alternate units \(k_0, k_1, k_2, k_3\).

Now we have to prove the linear independency of them all; namely to prove that \(\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3 + \beta_0 k_0 + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3\) \((\alpha, \beta: \text{complex numbers})\) is equal to zero when and only when \(\alpha\) and \(\beta\) are all zero.
First suppose that the above alternate-octonary number is zero, then we must have a biquaternion \( \alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3 \), which is equal to an alternate number \(- (\beta_0 k_0 + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3)\). But by Def. (A) of our system, any biquaternion \((\mathfrak{A}, \mathfrak{B})\) cannot be equal to an alternate number \((0, 0)\), unless \(\mathfrak{A}\) and \(\mathfrak{B}\) are both zero. Therefore, in order that an alternate-octonary number \((\mathfrak{A}, \mathfrak{B})\) may be zero, it is necessary that its constituent elements \(\mathfrak{A}\) and \(\mathfrak{B}\) are both zero. And again, by the fundamental theorem (B) of biquaternion and Theorem 2 of alternate number, in order that \(\mathfrak{A}\) and \(\mathfrak{B}\) may be both zero, it is necessary that \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) and \(\beta_0, \beta_1, \beta_2, \beta_3\) are all zero. Thus the necessary condition that the number \((\mathfrak{A}, \mathfrak{B}) = \alpha_0 + \sum_{r=1}^{3} \alpha_r j_r + \sum_{s=0}^{3} \beta_s k_s\) may be zero is that all of its coefficients are zero.

Next, that the above condition is sufficient for vanishing of the number \((\mathfrak{A}, \mathfrak{B})\) is clear.

Thus the linear independency of 8 units \(1, j_1, j_2, j_3, k_0, k_1, k_2, k_3\) is established.

Theorem 7. Relations of units.

(a) Eight units of our number-system \(1, j_1, j_2, j_3, k_0, k_1, k_2, k_3\) and those having negative sign and the fundamental number zero form a closed system with respect to the multiplication. For, the result obtained by multiplying any ones of the number-system \(0, \pm 1, \pm j_1, \pm j_2, \pm j_3, \pm k_0, \pm k_1, \pm k_2, \pm k_3\) always belongs to this number-system also.

(b) This closed system may be constructed from the five elements \(0, 1, j_1, j_2, k_0\) as follows.

By the fundamental properties of \(j(j_1 j_2 = j_3, j_1^2 = -1)\), we have

\[ j_1 1 = -1, j_1 j_1 = -j_1, j_1 j_2 = -j_2, j_1 j_3 = -j_3, \]

and thus we have 9 numbers

\[ 0, \pm 1, \pm j_1, \pm j_2, \pm j_3. \]

Next by multiplying these numbers by \(k_0\), we obtain

\[ \pm k_0 1 = \pm k_0, \pm k_0 j_1 = \pm k_1, \pm k_0 j_2 = \pm k_2, \pm k_0 j_3 = \pm k_3. \]

Therefore 17 numbers forming a closed system are generated from 5 numbers.

(c) This closed system has two maximal subsystems \((0, \pm 1, \pm j_1, \pm j_2, \pm j_3)\) and \((0, \pm k_0, \pm k_1, \pm k_2, \pm k_3)\). This former subsystem again has three subsystems which are cyclical ones having the order 4, namely \(j_r, j_r^2, j_r^3, j_r^4 = (\pm 1, \pm j_r), r = 1, 2, 3; \) and moreover a subsystem
The latter subsystem also has 4 subsystems having the order 2, namely, \((0, k_r), r=0, 1, 2, 3\); and moreover a subsystem \((0, \pm k_1^2, \pm k_2^2, \pm k_3^2)\).

**Theorem 8.** Our system of numbers satisfies seven fundamental laws of algebra, but not three of them, which was given by E. V. Huntington as the basis of operations of the ordinary number-system.

1. If \(a\) and \(b\) are elements of the system, then \(a + b\) is likewise an element of the system.
2. The associative law for addition holds good in the system of numbers.
3. The commutative law for addition holds good in the system of numbers.
4. If \(a + x = a + y\), then \(x = y\).
5. If \(\mu x = \mu y\), where \(\mu\) is any positive integer, then \(x = y\).
6. If \(a\) and \(b\) are elements of the system, then \(a \times b\) is likewise an element of the system.
7. The distributive law holds good in the system of numbers.

