A Class of Solutions of Einstein's Gravitational
Equations in Continuous Matter,

by

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I. Introduction.

§1. The gravitational equations of Einstein in the most general
form are

\[ k_{pq} - \frac{1}{2} g_{pq} k + \beta g_{pq} = -8\pi T_{pq}, \]  

where

\[ k_{pq} = \text{the contracted Riemann-Christoffel tensor} \]

and

\[ T_{pq} = \text{the material-energy-tensor}. \]

For the radially symmetric field, the metric of space-time may
be taken as

\[ ds^2 = -e^{-\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 + e^\mu dt^2, \]

where \((r, \theta, \varphi, t)\) are the coordinates and the velocity of light is taken
as unity. \(\lambda\) and \(\mu\) are functions of \(r\) only. We then have \(g_{pq} = k_{pq} = T_{pq} = 0\) \((p \neq q)\) and the equations (1.1) corresponding to the
metric (1.2) are

\[ -8\pi \left( T_{11}^{1} + \frac{\beta}{8\pi} \right) = e^{-\lambda} \left( \frac{\nu'}{r} - \frac{e^\lambda - 1}{r^2} \right), \]  

\[ -8\pi \left( T_{22}^{2} + \frac{\beta}{8\pi} \right) = e^{-\lambda} \left\{ \frac{1}{2} \nu'' - \frac{1}{2} \lambda' \nu' + \frac{1}{2} \nu'^2 + \frac{1}{2} (\nu' - \lambda) \right\} / r, \]  

\[ T_{3}^{3} = T_{2}^{2}, \]  

\[ -8\pi \left( T_{33}^{4} + \frac{\beta}{8\pi} \right) = e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{e^\lambda - 1}{r^2} \right), \]

the dashes denoting differentiation with respect to \(r\).

§2. The above equations have been solved in a few particular
cases, viz., the field of an isolated particle\(^{(1)}\), the field of empty
space\(^{(2)}\), the field occupied by homogeneous incoherent matter filling
all space\(^{(3)}\), and the field occupied by what Schwarzschild defines

as an incompressible fluid(1). A case of heterogeneous fluid(2) and also a case of an ideal fluid(3) have been discussed by the present writer.

§ 3. The object of the present paper is to investigate certain types of exact solutions of the above equations for fields occupied by matter possessing internal stresses. When the field is symmetrical about the origin, the two transverse stresses are identically equal. This is implied in equation (1.5). The radial and transverse components may be given as functions of $r$, but, in general, a relation between them is sufficient for the determination of the field when $T_4^4$ is given. We assume that $T_3^3$ and $T_4^4$ are connected by a linear relation. It is the simplest relation between the two components of stress and probably the only one which leads to exact solutions of the above equations.

II. Solutions of the First Type.

§ 4. In accordance with the remarks of Art. 3, we assume that $T_1^1$ and $T_2^2$ are connected by the relation

$$T_2^2 = mT_1^1 + n.$$  

(4.1)

Eliminating $T_2^2$ and $T_1^1$ from (1.3), (1.4) and (4.1), we get

$$e^{-\lambda} \left\{ \frac{1}{r} \nu' + \frac{1}{2} \lambda' \nu' + \frac{1}{2} v'^2 + \frac{1}{2} (\nu' - \lambda')/r \right\}$$

$$= me^{-\lambda} \left( \frac{\nu'}{r} - \frac{e^\lambda - 1}{r^2} \right) + (m - 1)\beta - 8\pi n$$

or

$$\frac{1}{2} \nu'' - \frac{1}{2} \lambda' \nu' + \frac{1}{2} v'^2 + \frac{1}{2} (\nu' - \lambda')/r - m \left( \nu'/r - \frac{e^\lambda - 1}{r^2} \right) - Ne^\lambda = 0,$$  

(4.2)

where

$$N = (m - 1)\beta - 8\pi n.$$  

(4.3)

From (1.6), we get

$$8\pi T_4^4 = e^{-\lambda} \left( \frac{\lambda'}{r} + \frac{e^\lambda - 1}{r^2} \right) - \beta.$$  

(4.4)

§ 5. The equation (4.2) is satisfied if the following two equations are satisfied:

$$- \frac{1}{2} \frac{\lambda'}{r} + \frac{m(e^\lambda - 1)}{r^2} - Ne^\lambda = 0,$$  

(5.1)
\[ \frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 + \nu' \left( \frac{1}{2r} - \frac{m}{r} - \frac{1}{4} \lambda' \right) = 0. \] (5.2)

From (5.1), we have
\[ e^{-\lambda} \cdot \lambda' - \frac{2m}{r} (1 - e^{-\lambda}) + 2N r = 0, \]
or, putting
\[ 1 - e^{-\lambda} = z, \]
\[ z' - \frac{2m}{r} z + 2N r = 0, \]
whence
\[ z = \frac{N}{m-1} r^2 + Ar^{2m}, \]
where \( A \) is a constant of integration.

