A New Method of Finding Moments of Moments,

by

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I. Introduction.

1. The Origin of the Problem.
The usefulness of the method of moments, as a method of curve fitting as well as a means of interpolation, is already acknowledged by authorities in statistics and this method is now practical to use.

Especially with the introduction of the theory of sampling, an important theoretical subject in statistics, the moments of moments have also become necessary, of which some simpler and important formulae have already been deduced by some specialists.

But the recent advancement in the theory of sampling has necessitated much higher moments of moments and yet their deduction suddenly becomes more complicated and difficult as they are of a little higher order.

Hence the deduction of higher moments of moments itself is now a study of importance.
For example, when, in my work(1), I tried to deduce the first four moments of sampling distribution of standard deviations, I found the moments $2M_n$ for the variance to be necessary, not only $2M_1'$ and $2M_n$ ($n=2,3,4$) which had been deduced by Tchouproff, but also $2M_n$ ($n=5,6,7,8$).

Furthermore as, at that time, $2M_n$ ($n=5,6,7,8$) were unknown, I was confronted by an obstacle in the way of study; so I was working at their deduction when Mr. A. Fisher, F.R.S. published his study. By the aid of Fisher's formula for the so-called $K(2^a)$-function and of the theory of semi-invariants, I was able to deduce all moments $2M_n$, which were necessary for my study at that time.

This fact was the very motive of this study of mine and I obtained a new method of finding moments of moments in sampling.

2. Brief Outline of the Study of this Problem.

The first person to attack this problem successfully was the Russian scholar Tchouproff(2). His theory has a very general nature, but his principle and notations are both too complicated to be of practical use; and his successors are limited to a very small number of Russian scholars. As an application of his method, he himself deduced the first four moments of the sampling distribution of $\mu_2$; $2M_1'$; $2M_2$, $2M_3$ and $2M_4$, of which the formula for $2M_4$ was found to be an error by Dr. Church in 1926 and was corrected by him(3).

This fact sufficiently endorses the criticism I mentioned above.

Church's method of deduction, as the "Student"'s method(4), has no special device and depends almost upon inspection. Therefore it is not exaggerating to say that it is almost impossible to deduce, by his method, higher moments of moments.

Recently two studies of this kind have been published: one by Dr. C.C. Craig, an American,(5) the other by Fisher, an Englishman(6).

Craig deduced moments of moments by his theory of semi-

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invariants but moments deduced really by him are not of so much higher order and it is doubtful, if his method is sufficiently practical.

Fisher introduced some new functions, the so-called $k$-functions; with the aid of these and semi-invariants, Fisher found a new method of deduction. His method, I think, is more practical than any other and he himself deduced many new formulae for higher moments of moments, but so far as he has published, his method lacks in generality.

The so-called "Symbolical Figures" in his method of deduction are found by mere inspection and its usefulness differs according to its users which is a drawback.

II. Symbolical Equations.

3. Fundamental Conditions and Relations.

At the starting-point of the study of this problem, there are two fundamental conditions which we have first to consider:

(a) Is the sampled population finite or infinite?

(b) Do the observed variables consist of one or more than one? If the population is infinite, any observed value may be considered independent of the others. The second question is the branching-point whether we have to discuss frequency curve or correlated surfaces. In this paper, it is assumed that (1) the sampled population is infinite and (2) the observed variable is only one.

Now let $x_1, x_2, \ldots, x_N$ be the observed values of a variable $x$ in the order of observations, $N$ being the sample size, and let us, as usual, use the following notations,

\[ s_r = x_1^r + x_2^r + \ldots + x_N^r : \text{symmetric functions of order } r, \]

\[ \mu_r = \frac{1}{N} s_r : \text{r th moment about a fixed origin}, \]

\[ \mu_r : \text{r th moment about the mean}, \]

\[ mM_n' : \text{n th moment of the sampling distribution of } \mu_n, \text{about a fixed origin}, \]

\[ mM_n : \text{similar function about the mean}, \]

then $\mu_n'$ and $\mu_n$ are defined by the equations,

\[ \mu_n' = \int_{-\infty}^{\infty} \phi(x) x^m dx, \quad \mu_n = \int_{-\infty}^{\infty} \phi(x)(x-\bar{x})^m dx, \]

where $\phi(x)$ is the probability function of the variate $x$ and $\bar{x}$ the mean of $x$.

And we can easily show that
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\[
\mu_m = \int_{-\infty}^{\infty} \phi(x)(x-\bar{x})^m \, dx
\]
\[
= \sum_{t=0}^{m} (-1)^t \binom{m}{t} \mu_{m-t} \left( \mu_t \right)^t
\]
\[
= \sum_{t=0}^{m} (-1)^t \binom{m}{t} \frac{1}{\mu_{m-t}}.
\]

Consequently we have

\[
mM'_n = \text{mean of } \mu_{m^n}
\]
\[
= \text{mean} \left\{ \sum_{t=0}^{m} (-1)^t \binom{m}{t} \frac{1}{\mu_{m-t}} \right\}.
\]

and also

\[
mM_n = mM'_n - \sum_{t=1}^{n-2} \left( \binom{n}{t} \right) mM_{n-t} (mM'_t)^t - (mM'_t)^n.
\]

Therefore, in the case of single variable, the problem of finding moments of moments reduces to the deduction of the mean of

\[
\pi(A)
\]

which I propose to call "Product \(\pi(A)\)" hereafter.

Now, since all \(s_t\)'s in \(\pi(A)\) are homogeneous expressions of \(x\)'s and of order \(p\), if we expand \(\pi(A)\) in terms of \(x\)'s, we formally have

\[
s_t^{h} = \sum \{C_{\lambda}^{n_1,n_2,n_3,...,n_\lambda} x_1^{n_1} x_2^{n_2} ... x_\lambda^{n_\lambda}\},
\]

where evidently \(n_1 + n_2 + ... + n_\lambda = p_1 l_1 + p_2 l_2 + ... + p_\lambda l_\lambda\) and \(x_i\)'s are a combination of \(x\)'s out of \(x_1, x_2, ..., x_N\).

Moreover I wish to denote, by the suffixes \(\alpha_1, \alpha_2, ..., \alpha_\lambda\), the numbers of \(p\)'s in the indices \(n_1, n_2, ..., n_\lambda\) respectively.

To examine the index \(n\) more in detail, let us take the following product, which I wish to call \(\pi(\tau)\), instead of \(\pi(A)\) in the equation (4),

\[
s_q = s_{q_1} s_{q_2} ... s_{q_s}; \quad \tau = l_1 + l_2 + ... + l_\lambda
\]

then it is evident that

\[
n_\alpha = q_{i_1} + q_{i_2} + ... + q_{i_\alpha}
\]

provided \(q_i\)'s are \(\alpha q_i\)'s taken out of \(q_1, q_2, ..., q_s\).