That the laws 1, 2, 3, 4, 5, 6 are all true in our system may be seen at once. Therefore we shall prove the distributive law only.

In the first place, we have

\[
(1) \quad (A, B)(A', B') + (A'', B'')
\]

\[
= (A, B)(A' + A'', B' + B'') \quad \text{(Pos. I)}
\]

\[
= \{(A' + A''), AB + AB'' + B(A' + A'') + B(B' + B'')\} \quad \text{(Pos. II)}
\]

\[
= \{A + A'', AB + AB'' + B + B''\}
\]

(Property of biquaternion).

And on the other hand, we have

\[
(2) \quad (A, B)(A', B') + (A, B)(A'', B'')
\]

\[
= (A', AB + AB' + AB'') + (A'', AB'' + AB'' + BB'') \quad \text{(Pos. II)}
\]

\[
= \{A' + A', AB + AB + BB' + BB''\} \quad \text{(Pos. I)}
\]

\[
= \{A' + A', AB + AB'' + B + B''\}
\]

(Property of biquaternion).

But, by (viii) of Theorem 2, we have

\[
B(B' + B'') = BB' + BB''
\]

Therefore, from (1) and (2) we have at once

\[
(A, B)(A', B') + (A'', B'') = (A, B)(A', B') + (A, B)(A'', B'').
\]
Similarly, we have

\[ \{ \mathfrak{A}', \mathfrak{B}' \} + (\mathfrak{A}'', \mathfrak{B}'') (\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}', \mathfrak{B}'')(\mathfrak{A}, \mathfrak{B}) + (\mathfrak{A}'', \mathfrak{B}'')(\mathfrak{A}, \mathfrak{B}). \]

Remark. Both quaternions and alternate numbers do not satisfy the commutative law of multiplication, and moreover alternate numbers do not satisfy the law \(" if \( ax = ay \) \( a \neq 0 \), then \( x = y \) \). To see the validity of the latter proposition, take three alternate numbers \( x, y, a \), satisfying the relation \( y = x + a \), then we have the relation \( ax = ay \), since \( ay = a(x + a) = ax + a^2 = ax \). But in this case, \( y \) is not equal to \( x \) by the definition of equality. Therefore, here we must have the relation \( x \neq y \), even when the relation \( ax = ay \) exists, which clearly contradicts the above law.

From these discussions, it follows at once that our alternate-octonary number-system does not satisfy the above two laws. Next as to the relation of associative law for multiplication and our number-system, we shall discuss it in detail in the following section.

Alternate number and associative law for multiplication.

In Hankel's "Complexe Zahlensysteme," the \( n \) units of alternate numbers and accordingly alternate numbers themselves are defined to satisfy the following relations

\begin{align*}
(i) & \quad k_r^2 = 0, \quad \mathfrak{R}_r^2 = 0, \\
(ii) & \quad k_r k_s = - k_s k_r \neq 0, \quad \mathfrak{R}_r \mathfrak{R}_s = - \mathfrak{R}_s \mathfrak{R}_r \neq 0 \ (r \neq s).
\end{align*}

Moreover, the units and accordingly the alternate numbers themselves are treated as one obeying the associative and distributive laws, and the units \( k_1, k_2, \ldots, k_n \) are called the units of the first degree, and \( k_1 k_2, k_1 k_3, \ldots, k_2 k_3, \ldots \) those of the second degree, and \( k_1 k_2 k_3, k_1 k_2 k_4, \ldots \) those of the third degree, and so on. But, in this case, the units of the second degree and those of the higher ones are to be considered as the ones which cannot be expressed by the units of the first degree. In other words, a product of alternate numbers is not an alternate number in general. For example, the products \( k_r k_s \) and \( k_r k_s \) have not a characteristic property \( (\mathfrak{A}_r \mathfrak{A}_s = - \mathfrak{A}_s \mathfrak{A}_r) \) of alternate system as will be seen below.

\[ (k_r k_s)(k_r k_s) = k_r (k_s k_r) k_s \]  
(associative law)
\[ = k_r (- k_s k_r) k_s \]  
(by (ii))
\[ = - (k_r k_s)(k_s k_r) \]  
(associative law)
\[ = - (- k_r k_s)(- k_s k_r) \]  
(by (ii))
Thus the units of the second degree have not the characteristic property (ii) of alternate numbers, so that they are to be classed as numbers different from alternate ones.