Hence
\[ e^\lambda = \frac{1}{1 - \frac{N}{m-1} r^2 - Ar^{2m}}. \] (5.3)

From (5.2), we get
\[ e^{\frac{1}{2} \nu'} \left( \frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 \right) + \frac{1}{2} \nu' \cdot e^{\frac{1}{2} \nu'} \left( 1 - \frac{2m}{r} - \frac{1}{2} \lambda' \right) = 0, \]
or, putting
\[ e^{\frac{1}{2} \nu'} = u, \]
\[ u'' + u' \left( 1 - \frac{2m}{r} - \frac{1}{2} \lambda' \right) = 0, \]
whence
\[ u' = B e^{\frac{1}{2} \lambda'} r^{2m-1}, \]
where \( B \) is a constant, and therefore
\[ e^\nu = u^2 = \left[ C + \frac{\int B r^{2m-1} \, d\tau}{\left( 1 - \frac{N}{m-1} r^2 + Ar^{2m} \right)^{\frac{1}{2}}} \right]^2. \] (5.4)

### III. Solution of the Second Type.

§ 6. A second class of solutions may be obtained in the following manner. The equation (4.2) is satisfied if the following two are satisfied:

\[ \frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 + \nu' \left( \frac{1}{2r} - \frac{m}{r} \right) = 0, \] (6.1)

\[ -\lambda' \left( \frac{1}{4} \nu' + \frac{1}{2r} \right) + \frac{m(e^\lambda - 1)}{r^2} - N e^\lambda = 0. \] (6.2)

Let \( z = e^{\frac{1}{2} \nu} \). We have \( z' = \frac{1}{2} e^{\frac{1}{2} \nu} \cdot \nu' \), and \( z'' = e^{\frac{1}{2} \nu} \left( \frac{1}{2} \nu'' + \frac{1}{2} \nu'^2 \right) \), so that the equation (6.1) becomes
\[ z'' + z' \left( \frac{1 - 2m}{r} \right) = 0, \]
whence
\[ z = C + Br^{2m} \]

and
\[ e^r = (C + Br^{2m})^2. \]  

(6.3)

Substituting \( r' = \frac{4mBr^{2m-1}}{C + Br^{2m}} \) in (6.2), we get

\[ -e^{-\lambda} \left\{ \frac{mBr^{2m-1}}{C + Br^{2m}} + \frac{1}{2r} \right\} + m(1 - e^{-\lambda}) - N = 0. \]

Putting \( 1 - e^{-\lambda} = y \), the equation becomes

\[ y' - Py + Q = 0, \]  

(6.4)

where

\[ P = 2m \left\{ \frac{1}{r} - \frac{2mBr^{2m-1}}{C + (2m + 1)Br^{2m}} \right\}, \]

\[ Q = \frac{2Nr(C + Br^{2m})}{C + (2m + 1)Br^{2m}}. \]

The solution of (6.4) is obviously

\[ y = e^{-\int Pdr} \left[ -\int Qe^{-\int Pdr} + H \right], \]

where we have

\[ e^{-\int Pdr} = \frac{1}{r^{2m}} \left\{ C + (2m + 1)Br^{2m} \right\}^{2m+1}. \]

Thus,

\[ y = \frac{r^{2m}}{C + (2m + 1)Br^{2m}} \left[ H - \int_{r^{2m-1}}^{2N(C + Br^{2m})dr} \frac{1}{C + (2m + 1)Br^{2m}} \right], \]

whence

\[ e^{-\lambda} = 1 - \frac{r^{2m}}{C + (2m + 1)Br^{2m}} \left[ H - \int_{r^{2m-1}}^{2N + (CBr^{2m})dr} \frac{1}{C + (2m + 1)Br^{2m}} \right]. \]

(6.5)

In the important case, when \( T_{22}^2 \) and \( T_{11}^1 \) are proportional, i.e. \( n=0 \) and the natural curvature of space is negligible, i.e. \( \beta=0, N=0 \), the integral (6.5) is always exact.

**IV. Remarks.**

§7. In addition to various particular cases of the above sets of solutions, viz., (5.3), (5.4) and (6.3), (6.5), the solutions of Schwarzschild, Einstein and De Sitter, referred to in the foot-notes may also be deduced as special cases of the above solutions.
The two solutions obtained by the present writer in the papers already mentioned are also particular cases of the above solutions. It may be expected that the radially symmetrical solutions corresponding to other physical problems are also included in the above solutions.

§ 8. Einstein proposed(1) as the equations in an electromagnetic field the following instead of (1.1)—

$$k_{\nu \rho} - \frac{1}{2} g_{\nu \rho} k = - 8\pi T_{\nu \rho},$$

(8.1)

which, on transvection with $g^{\rho \sigma}$, gives $T = 0$, so that the scalar of the electromagnetic-energy-tensor vanishes in such a field. Assuming $T^2_{\nu} = MT_{\nu} + N$, the solutions of (8.1) corresponding to the two types are as follows:

**First Types.**

$$e^\lambda = \frac{1}{1 - \frac{n}{2m-1} r^2 - Ar^m},$$

$$e^\nu = \left[ C + \int \frac{B r^{2m} dr}{\sqrt{1 - \frac{n}{2m-1} r^2 - Ar^m}} \right]^2.$$

**Second Types.**

$$e^\nu = (C + Br^{2m+1})^2,$$

$$e^\lambda = 1 - r^m \left\{ 2mC + (4m+1)Br^{2m+1} \right\}^{\frac{1}{s}}$$

$$\times \left[ H + \int 2n (C + Br^{2m+1}) r^{1 - \frac{1}{m}} \left\{ 2mC + (4m+1)Br^{2m+1} \right\}^{\frac{1}{s}} dr \right].$$

In the above expressions, $m = \frac{M}{M+1}$, $n = \frac{16\pi N}{M+1}$ and $s = 4m+1$.

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(1) Mathematische Annalen, Dec., 1926.