But \(\pi(A)\) can be considered as a special case of \(\pi(\tau)\), where

\[
q_1 = q_2 = ... = q_N = p_1,
\]
Therefore, from the equation (5), we can easily show that $n_a$ in the equation (4) is given by the equation

$$n_a = p_1 t' + p_2 t' + \ldots,$$

where

$$t' + t' + \ldots = \alpha; \quad t' \leq t_i, \quad (i = 1, 2, 3 \ldots).$$

Now, since the sampled population is infinite, we have

$$\text{mean} (x_1^{n_1} x_2^{n_2} \ldots x_{\lambda}^{n_{\lambda}}) = \text{mean} (x_1^{n_1}) \text{mean} (x_2^{n_2}) \ldots \text{mean} (x_{\lambda}^{n_{\lambda}}) = \bar{x}_{n_1} \bar{x}_{n_2} \ldots \bar{x}_{n_{\lambda}},$$

where $\bar{x}_m$ denotes the $m$th moment of the parent distribution.

Therefore, for the case of infinite population and of single variable, we have

$$\text{mean} (s_1^{t_1} s_2^{t_2} \ldots s_{\mu}^{t_{\mu}}) = \sum C \bar{x}_{n_1} \bar{x}_{n_2} \ldots \bar{x}_{n_{\lambda}}$$

and if we could find a general expression for the coefficient $C$ and any concrete direction for the summation $\Sigma$, we should be able to deduce $\mu M$ without much difficulty.

But the expression for $C$ is not yet known and the summation $\Sigma$ in (6) is so complex that we can not treat it unless $v$s and $t$ are very small integers.

4. Symbolical expansion of $\pi(\tau)$.

Now let us expand $\pi(\tau)$ or $s_1 s_2 \ldots s_{\lambda}$, then, as discussed before, it is expressed as an algebraic sum of terms of the form

$$x_{i_1}^{n_1} x_{i_2}^{n_2} \ldots x_{i_{\lambda}}^{n_{\lambda}}, \quad \text{Term (B)}$$

where

$$n_1 + n_2 + \ldots + n_\lambda = q_1 + q_2 + \ldots + q_\tau,$$

and

$$\alpha_1 + \alpha_2 + \ldots + \alpha_\lambda = \alpha.$$

Here the power $x_{i_1}^{n_1}$, for example, shows that, to obtain this term, we have to select the same $x_{i_1}, \alpha_1$ times out of $\tau$ factors in $\pi(\tau)$.

Similarly $\alpha_2, \alpha_3, \ldots, \alpha_\lambda$ show us that we have to select the same $x_{i_2}, \alpha_2$ times out of a set of $s$-factors,
the same $x_i$ times out of another set of $s$-factors,

the same $x_{i_i}$ times out of the last set of $\alpha$ factors.

To see from another point of view, the set of integers

$$\alpha_1, \alpha_2, \ldots, \alpha_{\lambda},$$

where

$$\alpha_1 + \alpha_2 + \ldots + \alpha_{\lambda} = \tau; \quad \alpha_i \geq \alpha_{i+1}, \quad (i = 1, 2, \ldots),$$

is nothing more than a partition of the integer $\tau$ and this partition of $\tau$ shows an important peculiarity of the term (B) in the expansion of $\pi(\tau)$.

Thus we can classify all terms in the expansion of $\pi(\tau)$ by the way of selecting $s$-factors, or by the partition $\alpha_1, \alpha_2, \ldots, \alpha_{\lambda}$ of the integer $\tau$ and I wish to define the partition as the "type" of the term (B) and $\lambda$ as the "order" of this type.

Now let us denote by a symbol $[\alpha_1, \alpha_2, \ldots, \alpha_{\lambda}]$ the sum of this type; then we can expand $\pi(\tau)$ symbolically as follows:

$$\pi(\tau) = [\tau] + [\tau - 1, 1] + [\tau - 2, 2] + [\tau - 2, 1, 1] + \ldots + [1, 1, \ldots, 1]$$

or

$$\pi(\tau) = s_1s_2\ldots s_\tau = \sum_{i=1}^{\tau} [\alpha_1, \alpha_2, \ldots, \alpha_{\lambda}],$$

which I wish to call the "first form of symbolical equation" hereafter.

**Ex. (1).** Expand $s_1s_2s_3$ symbolically.

(Sol.) Here $\tau = 3$ and $\pi(3) = s_1s_2s_3$

\[
\begin{array}{ccc}
(x_1) & (x_2) & (x_3) \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{align*}
\pi(3) &= \sum_{i=1}^{3} x_i^1 \sum_{i=1}^{3} x_i^2 \sum_{i=1}^{3} x_i^3 \\
&= (\sum_{i=1}^{3} x_i^1)(\sum_{i=1}^{3} x_i^2)(\sum_{i=1}^{3} x_i^3) \\
&= (\text{terms of type } x_1^1x_2^1x_3^1) \\
&\quad + (\text{terms of type } x_1^2x_2^2x_3^2) \\
&\quad + (\text{terms of type } x_1^3x_2^3x_3^3).
\end{align*}
\]

Therefore

$$\pi(3) = s_1s_2s_3 = [3] + [2, 1] + [1, 1, 1].$$

5. **Symbolical Expansion of $\pi(A)$**.

In the second place, let us consider the product $\pi(A)$ or $s_{p_1}s_{q_1}\ldots s_{p_l}$, where $l_1 + l_2 + \ldots + l_i = \tau$. 

Now since we can distinguish the same \( s_p \) in \( s^l_p \) by arranging them in the order of factors as follows

\[
\begin{align*}
\frac{s^l_p}{l} &= s_{p_1} . s_{p_1} . s_{p_1} . . . s_{p_1} \\
\end{align*}
\]

without any effect on the results, the equation (7) applies to \( \pi(A) \) as well as to \( \pi(\tau) \) and we have

\[
(7')
\]

where

\[
\alpha_1 + \alpha_2 + \ldots + \alpha_\lambda = \tau \quad \text{and} \quad \alpha_i \geq \alpha_{i+1}, \quad (i = 1, 2, 3, \ldots).
\]

6. New Definition of Like Terms.

If we examine the terms of \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]\) more in detail, it is evident that the terms of the same type are not necessarily identical. They may differ by the values of the indices \( n_a \)'s and also by the suffixes \( i \) of \( x_i \).

For instance, in \([2, 1]\) for \( \pi(3) \), there are three ways of combination of \( q \)'s in the indices, namely

\[
x_{i_1}^{q_1+q_2} x_{i_2}^{q_3} \quad x_{i_1}^{q_1+q_4} x_{i_2}^{q_2} \quad \text{and} \quad x_{i_1}^{q_1+q_3} x_{i_2}^{q_4},
\]

which give us, at the end, three different means as follows:

\[
\begin{align*}
\tilde{p}_1 q_{1+q_2} \tilde{p}_{q_3}, \quad \tilde{p}_{q_1+q_2} \tilde{p}_{q_3}, \quad \text{and} \quad \tilde{p}_{q_2+q_3} \tilde{p}_{q_1}.
\end{align*}
\]

Such difference of terms comes from the combination of \( q \)'s in the product \( \pi(\tau) \) or of \( p \)'s and \( l \)'s for the product \( \pi(A) \).