In general, the numbers of the second degree of the form

\[(\sum_{r=0}^{3} \alpha_r k_r)(\sum_{s=0}^{3} \beta_s k_s) = \sum_{r,s=0}^{3} \alpha_r \beta_s k_r k_s \quad (r \neq s)\]

are not the alternate ones. For, they satisfy the commutative law for multiplication, while the alternate numbers do not.

\[(\sum_{r,s=0}^{3} \alpha_r \beta_s k_r k_s) (\sum_{r,s=0}^{3} \alpha_r \beta_s k_r k_s) = \sum_{r,s=0}^{3} \alpha_r \beta_s k_r k_s \quad \text{(distributive law)}\]

\[= \sum_{r,s=0}^{3} \alpha_r \beta_s k_r k_s \quad \text{(by the above proof)}\]

\[= (\sum_{r,s=0}^{3} \alpha_r \beta_s k_r k_s) (\sum_{r,s=0}^{3} \alpha_r \beta_s k_r k_s), \quad \text{(distributive law)}\]

Therefore, if we admit the consistency of associative law in the alternate number-system, the product of these numbers cannot be the alternate one in general, so that, in this case, we must sacrifice the following fundamental proposition:

"If \( A \) and \( B \) be any number of a system, then \( A \times B \) is also a number of that system (law of closed system),"

perhaps, a far more fundamental one than associative law, viewed from the standpoint of number-system.

From these considerations, we must conclude that the system of the following five propositions implies a contradiction:

(i) \[ (\sum_{r=0}^{3} \alpha_r k_r)^2 = 0. \]

(ii) \[ (\sum_{r=0}^{3} \alpha_r k_r)(\sum_{s=0}^{3} \beta_s k_s) = - (\sum_{s=0}^{3} \beta_s k_s)(\sum_{r=0}^{3} \alpha_r k_r). \]

(iii) The system of numbers satisfies the associative law for multiplication.

(iv) The system of numbers satisfies the distributive law.

(v) The product of numbers of the system also belongs to that
system of numbers.

Therefore any system of numbers satisfying the propositions (i), (ii), (iv), (v) cannot satisfy the proposition (iii), while any one satisfying the propositions (i), (ii), (iii), (iv) cannot satisfy the proposition (v). That is to say, in this case, we are obliged to sacrifice either associative law or law of closed system.

Now our system of numbers satisfies the propositions (i), (ii), (iv), (v), as was already shown. Therefore necessarily it cannot satisfy the associative law in general. But, in the special case, where the product consists of the factors \((\mathcal{A}_1, 0), (\mathcal{A}_2, 0), \ldots, (\mathcal{A}_n, 0), (\mathcal{A}, \mathcal{B})\) namely where actual alternate-octonary number \((\mathcal{A}, \mathcal{B})\) or pure alternate number \((0, \mathcal{B})\) occurs only once in the product, the associative law holds good always. For, in the product of units \(j\)'s and \(k\)'s, if \(k\) occur only once, then the associative law holds good always.

(b) Alternate-Octonary Number-System of The Second Kind.

As was already shown, though our alternate numbers possess the fundamental property (v), which the ordinary alternate numbers lack, yet, in return, they do not possess the associative property, so that they lack the interesting application. Therefore, here we shall try to construct another alternate-octonary number, whose alternate subsystem has associative property and many applications.

To construct a required system of numbers, we take four new units \(J_0, J_1, J_2, J_3\), instead of the four quaternion units \(1, j_1, j_2, j_3\), and assume that new number \((\mathcal{B} = \beta_0 J_0 + \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3, \beta: \text{complex number})\) and new unit \(J\) obey the same laws as those of biquaternion and its units, in their addition, multiplication; equality, and inequality, except that, in the product of the new numbers, the unit-product \(J_r J_s\) cannot be reduced to the number of the lower degree \(\alpha J_r\) or \(\alpha\), and it remains in the form of product evermore. In other words, in the unit-product of \(J\)'s, we admit the only relations \(J_r J_s = -J_s J_r\) and the associative law \((J_r J_s) J_t = J_r (J_s J_t)\). We shall call

\((1)\) See Hankel: Zahlensysteme p. 103.

