Some Inequalities in the Theory of Functions
of a Complex Variable,

by
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Let \( f(z) \) be regular in \( |z| < 1 \) and limited, that is, there is a constant \( M \) such that

\[
|f(z)| \leq M \quad (|z| < 1).
\]

Let the Taylor's expansion of \( f(z) \) be

\[
a_0 + a_1z + a_2z^2 + \ldots + a_nz^n + \ldots,
\]

then the inequality

\[
M^2 \geq |a_0|^2 + |a_1|^2 + \ldots + |a_n|^2 + \ldots.
\]

is well known and is used frequently. The object of this paper is to extend this inequality.

1. Suppose that

\[
|f(z)| < 1 \quad (0 \leq |z| < 1)
\]

and let

\[
f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \ldots = f(0) + z f_1(z).
\]

By the Carathéodory's theorem\(^{(1)}\), we have

\[
|f_1(z)| \leq \frac{1 - |f(0)|^2}{1 - |f(0)| \cdot |z|} < 1 + |f(0)| \quad (0 < |z| < 1)
\]

and further

\[
|f_1(z) - f_1(0)| \leq |z| \frac{(1 + |f(0)|^2) - |f_1(0)|^2}{1 + |f(0)| - |z| \cdot |f_1(0)|} \quad (0 < |z| < 1),
\]

that is,

\[
|f(z) - f(0) - \frac{f'(0)}{1!} z| \leq |z|^2 \frac{(1 + |f(0)|^2) - |f_1(0)|^2}{1 + |f(0)| - |z| \cdot |f_1(0)|} \quad (0 < |z| < 1).
\]

Thus, in general, we have

\(^{(1)}\) Pólya u. Szegő: Aufgaben und Lehrsätze I (1926).
In the narrower domain we get better inequalities. Let \(0<|z|<|f(0)|\) \((<1)\), then we have

\[
|f(z) - f(0) - f'(0)z| \leq |z|^2 \frac{41 - |f'(0)|^2}{21 - |z| \cdot |f'(0)|}.
\]

In general we have

\[
|f(z) - \sum_{r=0}^{n} \frac{f^{(r)}(0)}{r!} z^r| \leq |z|^{n+1} \frac{\sqrt{1 - \frac{|f^{(n)}(0)|}{n!}}^2}{\sqrt{1 - |z| \cdot \frac{|f^{(n)}(0)|}{n!}}}.
\]

in the domain

\(0<|z|<\text{Min} \left( |f(0)|, \frac{|f'(0)|}{1!}, \ldots, \frac{|f^{(n)}(0)|}{n!} \right) < 1\)

2. Carathéodory's theorem gives, putting \(z=0\)

\[|f'(0)| \leq 1 - |f(0)|^2.\]

Instead of (1), we will take

\[|f_1(z)| < \frac{1 - |f(0)|^2}{1 - \varepsilon |f(0)|} \quad (0<|z|<\varepsilon<1).\]

Then

\[
|f_1(z) - f'(0)| \leq |z| \left\{ \left( \frac{1 - |f(0)|^2}{1 - \varepsilon |f(0)|} \right)^2 - |f'(0)|^2 \right\}
\]

\[= \left\{ \frac{1 - |f(0)|^2}{1 - \varepsilon |f(0)|} - |z| \cdot |f'(0)| \right\}.
\]

Letting \(\varepsilon\) and then \(z\) tend to zero, we have

\[
\frac{1}{2!} |f''(0)| \leq 1 - |f(0)|^2 - \frac{|f'(0)|^2}{1 - |f(0)|^2}.
\]

Similarly

\[
\frac{1}{3!} |f'''(0)| \leq 1 - |f(0)|^2 - \frac{|f'(0)|^2}{1 - |f(0)|^2} - \frac{|f''(0)|^2 / 2!}{1 - |f(0)|^2 - \frac{|f'(0)|^2}{1 - |f(0)|^2}}.
\]
Thus we can calculate the magnitude of \(|f^{(n)}(x)|\).

3. Let

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \]

and

\[ |f(z)| \leq M \quad (|z| \leq 1). \]

Then we have

(2) \[ 0 \leq |a_n| \leq M - \frac{|a_0|^2}{M} - \frac{|a_1|^2}{M} - \cdots, \]

that is,

(2) \[ 0 \leq M - \frac{1}{M} (|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 + \cdots), \]

that is,

(3) \[ M^2 \geq |a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 + \cdots. \]

This is the well known inequality. (2) gives us the more general inequalities:

\[ 0 \leq M - \frac{|a_0|^2}{M} - \frac{1}{M} (|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 + \cdots), \]

that is,

(3) \[ \left( M - \frac{|a_0|^2}{M} \right)^2 \geq |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 + \cdots, \]

This is better than (3), for

\[ M^2 - |a_0|^2 \geq \left( M - \frac{|a_0|}{M} \right)^2. \]

Quite similarly we have

\[ \left( M - \frac{|a_0|^2}{M} - \frac{|a_1|^2}{M} \right)^2 \geq |a_2|^2 + |a_3|^2 + \cdots, \]

and

\[ \left( M - \frac{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}{M} \right)^2 \geq |a_{n+1}|^2 + |a_{n+2}|^2 + \cdots. \]