Terms with the same set of indices, of the same type, may also differ by the suffixes of \( x_i \)'s.

For instance the suffixes \( i \) and \( j \) in the term \( x_{i_1}^{q_1+q_2} x_{j_1}^{q_4} \) of \( \pi(3) \) may be any permutation of two integers out of 1, 2, 3, \ldots, \( N \) and there are \( N(N-1) \) such terms in this case.

Generally, in the expansion of \( \pi(A) \) or \( \pi(\tau) \), with a definite set of indices \( n_{a_1}, n_{a_2}, \ldots, n_{a_\lambda} \), there are \( Np_\lambda \) terms of the form

\[
x_{i_1}^{n_{a_1}} x_{i_2}^{n_{a_2}} \ldots x_{i_\lambda}^{n_{a_\lambda}};
\]

\[
np_\lambda = N(N-1)(N-2)(N-3)\ldots(N-\lambda+1), \quad (8)
\]

which, for simplicity, I wish to denote by a symbol \( N_\lambda \) hereafter.

Now, since mean \( (x_i^a x_i^b) = \tilde{p}_2 \tilde{p}_3 = \text{mean} (x_i^a x_i^b) = \text{mean} (x_i^a x_i^b) \) for an infinite population, it is convenient, in the theory of moments, to treat those terms alike which give us at the end the same mean.

Therefore I propose, in this paper, to say “two terms are alike when they are of the same type and their indices are the same com-
bination of $q$'s for $\pi(\tau)$, or of $q$'s and $l$'s for the product $\pi(A)$.

By this new definition of "like terms", we can say that for any term of $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$, there are $N_\lambda$ like terms in $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$.

**Note.** Like terms always give us the same mean at the end, but we must be careful to note that the inverse is not necessarily true.

**7. Second Form of Symbolical Equations.**

Let us examine $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$ again, especially with regard to the indices $n_\alpha$'s.

By the new definition of "like term", the number of different terms in $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$ becomes the number of possible sets of $n_\alpha$'s and the latter, in the case of $\pi(\tau)$, is nothing but the number of possible combinations of $q$'s when they are grouped into $\lambda$ groups, the numbers of $q$'s in groups being $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ respectively.

If we denote this number by $C_{\tau; a_1, a_2, \ldots, a_\lambda}$ or simply by $C$, we can easily show that

$$C_{\tau; a_1, a_2, \ldots, a_\lambda} = \frac{|\tau|}{|\alpha_1| \alpha_2 \ldots |\alpha_\lambda|}$$

provided that all $\alpha$'s are different.

If there are equal $\alpha$'s in $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$, we can, also without much difficulty, show that

$$C_{\tau; a_1, a_2, \ldots, a_\lambda} = \frac{|\tau|}{|\alpha_1| \alpha_2 \ldots |\alpha_\lambda| s_1 s_2 \ldots}$$

where $s$'s are the number of equal $\alpha$'s.

Thus the following theorem is obtained.

**Theorem I.** The total number $M_{\tau; a_1, a_2, \ldots, a_\lambda}$, or $M$ simply, of the terms of $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$ is given by

$$M = N_\lambda \cdot C_{\tau; a_1, a_2, \ldots, a_\lambda}$$

From this theorem we get a more significant symbolical equation than (7) as follows:

$$\pi(\tau) = \sum_{\lambda=1}^{\lambda} M_{\tau; a_1, a_2, \ldots, a_\lambda} [\alpha_1, \alpha_2, \ldots, \alpha_\lambda],$$

where

$$\alpha_1 + \alpha_2 + \ldots + \alpha_\lambda = \tau.$$

I propose to call this equation "the second form of symbolical equation".

**Note (i).** This theory, discussed with regard to $\pi(\tau)$, holds
true, also, of the product \( \pi(A) \) and we have

\[
\gamma_{\alpha_1}^{t_{\alpha_2}} \cdots \gamma_{\alpha_3}^{t_{\alpha_4}} = \sum_{t=1}^{T} M_{t_1, t_2, \ldots, t_{\alpha}} [\alpha_1, \alpha_2, \ldots, \alpha_{\alpha}]
\]

provided

\[
\alpha_1 + \alpha_2 + \ldots + \alpha_{\alpha} = l_1 + l_2 + \ldots + l_3 = \tau.
\]

**Note (ii).** In the equations (11) or (11)', we must pay attention to the symbolical coefficient \( M \). It does not mean to multiply each term of \([\alpha_1, \alpha_2, \ldots, \alpha_{\alpha}]\) by \( M \) as ordinary coefficients mean, but \( M \) means only that the total number of terms in \([\alpha_1, \alpha_2, \ldots, \alpha_{\alpha}]\) is given by \( M \) or \( N_{\alpha} \cdot C \), where \( N_{\alpha} \) is the number of like terms and \( C \) the number of different terms in \([\alpha_1, \alpha_2, \ldots, \alpha_{\alpha}]\).

However, when all \( q \)'s in \( \pi(\tau) \) are 1, the coefficient \( M \) becomes an ordinary coefficient. For instance, from the equation (11), we have

\[
\pi(3) = s_{q_1} s_{q_2} s_{q_3} = N[3] + 3N[2, 1] + N[1, 1, 1]
\]

and from this second form of the symbolical equation, at once, we have

\[
\text{mean } s_1^3 = N \text{ mean } x_1^3 + 3N_2 \text{ mean } (x_1^2 x_2) + N_3 \text{ mean } (x_1 x_2 x_3)
\]

\[
= N \bar{x}_3 + 3N_2 \bar{x}_2 \bar{x}_1 + N_3 \bar{x}_1^3.
\]

But \( \bar{x}_1 = 0 \). And consequently

\[
\text{mean } s_1^3 = N \bar{x}_3.
\]

**8. Some Applications of Symbolical Equations.**

**Ex. (2).** Find the mean \([2, 1]\) for the product \( s_{q_1} s_{q_2} s_{q_3} \).

**(Sol.)** From the equation (10)

\[
M_{3; 2, 1} = N_2 \times \frac{3}{2 \cdot 1} = 3N(N-1)
\]

and three different forms of the terms in \([2, 1]\) are

\[
\gamma_{x_1^2 + q_1} \gamma_{x_2^2}, \ x_1^3 \gamma_{x_2^3} \gamma_{x_2^2} \text{ and } x_1^{2+q_2} \gamma_{x_2^2}^2.
\]

Consequently we have

\[
\text{mean } [2, 1] ; \ s_{q_1} s_{q_2} s_{q_3}
\]

\[
= N_2 \{ \text{mean } x_1^2 + q_2 \gamma_2^2 + \text{mean } x_1^{2+q_2} x_2^2 + \text{mean } x_1^{2+q_2} \}
\]

\[
= N(N-1) \{ \bar{p}_{q_1+q_2} \bar{p}_{q_3} + \bar{p}_{q_1+q_2} \bar{p}_{q_2} + \bar{p}_{q_2+q_3} \bar{p}_{q_1} \}.
\]

Thus if the product \( \pi(A) \) or the product \( \pi(\tau) \) are very simple, we can find their means at once by the aid of symbolic equations, but if \( \pi(A) \) or \( \pi(\tau) \) become a little more complex, the symbolical equations alone are not of much use.
However the equation (11) or the equation (11)' becomes very powerful when they are used with the "Reduction Formulae" in the next chapter.