\((2)\) As to the product of new number \(\mathcal{B} = \sum_{r=0}^{3} \beta_r J_r\) and biquaternion \(\mathcal{A} = \alpha_0 + \sum_{r=1}^{3} \alpha_r J_r\), we impose the same laws to it, and so it is equivalent to neither biquaternion nor the new number in general. But if there be two or more biquaternion factors in the product, then it may occur the case where it is equivalent to a new number, for example, \(j_r j_r J_s = -J_s\).
this number \( \sum_{r=0}^{3} \beta_r J_r \) a pseudo-biquaternion.

With this pseudo-biquaternion \( B \) and biquaternion \( A \), we again construct a new number \( (A,B) \), and lay down the following definitions and postulates.

**Definitions of equality and inequality are the same as those of the alternate-octonary numbers given before.**

**Postulate (I).** \( (A,B)+(A',B')=(A+A',B+B') \).

**Postulate (II).** \( (A,B) \times (A',B')=(AA',AB'+BA'+BB') \).

Here by \( \overline{BB'} \) is meant the following number

\[
(\alpha) \quad \overline{BB'} = \sum_{r,s=0}^{3} \beta_r \beta_s J_r J_s \quad (r \neq s, r, s=0, 1, 2, 3)
\]

\[
= \sum (\beta_r \beta_s' - \beta_s \beta_r') J_r J_s \quad \text{since } J_r J_s = -J_s J_r,
\]

where \( r, s \) take 6 pairs of values \((0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\).

**Remark 1.** The above underlined product means that it lacks the terms containing the same unit \( J \) twice or more in the product obtained by usual method. This convention is to be held also in the product of \( A, B \) whose factors are more than two. For example,

\[
(\overline{BB'})B'' = (\overline{BB'})B'' = \sum_{r,s,t=0}^{3} (\beta_r \beta_s' \beta_t'' (J_r J_s) J_t
\]

\( (r, s, t \) are different from one another).

\[
(\overline{AB})B'' = \sum_{r,s,t=0}^{3} (\alpha_r \beta_s') \beta_t'' (J_r J_s) J_t
\]

\( (s, t \) are different from each other; and \( j_0 \) is to be taken as 1).

\[
(\overline{BB'})B'' = \sum_{r,s,t=0}^{3} (\beta_r \alpha_s') \beta_t'' (J_r J_s) J_t
\]

\( (r, t \) are different from each other; and \( j_0 \) is to be taken as 1).

**Remark 2.** According to our law of multiplication, the product containing \( B \)'s twice or more as factors is always underlined product; namely, the products such as \( AB''B'' \), \( BA''B'' \) cannot occur in the form as it is, but always in the form \( \overline{AB''B''} \), \( \overline{BA''B''} \).

Here we shall mention certain principal properties of this new number \( (A,B) \).

**Theorem 1.** When \( B=0 \), the new number \( (A,B) \) is equivalent to the biquaternion \( A \).

The proof is the same as that in the case of previous alternate-octonary number.
Theorem 2. When \( \mathfrak{A} = 0 \), the new number \((\mathfrak{A}, \mathfrak{B})\) has all fundamental properties of the ordinary alternate number.

(i) \((0, \mathfrak{B})^2 = 0\).

\[ \therefore (0, \mathfrak{B})(0, \mathfrak{B}) = (0, \mathfrak{BB}). \]  
(Proposition II)

\[ \mathfrak{BB} = \sum 0J_rJ_r = 0. \]  
(Proposition II(a))

\[ \therefore (0, \mathfrak{B})(0, \mathfrak{B}) = (0, 0) = 0. \]  
(Proposition I)

(ii) \((0, \mathfrak{B})(0, \mathfrak{B'}) = -(0, \mathfrak{B'})(0, \mathfrak{B})\).

\[ \therefore (0, \mathfrak{B})(0, \mathfrak{B'}) = (0, \mathfrak{BB'}), \quad (0, \mathfrak{B'})(0, \mathfrak{B}) = (0, \mathfrak{BB}). \]