Now any distribution law of a variate $x$ can be defined by the moment coefficients for this distribution. If we can find the first four moments, or in usual notations, $\mu_1, \mu_2, \mu_3, \mu_4$, we have as a rule enough informations to define the frequency distribution with sufficient accuracy for practical purposes.

From this point of view, with regard to $mM_n$, we can assume $m\leq 4$, without losing the generality of this theory and, consequently, we can assume that $s$ in $\pi(A)$ is $s_1, s_2, s_3$ or $s_4$. Moreover, by the reduction formulae in the next chapter, we can express the mean $[\alpha_1, \alpha_2, \ldots, \alpha_n]$; $s_{p_1}^{\alpha_1} s_{p_2}^{\alpha_2} \ldots s_{p_s}^{\alpha_s}$ in terms of the following means

$\text{mean } [\alpha_1]$; $s_{p_1}^{\alpha_1} s_{p_2}^{\alpha_2} \ldots s_{p_s}^{\alpha_s}$ $\text{mean } [C]$ or

$\text{mean } [\alpha_1, \alpha_2, \ldots, \alpha_n]$; $s_{p_1} s_{p_2} \ldots s_{p_s}$ $\text{mean } [D]$ of which we may assume $s\leq 4$ for the reason mentioned just before.

Therefore it is important and necessary to deduce these means for a foundation of the theory of reduction in the next chapter.

**Ex. (3).** Find the mean $[\alpha_1]$; $s_{p_1}^{\alpha_1} s_{p_2}^{\alpha_2} \ldots s_{p_s}^{\alpha_s}$.

*(Sol.)* $\text{mean } [\alpha_1]$ for the product $s_{p_1}^{\alpha_1} s_{p_2}^{\alpha_2} \ldots s_{p_s}^{\alpha_s}$

or

$\text{mean } N[\alpha_1]$; $s_{p_1}^{\alpha_1} s_{p_2}^{\alpha_2} \ldots s_{p_s}^{\alpha_s}$

$=$ sum of $N$ means of the power $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s}$

$= \text{mean } (x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s})$

$\ldots$ mean $[\alpha_1]$; $s_{p_1}^{\alpha_1} s_{p_2}^{\alpha_2} \ldots s_{p_s}^{\alpha_s} = N\bar{\mu}_{p_1} \bar{\mu}_{p_2} \ldots \bar{\mu}_{p_s}$. \hspace{1cm} (12)

**Ex. (4).** Find the mean $[\alpha_1, \alpha_2, \ldots, \alpha_n]$; $s_{p_1} s_{p_2} \ldots s_{p_s}$ provided $s\leq 4$.

*(Sol.)* If $s=1$ the mean in question becomes mean $[\alpha_1]$; $s_{p_1}$ which is a special case of the mean $[C]$ and we have only the following three cases to discuss

$s=2$, $s=3$ and $s=4$.

When $s=2$, $[1, 1]$; $s_{p_1} s_{p_2}$ is the only possible case and we have

$\text{mean } [1, 1]$; $s_{p_1} s_{p_2} = N_2 \text{mean } x_1^{\alpha_1} x_2^{\alpha_2} = N(N-1) \bar{\mu}_{p_1} \bar{\mu}_{p_2}$. \hspace{1cm} (13)

When $s=3$, there are two possible cases

$[2, 1]$; $s_{p_1} s_{p_2} s_{p_3}$, $[1, 1, 1]$; $s_{p_1} s_{p_2} s_{p_3}$

and, as already shown, we have
mean $[2, 1]; s_{p_1}s_{p_2}s_{p_3} = N(N-1)(\mu_{p_1+p_2} + \mu_{p_1+p_3} + \mu_{p_2+p_3}) (14)a$

and we can easily show that

mean $[1, 1, 1]; s_{p_1}s_{p_2}s_{p_3} = N(N-1)(N-2) \mu_{p_1} \mu_{p_2} \mu_{p_3}. (14)b$

Similarly, when $s=4$, we obtain the following results:

mean $[3, 1]; s_{p_1}s_{p_2}s_{p_3}s_{p_4} = N(N-1)[\mu_{p_1} + \mu_{p_2+p_3+p_4} + \mu_{p_2} \mu_{p_1+p_3+p_4} + \mu_{p_3} \mu_{p_1+p_2+p_4} + \mu_{p_4} \mu_{p_1+p_2+p_3}] (15)a$

[check] $C_4; 3, 1 = \frac{4}{13} = 4. (15)b$

mean $[2, 2]; s_{p_1}s_{p_2}s_{p_3}s_{p_4} = N(N-1)[\mu_{p_1+p_2} \mu_{p_3+p_4} + \mu_{p_1+p_3} \mu_{p_2+p_4} + \mu_{p_1+p_4} \mu_{p_2+p_3}] (15)c$

[check] $C_4; 2, 2 = \frac{4}{22} = 3.$

mean $[2, 1, 1]; s_{p_1}s_{p_2}s_{p_3}s_{p_4} = N(N-1)(N-2)[\mu_{p_1+p_2} \mu_{p_3} \mu_{p_4} + \mu_{p_1+p_3} \mu_{p_2+p_4} \mu_{p_3} + \mu_{p_1+p_4} \mu_{p_2+p_3} \mu_{p_4} + \mu_{p_2+p_3} \mu_{p_1+p_4} \mu_{p_3} + \mu_{p_2+p_4} \mu_{p_1+p_3} \mu_{p_4} + \mu_{p_3+p_4} \mu_{p_1+p_2} \mu_{p_4}] (15)d$

[check] $C_4; 2, 1, 1 = \frac{4}{22} = 6.$

mean $[1, 1, 1, 1]; s_{p_1}s_{p_2}s_{p_3}s_{p_4} = N(N-1)(N-2)(N-3) \mu_{p_1} \mu_{p_2} \mu_{p_3} \mu_{p_4}. (15)e$

**Note.** In the solutions of these two examples we can find all the means which are necessary for a foundation of the theory in the following chapter.

**III. Reduction Formulae.**

9. *Reduction Formula for $s_{p_1}^i s_{p_2}^j$.***

If we can find, by inspection, the following mean

mean $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]; s_{p_1}^i s_{p_2}^j$

where

$\alpha_1 + \alpha_2 + \ldots + \alpha_\lambda = l_1 + l_2,$

mean (E)

as those in the examples in Art. (8), then we can find the mean $(s_{p_1}^i s_{p_2}^j)$ easily by the aid of the symbolical equations (11) or (11)'.

But the mean (E) can not easily be found, unless $l_1, l_2$ and $\lambda$ are very small integers.

Therefore the symbolical equations alone are not of much use, unless some special method of deduction for the mean $[E]$ can be devised.
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Now the terms of $[\alpha_1, \alpha_2, \ldots, \alpha_n]; s_{p_1}^{a_{p_1}} s_{p_2}^{a_{p_2}}$, are all of the form $x_{i_1}^{a_{i_1}} x_{i_2}^{a_{i_2}} \ldots x_{i_\lambda}^{a_{i_\lambda}}$, which may be considered as the product of a power $x_{i_\lambda}^{a_{i_\lambda}}$ and a factor

$F = x_{i_1}^{a_{i_1}} x_{i_2}^{a_{i_2}} \ldots x_{i_\lambda}^{a_{i_\lambda-1}}$.

Of the powers $x_{i_\lambda}^{a_{i_\lambda}}$, some $x_{i_\lambda}$ in it must be selected from $s_{p_1}$, say $l'_1$ times and the other $x_{i_\lambda}$ from $s_{p_2}$, say $l'_2$ times, provided

$l'_1 + l'_2 = \alpha_\lambda; \quad l'_1 \leq l_1, \quad l'_2 \leq l_2$.

If all the powers $x_{i_\lambda}^{a_{i_\lambda}}$, thus obtained, be multiplied by the factor $(F)$, we get all the terms of $[\alpha_1, \ldots, \alpha_\lambda]; s_{p_1}^{l'_1} s_{p_2}^{l'_2}$ without omission and repetition and the factor $(F)$ is nothing but the terms of

$[\alpha_1, \alpha_2, \ldots, \alpha_\lambda-1]; s_{p_1}^{l'_1} s_{p_2}^{l'_2-t}$

where

$\alpha_1 + \alpha_2 + \ldots + \alpha_{\lambda-1} = l_1 + l_2 - \alpha_\lambda$.

Case (G)

Here, in the case (G), we must note that all $x_{i_\lambda}$'s in (G) are different from $x_{i_\lambda}$ and they can be any one of $x_1, x_2, \ldots, x_n$ except $x_{i_\lambda}$ and therefore the symbolical coefficients $M', N'_\lambda$, and $C'$ for case (G) are given as follows:

$M' = N'_{\lambda-1}, C'_{\tau; \alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}}$ (16)

and

$N'_{\lambda-1} = (N-1) (N-2) \ldots (N-\lambda+1)$ (17)

$C' = C'_{\tau; \alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1} = \sum_{\tau} \frac{|\tau|}{\alpha_1 \alpha_2 \ldots \alpha_{\lambda-1} l_1 l_2 \ldots}$ (18)

where $\alpha_1 + \alpha_2 + \ldots + \alpha_{\lambda-1} = l_1 + l_2 - \alpha_\lambda = \tau$ and $t's$ are the numbers of equal $\alpha$'s in $[\alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}]$.

Thus we get the following theorem.

**Theorem II.** The $M$ terms of $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ for $s_{p_1}^{l'_1} s_{p_2}^{l'_2}$ consist of $N'_{\lambda-1} C'$ terms of $[\alpha_1, \ldots, \alpha_{\lambda-1}]$ for $s_{p_1}^{l'_1-t} s_{p_2}^{l'_2-t}$ each being multiplied by $N \cdot t_i C_{i_1} \ldots t_i C_{i_\nu}$ powers of the form $x_{i_1}^{l'_1} x_{i_2}^{l'_2}$, provided $l'_1 + l'_2 = \alpha_\lambda$ and $\alpha_{\lambda-1} > \alpha_\lambda$.

From this theorem we can at once deduce the following equation between $C$ and $C'$ which is very useful as a means of checking results at each step of deduction for the mean $[\alpha_1, \alpha_2, \ldots, \alpha_n]; s_{p_1}^{l'_1} s_{p_2}^{l'_2}$

$C = C'(\sum_{t_1} s_{i_1} C_{i_1} \times t_i C_{i_\nu})$, (19)

where

$l'_1 + l'_2 = \alpha_\lambda$.

Finding the mean $[E]$, by the aid of the theorem II, we obtain also the following theorem.
Theorem III. If $\alpha_{\lambda-1} > \alpha_\lambda$, then the mean $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$; $s_{p_1}^{t_1}, s_{p_2}^{t_2}$ becomes
\[
N\sum_{i_1' i_2'} \{ i_1 C_{i_1'} \times i_2 C_{i_2'} \bar{P}_{p_1 i_1' + p_2 i_2'} \text{ mean } [\alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}]; s_{p_1}^{t_1-i'}, s_{p_2}^{t_2-i'} \}
\]
provided
\[
l_1 + l_2' = \alpha_\lambda \quad \text{and} \quad \alpha_1 + \alpha_2 + \ldots + \alpha_{\lambda-1} = l_1 + l_2 - \alpha_\lambda.
\]

Ex. (1). Find
mean $[4, 3]$; $s_{[4]}^3 s_{[3]}^4$.
(Sol.) From the theorem III, we have
\[
\begin{array}{|c|c|c|c|}
\hline
\alpha_\lambda = 3 & l_1' & l_2' & s_{C_{i_1'} \times i_2 C_{i_2'}} \\
\hline
3 & 0 & 1 & \\
2 & 1 & 12 & \\
1 & 2 & 18 & \\
0 & 3 & 4 & \\
\hline
\end{array}
\]
\[
= N\sum_{i_1' i_2'} \{ i_1 C_{i_1'} \times i_2 C_{i_2'} \bar{P}_{p_1 i_1' + p_2 i_2'} \text{ mean } [4], s_{4-i'} s_{i'}^1\}
\]
\[
= N \{ s_3 C_{i} \bar{P}_s \text{ mean } [4]; s_{[4]}^1 \}
\]
\[
= \sum C_{i} \bar{P}_s \text{ mean } [4]; s_{[4]}^2 s_{[3]}^3
\]
\[
= \sum C_{i} \bar{P}_s \text{ mean } [4]; s_{[4]}^2 s_{[3]}^3
\]
\[
= N(N-1)(\bar{P}_4 + 12\bar{P}_5 + 18\bar{P}_6 + 4\bar{P}_7)
\]
\[
[\text{check}] \quad C_{7,4,3} = \frac{7}{3} = 35,
\]
\[
35 = 1 + 12 + 18 + 4
\]
Therefore mean $[4, 3]$; $s_{[4]}^3 s_{[3]}^4 = N(N-1) (4\bar{P}_7 + 19\bar{P}_6 + 12\bar{P}_5 + 4\bar{P}_4)$

10. Reduction Formula for $s_{p_1}^{t_1}, s_{p_2}^{t_2}$ (continued).

The theorem III has been deduced assuming $\alpha_{\lambda-1} > \alpha_\lambda$, but $\alpha_\lambda$ may be equal to $\alpha_{\lambda-1}$ or more generally the last $k$, $\alpha$'s in $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$ may sometimes be equal and we have such cases to examine more.