But, by Postulate (II), we have

\[ \mathfrak{BB'} = -\mathfrak{BB}. \]

\[ \therefore (0, \mathfrak{BB'}) = (0, -\mathfrak{BB}) = (-1, 0)(0, \mathfrak{BB'}) = -(0, \mathfrak{BB'}). \]

\[ \therefore (0, \mathfrak{B})(0, \mathfrak{B'}) = -(0, \mathfrak{B'})(0, \mathfrak{B}). \]

(iii) Put \( K_0 = (0, J_0), \quad K_1 = (0, J_1), \quad K_2 = (0, J_2), \quad K_3 = (0, J_3) \), then, from (i) and (ii), we have

\[ K_r^2 = 0, \quad K_r K_r = -K_r K_r; \]

and moreover

\[ (0, \mathfrak{B}) = (0, \beta_0 J_0 + \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3) = (0, \beta_0 J_0) + (0, \beta_1 J_1) + (0, \beta_2 J_2) + (0, \beta_3 J_3) = (\beta_0, 0)(0, J_0) + (\beta_1, 0)(0, J_1) + (\beta_2, 0)(0, J_2) + (\beta_3, 0)(0, J_3) = \beta_0 K_0 + \beta_1 K_1 + \beta_2 K_2 + \beta_3 K_3, \]

and

\[ (0, \beta J_r J_s) = (0, \beta J_r)(0, J_s) = \beta K_r K_s, \quad r \neq s. \]

(iv) The four units \( K_0, K_1, K_2, K_3 \) are linearly independent.

Since we have assumed that the units \( J_0, J_1, J_2, J_3 \) have the same properties (except one) as quaternion units, they are linearly independent, and so \( K_0, K_1, K_2, K_3 \) are also linearly independent. (See p. 35).

(v) The sum and product of any two numbers \((0, \mathfrak{B})\) and \((0, \mathfrak{B'})\) of our system are equal to those obtained by applying the ordinary addition and multiplication of algebra to the corresponding numbers \( \sum_{r=0}^{3} \beta_r K_r \) and \( \sum_{r=0}^{3} \beta'_r K_r \).

Proof. (a) Addition.

\[ (0, \mathfrak{B}) + (0, \mathfrak{B'}) = (0, \mathfrak{BB'}) = (0, \sum_{r=0}^{3} \beta_r J_r + \sum_{r=0}^{3} \beta'_r J_r) = (0, \sum_{r=0}^{3} (\beta_r + \beta'_r) J_r) = \sum_{r=0}^{3} (\beta_r + \beta'_r) K_r = \sum_{r=0}^{3} \beta_r K_r + \sum_{r=0}^{3} \beta'_r K_r. \]
(b) Multiplication.

\((0, \mathcal{B})(0, \mathcal{B}') = (0, \mathcal{B}'\mathcal{B}) = \{0, \sum_{r=0}^{3} \beta_r \beta_r' K_r K_s\} = \{0, \sum_{r=0}^{3} \beta_r \beta_r' K_r K_s\} + \{0, \sum_{r=0}^{3} \beta_r \beta_r' K_r^2\} = (\sum_{r=0}^{3} \beta_r K_r)(\sum_{r=0}^{3} \beta_r' K_r) = 0\) (since \(K_r^2 = 0\)).

(vi) The number \((0, \mathcal{B})\) obeys the distributive law.
(vii) The number \((0, \mathcal{B})\) obeys the commutative and associative laws for addition.

These may be proved in the same way as in the former system.
(viii) The number \((0, \mathcal{B})\) and especially the unit \(K\) obey the associative law for multiplication.

\[
\{(0, \mathcal{B})(0, \mathcal{B}')\}(0, \mathcal{B}) = (0, \mathcal{B}'\mathcal{B})(0, \mathcal{B}) = \{0, (\mathcal{B}'\mathcal{B})\mathcal{B}\} = \{0, \sum (\beta_r \beta_r') \beta_r''(J_r J_s J_t)\},
\]

and \(\{(0, \mathcal{B})\}(0, \mathcal{B}'\mathcal{B}') = (0, \mathcal{B})(0, \mathcal{B}'\mathcal{B}') = \{0, \mathcal{B}(\mathcal{B}'\mathcal{B}')\} = \{0, \sum (\beta_r \beta_r''\beta_r''')(J_r J_s J_t)\},
\]

where \(r, s, t\) are different from one another.