In the first place, let us consider the case where
$\alpha_\lambda = \alpha_{\lambda-1}$, but $\alpha_{\lambda-1} < \alpha_{\lambda-2}$,
and examine this case, repeating the discussion in Art. 9 from the beginning, then we can easily ascertain that the theory in Art. 9 is also applicable in this case and the only different point is that the possible combinations given by the following rule become identical two by two:
\[
\text{mean } [\alpha_1, \alpha_2, \ldots, \alpha_\lambda]; s_{p_1}^{t_1}, s_{p_2}^{t_2}
\]
\[
= N\sum_{i_1' i_2'} \{ i_1 C_{i_1'} \times i_2 C_{i_2'} \bar{P}_{p_1 i_1' + p_2 i_2'} \text{ mean } [\alpha_1, \ldots, \alpha_{\lambda+1}]; s_{p_1}^{t_1-i'}, s_{p_2}^{t_2-i'} \},
\]
Rule (I)

which is nothing but the theorem III.
This sole difference comes from the fact that, in this case, p's in \(\alpha\)-group and those in \(\alpha_{\lambda-1}\)-group can be interchanged without getting any new combination.

Similarly if the last \(k\) \(\alpha\)'s in \([\alpha_1, \alpha_2, \ldots, \alpha_{\lambda}]\) be equal and if we count the possible different terms in \([\alpha_1, \ldots, \alpha_\lambda]\) by the rule (I), we get \(k\) times the number of really possible different terms.

Thus we obtain the following general theorem which contains the theorem III. as its special case.

**Theorem IV.** If, in \([\alpha_1, \alpha_2, \ldots, \alpha_{\lambda}]\), the last \(k\) \(\alpha\)'s are equal, then

\[
\text{mean } [\alpha_1, \alpha_2, \ldots, \alpha_{\lambda}] ; s_{\lambda}^4 s_{\lambda}^2 = \sum_{l_1+l_2 = \alpha_\lambda} \left\{ \frac{N}{k} \sum_{i_1, i_2} \left[ i_1 C_{l_1} \times i_2 C_{l_2} \bar{\mu}_{i_1} \bar{\mu}_{i_2 + l_2} \right] \text{mean } [\alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}] ; s_{l_1}^{l_1-l_2} s_{l_2}^{l_2-l_2} \right\},
\]

where \(\alpha_1 + \alpha_2 + \ldots + \alpha_\lambda = l_1 + l_2\); \(l_1 + l_2 = \alpha_\lambda\). Rule (II)

From this theorem, we have

\[U = \frac{N'}{k} \left( \sum_{i_1, i_2} \left[ i_1 C_{l_1} \times i_2 C_{l_2} \right] \right) ; l_1 + l_2 = \alpha_\lambda. \quad (20)\]

The above rule and relation are very important as a means of reducing the order \(\lambda\) and \(\tau\) of the mean (E), expressing the mean (E) in terms of

\[
\text{mean } [\alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}] ; s_{l_1}^{l_1-l_2} s_{l_2}^{l_2-l_2},
\]

whose order is \(\lambda - 1\) and where \(\tau = l_1 + l_2 - \alpha_\lambda\).

Therefore we may call this rule a "reduction rule" or "reduction formula".

**Ex. (2).** Find the mean \([3, 2, 1, 1]\); \(s_3^4 s_1^2\).

(Sol.) From the rule (II), we have

\[
\text{mean } [3, 2, 1, 1] ; s_3^4 s_1^2 = \frac{N}{2} \sum_{i_1, i_2} \left[ 6 C_{i_1} \times 2 C_{i_2} \bar{\mu}_{i_1+i_2} \right] \text{mean } [3, 2, 1] ; s_{i_1}^{i_1-i_2} s_{i_2}^{i_2-i_2} (l_1 + l_2 = 1)
\]

\[= \frac{N}{2} (5 \bar{\mu}_2 \text{ mean } [3, 2, 1]; s_3^4 s_1^2 + 2 \bar{\mu}_1 \text{ mean } [3, 2, 1]; s_3^5 s_1).\]

But \(\bar{\mu}_1 = 0\). And, after applying rule (I) repeatedly, we have

\[
\text{mean } [3, 2, 1] ; s_3^4 s_1^2 = 4(N-1)(N-2)(N-3) \bar{\mu}_2 (3 \bar{\mu}_4^2 + 6 \bar{\mu}_2 \bar{\mu}_5 + 6 \bar{\mu}_3 \bar{\mu}_5 + \bar{\mu}_2 \bar{\mu}_6).\]

Therefore, we obtain the following result:
mean \[3, 2, 1, 1]\]; 
\[s_2^5 s_1\]

\[= 10N(N-1)(N-2)(N-3)\rho_2^2(\rho_0 \rho_2 + 6\rho_0 \rho_3 + 3\rho_4^2),\]

[check] \[C_7; 3, 2, 1, 1 = 210, \quad C' = C'_0; 3, 2, 1 = 60.\]

\[\therefore \quad C_7; 3, 2, 1, 1 = \frac{1}{2} \times (5 + 2) C' = 210.\]

and moreover

\[210 \times \frac{s C_1 \times s C_1}{C_1} = 100; \quad 100 = 10 \times (1 + 6 + 3).\]

11. Reduction Formulae for \(s_{p_1}^{l_1} s_{p_2}^{l_2} \ldots s_{p_s}^{l_s}\); \(s = 3\) or \(4\).

The theory in the two foregoing articles is also applicable to

the deduction of the mean \(\left[\alpha_1, \alpha_2, \ldots, \alpha_s\right]; \left(s_{p_1}^{l_1} s_{p_2}^{l_2} \ldots s_{p_s}^{l_s}\right)\). The only

point we have to consider now is the possible combinations for the

power \(x^{l_1}_1, x^{l_2}_2\) in this case being selected from \(s_{p_1}, s_{p_2}, \ldots, s_{p_s}\) instead

of \(s_{p_1}\) and \(s_{p_2}\) and we can prove, without much difficulty, the

Theorem V. If, in \(\left[\alpha_1, \alpha_2, \ldots, \alpha_s\right]\), the last \(k\) \(\alpha\)'s are equal,

then

\[
\text{mean} \left[\alpha_1, \alpha_2, \ldots, \alpha_s\right]; s_{p_1}^{l_1} s_{p_2}^{l_2} s_{p_3}^{l_3} \\
= \frac{N}{k} \sum_{l_1', l_2', l_3'} \left[ l_1 C_{l_1'} \times l_2 C_{l_2'} \times l_3 C_{l_3'} \rho_{l_1}^{l_1'} l_1 + \rho_{l_2}^{l_2'} l_2 + \rho_{l_3}^{l_3'} l_3 \right] \text{mean} \left[\alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}\right]; s_{p_1}^{l_1-l_1'} s_{p_2}^{l_2-l_2'} s_{p_3}^{l_3-l_3'},
\]

where

\[l_1' + l_2' + l_3' = \alpha_\lambda\]

and

\[\alpha_1 + \alpha_2 + \ldots + \alpha_{\lambda} = l_1 + l_2 + l_3. \quad \text{Rule (III)}\]

And so is:

Theorem VI. If, in \(\left[\alpha_1, \alpha_2, \ldots, \alpha_s\right]\), the last \(k\) \(\alpha\)'s are equal,

then

\[
\text{mean} \left[\alpha_1, \alpha_2, \ldots, \alpha_s\right]; s_{p_1}^{l_1} s_{p_2}^{l_2} s_{p_3}^{l_3} s_{p_4}^{l_4} \\
= \frac{N}{k} \sum_{l_1', l_2', l_3', l_4'} \left[ l_1 C_{l_1'} \times l_2 C_{l_2'} \times l_3 C_{l_3'} \times l_4 C_{l_4'} \rho_{l_1}^{l_1'} l_1 + \rho_{l_2}^{l_2'} l_2 + \rho_{l_3}^{l_3'} l_3 + \rho_{l_4}^{l_4'} l_4 \right] \text{mean} \left[\alpha_1, \ldots, \alpha_{\lambda-1}\right]; s_{p_1}^{l_1-l_1'} s_{p_2}^{l_2-l_2'} s_{p_3}^{l_3-l_3'} s_{p_4}^{l_4-l_4'},
\]

where

\[l_1' + l_2' + l_3' + l_4' = \alpha_\lambda\]

and

\[\alpha_1 + \alpha_2 + \ldots + \alpha_{\lambda} = l_1 + l_2 + l_3 + l_4. \quad \text{Rule (VI)}\]

These theorems are only the extension of the theorem IV for

the case of \(s_{p_1}^{l_1} s_{p_2}^{l_2} s_{p_3}^{l_3}\) or \(s_{p_1}^{l_1} s_{p_2}^{l_2} s_{p_3}^{l_3} s_{p_4}^{l_4}\) and, if these reduction rules and
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rule (II) be properly applied to the deduction of mean \([\alpha_1, \alpha_2, \ldots, \alpha_n]\); 
\[s_1^2, s_2^2, \ldots, s_n^2; \quad s = 4\] again and again, then the mean in question reduces
to the case of mean \([C]\) or \([D]\) in Art. 8 and we can always find
such means, if necessary.

**Ex. (3).** Find mean \([3, 2, 1]\); \[s_3^2, s_2^2, s_1^2\].

(Sol.) From the rule (III) and, since \(\mu_1 = 0\), we have
mean \([3, 2, 1]\); \[s_3^2, s_2^2, s_1^2\]
\[= N(2C_1 \cdot 2C_0 \cdot 2C_0 \bar{\mu}_3 \text{ mean } [3, 2]; \quad s_3 s_2^2 s_1^2 \]
\[+ 2C_0 \cdot 2C_1 \cdot 2C_0 \bar{\mu}_2 \text{ mean } [3, 2]; \quad s_3^2 s_2 s_1^2)\].

But
\[2C_1 \cdot 2C_0 \cdot 2C_0 \bar{\mu}_3 \text{ mean } [3, 2]; \quad s_3 s_2^2 s_1^2\]
\[= 2(N - 1) \bar{\mu}_3 (1C_1 \cdot 2C_1 \cdot 2C_0 \bar{\mu}_5 \text{ mean } [3]; \quad s_2 s_1^2 \]
\[+ 1C_0 \cdot 2C_0 \cdot 2C_1 \bar{\mu}_4 \text{ mean } [3]; \quad s_3 s_1^2 \]
\[+ 1C_0 \cdot 2C_2 \cdot 2C_0 \bar{\mu}_4 \text{ mean } [3]; \quad s_3 s_2 s_1 \]
\[+ 1C_0 \cdot 2C_1 \cdot 2C_1 \bar{\mu}_3 \text{ mean } [3]; \quad s_3 s_2 s_1 \]
\[= 2(N - 1) (N - 2) \bar{\mu}_3 (2\bar{\mu}_5 \bar{\mu}_4 + 2\bar{\mu}_4 \bar{\mu}_3 + 4\bar{\mu}_3 \bar{\mu}_2 + \bar{\mu}_2 \bar{\mu}_1)\]
\[= 2(N - 1) (N - 2) \bar{\mu}_3 (\bar{\mu}_7 \bar{\mu}_2 + 4\bar{\mu}_6 \bar{\mu}_3 + 5\bar{\mu}_5 \bar{\mu}_4).\]

Similarly
\[2C_0 \cdot 2C_1 \cdot 2C_0 \bar{\mu}_4 \text{ mean } [3, 2]; \quad s_3^2 s_2 s_1^2\]
\[= 2(N - 1) (N - 2) \bar{\mu}_4 (\bar{\mu}_8 \bar{\mu}_2 + 2\bar{\mu}_7 \bar{\mu}_3 + 5\bar{\mu}_6 \bar{\mu}_4 + 2\bar{\mu}_3^3).\]

\[\therefore \quad \text{mean } [3, 2, 1]; \quad s_3^2 s_2 s_1^2\]
\[= 2N(N - 1) (N - 2) (\bar{\mu}_8 \bar{\mu}_2^2 + 3\bar{\mu}_7 \bar{\mu}_3 \bar{\mu}_2 + 5\bar{\mu}_6 \bar{\mu}_4 \bar{\mu}_2\]
\[+ 4\bar{\mu}_5 \bar{\mu}_3^2 + 2\bar{\mu}_5 \bar{\mu}_2 + 5\bar{\mu}_6 \bar{\mu}_4 \bar{\mu}_3).\]

[check] \[C_3; s_3 s_1 \times \frac{4C_1}{s_0} = 40; \quad 40 = 2 \times (1 + 3 + 5 + 4 + 2 + 5).\]

12. Some Useful Theorems for Simplification.

The symbolical equations (11) or (11)', if used together with the
reduction rules, are very powerful in the deduction of mean (\(s_1^4, s_2^4\)
\[\ldots s_n^4\)). But symbolical equations themselves are generally not so
simple. The terms of symbolical equations become numerous, if the
integer \(r\) is not small and it is quite a useful device to neglect, if
possible, their unnecessary terms at the beginning.

For instance, if \(r = 9\), the required symbolical equation, in the
first form, is as follows, the total number of terms being thirty:
But all of these terms are not always necessary. By reason of $\mu_1 = 0$, or by the special nature of the product $\pi(A)$ or the partition $\alpha_1, \alpha_2, \ldots, \alpha_k$, or according to the purposes of our deduction, there are many terms we need not retain. Now if the product $\pi(A)$ contain the factor $s_1$ or is of the form $\text{Product}(H)$ and, at the same time, if the type is of the form $\text{Type}(I)$ then those terms, in which the $x$ for 1 in $[\alpha_1, \ldots, \alpha_k, 1, 1, \ldots, 1]$ be selected from $s_1$, give us no mean at the end for in the means of such terms, there is at least one $\mu_1$ as a factor and consequently their means become all zero.