But \((\beta_r \beta_r') \beta_r''\) is equal to \(\beta_r (\beta_r' \beta_r'')\) by the property of complex number, and \(J_r J_s J_t\) is equal to \(J_s (J_r J_t)\) by the property of \(J\).

\[
\{(0, \mathcal{B})(0, \mathcal{B}')\}(0, \mathcal{B}) = (0, \mathcal{B})(0, \mathcal{B}'\mathcal{B}) = \{0, \mathcal{B}(\mathcal{B}'\mathcal{B})\} = \{0, \sum (\beta_r \beta_r' \beta_r'' \beta_r''')(J_r J_s J_t)\}.
\]

Cor. 1.

\((0, \mathcal{B})(0, \mathcal{B}')(0, \mathcal{B}) = \{0, \sum \beta_r \beta_r' \beta_r'' J_r J_s J_t\} = \sum \beta_r \beta_r' \beta_r'' K_r K_s K_t,\)

where \(r, s, t\) are different from one another.

Similarly\n
\[(0, \mathcal{B})(0, \mathcal{B}') (0, \mathcal{B}')(0, \mathcal{B}'') = \sum \beta_r \beta_r' \beta_r'' \beta_r''' K_r K_s K_t K_u,\]

where \(r, s, t, u\) are different from one another.

Cor. 2. In the product of the numbers of the form \((0, \mathcal{B})\), if the same factor occurs more than once, then the product is zero.

For example, \((0, \mathcal{B})(0, \mathcal{B}')\) is zero. For,
\[
(0, \mathcal{B})(0, \mathcal{B}') = (0, \mathcal{B})\{0, \mathcal{B}'(0, \mathcal{B})\} = (0, \mathcal{B})\{- (0, \mathcal{B})(0, \mathcal{B}')\} = -(0, \mathcal{B})(0, \mathcal{B}') = 0(\mathcal{B}').
\]

\((1)\) Of course, the direct operation gives the same result.
Cor. 3. *Product of more than 4 factors is always zero.*

For, every term of number of more than 4 degree contains more than 4 units, and so the same units must occur in it at least twice. Therefore, every term, and so the number itself must be zero.

Therefore our number \((0, \mathcal{B}) = \alpha_0 K_0 + \alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3\) has all essential properties of ordinary alternate numbers.

1. \((0, \mathcal{B})^2 = 0.\)
2. \((0, \mathcal{B})(0, \mathcal{B'}) = -(0, \mathcal{B'})(0, \mathcal{B}).\)
3. *Associative law for multiplication.*
4. *Associative and commutative laws for addition.*
5. *Distributive law.*

Therefore it may be applied to the solution of linear simultaneous equations, and to the establishment of theorems of determinant(1).

**Theorem 3.** Our number \((\mathcal{A}, \mathcal{B})\) may be expressed as the sum of biquaternion and alternate number.

\[
(\mathcal{A}, \mathcal{B}) = (\mathcal{A}, 0) + (0, \mathcal{B})
\]

\[
= (\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3) + (\beta_0 K_0 + \beta_1 K_1 + \beta_2 K_2 + \beta_3 K_3).
\]

**Theorem 4.** The sum and product of any two numbers \((\mathcal{A}, \mathcal{B})\) and \((\mathcal{A}', \mathcal{B}')\) of our system are equal to those obtained by applying the ordinary addition and multiplication of algebra to the corresponding numbers \(\sum \alpha_r j_r + \sum \beta_r K_r\) and \(\sum \alpha'_r j_r + \sum \beta'_r K_r.\)

**Lemma.**

\[
\{0, (\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)(\beta_0 J_0 + \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3)\}
\]

\[
= (\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)(\beta_0 K_0 + \beta_1 K_1 + \beta_2 K_2 + \beta_3 K_3).
\]

Here right-hand side is to be calculated by the ordinary multiplication-formula.

**Left-hand side** = \(\{0, \sum_{r=0}^{3} \alpha_r j_r, \sum_{r=0}^{3} \beta_r j_r, J_0\}\)

\[
= \sum_{r=0}^{3}(0, \alpha_0 \beta_r, J_r) + \sum_{r=0}^{3}(0, \alpha_r \beta_r, J_r)
\]

\[
= \sum_{r=0}^{3}\{0, (\alpha_0 \beta_r, 0, J_r)\} + \sum_{r=0}^{3}\{0, (\alpha_r \beta_r, J_r, 0)\}
\]

\[
= \sum_{r=0}^{3}(0, \alpha_0 \beta_r, J_r) + \sum_{r=0}^{3}(0, \alpha_r \beta_r, J_r)
\]

(1) For the method of application, see Hankel: Zahlensysteme.
Similarly,
\[
\{0, (\beta_0 J_0 + \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3)(\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3)\}
\]
\[
=(\beta_0 K_0 + \beta_1 K_1 + \beta_2 K_2 + \beta_3 K_3)(\alpha_0 + \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3).
\]

From this lemma, the theorem may be proved as that of Theorem 5 of the previous system.

**Theorem 5.** Our number \((\mathbb{A}, \mathbb{B})\) satisfies the following fundamental laws of operations.

(i) Commutative and associative laws for addition.

(ii) Associative law for multiplication.

(iii) Distributive law.

The proof of (i) and (iii) may be effected as the corresponding ones of the previous system. Therefore we shall prove the second property only. Here we take the case of 3 factors.

\[
((\mathbb{A}, \mathbb{B})(\mathbb{A}', \mathbb{B}')(\mathbb{A}'', \mathbb{B}'')) = (\mathbb{A}, \mathbb{B})((\mathbb{A}'', \mathbb{B}'') + (\mathbb{B}' + \mathbb{A}' + \mathbb{B}'')\mathbb{A}'')
\]
\[
= (\mathbb{A}'', \mathbb{B}'')((\mathbb{A}', \mathbb{B}')\mathbb{A}'')
\]
\[
+ (\mathbb{B}' + \mathbb{A}' + \mathbb{B}'')\mathbb{A}''.
\]

But, by the properties of \(\mathbb{A}'\)'s, \(\mathbb{B}'\)'s and underlined product, we have
\[
\mathbb{B}(\mathbb{A}' + \mathbb{A}' + \mathbb{B}''') = (\mathbb{A}' + \mathbb{A}' + \mathbb{B}''')\mathbb{B}
\]
\[
= (\mathbb{A}' + \mathbb{A}' + \mathbb{B}''')\mathbb{B}
\]
\[
+ (\mathbb{B}' + \mathbb{A}' + \mathbb{B}'')\mathbb{A}''.
\]

\[
A(\mathbb{A}' + \mathbb{A}' + \mathbb{B}''') + B(\mathbb{A}' + \mathbb{A}' + \mathbb{B}''')
\]
\[
= A(\mathbb{A}' + \mathbb{A}' + \mathbb{B}''') + B(\mathbb{A}' + \mathbb{A}' + \mathbb{B}''')
\]
\[
+ (\mathbb{B}' + \mathbb{A}' + \mathbb{B}'')\mathbb{A}''.
\]

\[
= A(\mathbb{A}' + \mathbb{A}' + \mathbb{B}''') + B(\mathbb{A}' + \mathbb{A}' + \mathbb{B}''')
\]
\[
+ (\mathbb{B}' + \mathbb{A}' + \mathbb{B}'')\mathbb{A}''.
\]

\[
= (\mathbb{A}' + \mathbb{A}' + \mathbb{B}''') + (\mathbb{B}' + \mathbb{A}' + \mathbb{B}'')\mathbb{A}''.
\]

\[
= (\mathbb{A}' + \mathbb{A}' + \mathbb{B}''') + (\mathbb{B}' + \mathbb{A}' + \mathbb{B}'')\mathbb{A}''.
\]

**Remark.** A relation between the quaternion and alternate numbers.

In the multiplication of quaternions, if we abandon the law that the product of them is also a quaternion \((j, j^2 = \pm j, j^3 = -1)\), and, in
return, maintain the relation \( j_r j_s = -j_s j_r \) without exception (even when \( r = s \)), then we have an alternate system of numbers satisfying the relations

\[
\begin{align*}
j_r j_s &= -j_s j_r \quad (r \neq s), \\
j_r j_r &= -j_r j_r \quad \text{or} \quad j_r^2 = 0.
\end{align*}
\]