Moreover if $m > l_1 + l_2 + \ldots + l_{i-1}$, then there is at least one $x$ for 1 of the type, which is taken from the factor $s_1$ in every term of $[\alpha_1, \alpha_2, \ldots, \alpha_k, 1, 1, \ldots, 1]$. Therefore, in such cases, all the terms of $[\alpha_1, \ldots, \alpha_k, 1, 1, \ldots, 1]$ become zero at the end and consequently we can neglect such types in the symbolical equation from the beginning.

Thus we obtain the following theorem.

**Theorem VII.** If the number of 1 in $[\alpha_1, \alpha_2, \ldots, \alpha_k, 1, 1, \ldots, 1]$ is greater than the number of $s_p$-factors, whose suffixes $p$'s are not 1, in the product $(H)$, then

$$\text{mean} [\alpha_1, \ldots, \alpha_k, 1, 1, \ldots, 1]; s_{p_1}^{l_1} s_{p_2}^{l_2} \ldots s_{p_{i-1}}^{l_{i-1}} s_i^{l_i} = 0.$$ 

For instance, if $m > l_2 + l_3 + l_4$, then we have

$$\text{mean} [\alpha_1, \ldots, \alpha_k, 1, 1, \ldots, 1]; s_1^{l_1} s_2^{l_2} s_3^{l_3} s_4^{l_4} = 0.$$ 

The above theorem is very useful as a means of simplifying symbolical equations.
Ex. (4). What terms of the symbolical equation for the mean $s_2 s_1^3$ must be retained?

(Sol.) The mean $s_2 s_1^3$ is necessary also as a term of $2M'$ and $\tau = 1 + 8 = 9$.

Now, since there is only one factor $s_2$ whose suffix is not 1, any type which contains 1 more than once becomes negligible and the necessary symbolical equation $\tau = 9$, which contains thirty terms as shown before, becomes very simple as follows:

$$\pi(9) = [9] + [8, 1] + [7, 2] + [6, 3] + [5, 4] + [6, 2, 1] + [5, 3, 1]$$
$$+ [5, 2, 2] + [4, 4, 1] + [4, 3, 2] + [3, 3, 3] + [4, 2, 2, 1]$$
$$+ [3, 3, 2, 1] + [\ell, 2, 2, 2] + [2, 2, 2, 2, 1].$$ (22)

I have also deduced the following short method of reduction, which can easily be obtained by applying the reduction rules to the type (I) repeatedly.

Theorem VIII.

$$\text{mean} \left[ \alpha_1, \alpha_2, \ldots, \alpha_k, 1, 1, \ldots, 1 \right]; s_1^t s_2^s s_3^t s_4^t$$

$$= \sum_{a, b, y} \left( \frac{N_a}{m} \right) \left[ \frac{l_2}{l_2 - \alpha} \frac{l_3}{l_3 - \beta} \frac{l_4}{l_4 - \gamma} \bar{p}_2^a \bar{p}_3^b \bar{p}_4^y \right] \times \text{mean} \left[ \alpha_1, \ldots, \alpha_k \right]; s_1^t s_2^{t-a} s_3^{t-b} s_4^{t-y},$$

where

$$\alpha + \beta + \gamma = m. \quad \text{Rule (V)}$$

Especially, if $l_3 = l_4 = 0$ in the rule (V), then we have

$$\text{mean} \left[ \alpha_1, \ldots, \alpha_k, 1, 1, \ldots, 1 \right]; s_1^t s_2^t$$

$$= \left( \frac{l_2}{l_2 - m} \right) \frac{N_a}{m} \text{mean} \left[ \alpha_1, \ldots, \alpha_k \right]; s_1^t s_2^{t-m}. \quad (23)$$

Ex. (5). Reduce the mean $[4, 2, 1, 1]; s_1^t s_2^t s_3^4$ to the form $[4, 2]$.

(Sol.) From the rule (V), we have

$$\text{mean} \left[ 4, 2, 1, 1 \right]; s_1^t s_2^t s_3^4$$

$$= \sum_{a, b} \left( \frac{N_a}{2} \right) \left( \frac{2}{2 - \alpha} \frac{4}{4 - \beta} \bar{p}_2^a \bar{p}_3^b \text{mean} \left[ 4, 2 \right]; s_1^t s_2^{t-a} s_3^{1-b} \right),$$

where

$$\alpha + \beta = 2.$$
IV. Summary of the Results obtained.

13. Method of Deduction of Moment \( m \) \( M_n \).

The theories and processes, mentioned hitherto, make a perfect system as a general and systematic method of deduction for the moments of moments \( m \) \( M_n \), \( (m \leq 4) \).

The deduction is generally done in the following six steps:

(1) **Determination of necessary products** \( s_{l_1} s_{l_2} \ldots s_{l_t} \), expressing the required moment \( m M_n' \) in terms of mean \( (s_{l_1} \ldots s_{l_t}) \).
   
   This step can easily be taken by the aid of the equation (2).

(2) **Determination of types \( [\alpha_1, \alpha_2, \ldots, \alpha_\lambda] \)**, necessary to be retained in the symbolical equation for the product \( \pi(A) \) in question.

   The symbolical equation for \( \pi(A) \) is determined at once from the equation
   
   \[
   \tau = l_1 + l_2 + \ldots + l_t,
   \]

   and its terms or possible partitions for the integer \( \tau \) can rather easily and systematically be found.

   It is the best way to neglect those terms here, which, by the theorem VII, become negligible.

(3) **Deduction of necessary mean** \( [\alpha_1, \alpha_2, \ldots, \alpha_\lambda] \); \( s_{l_1} s_{l_2} \ldots s_{l_t} \).

   We can find these means by the aid of reduction formulae, applying them repeatedly. Here we must be careful to utilize results already obtained.

(4) **Deduction of necessary mean** \( (s_{l_1}^{\prime} s_{l_2}^{\prime} \ldots s_{l_t}^{\prime}) \).

   The mean \( (s_{l_1}^{\prime} \ldots s_{l_t}^{\prime}) \) is obtained by substituting mean \( [\alpha_1, \alpha_2, \ldots, \alpha_\lambda] \), already prepared, into the symbolical equation for the product \( s_{l_1}^{\prime} s_{l_2}^{\prime} \ldots s_{l_t}^{\prime} \) and this process is simply an algebraic operation.

(5) **Deduction of moments** \( m M_n' \).

   This process is also an algebraic operation, substituting the mean \( (s_{l_1}^{\prime} \ldots s_{l_t}^{\prime}) \) above into the equation (2).

(6) **Determination of moment** \( m M \) in question.

   Now we can easily deduce the moment \( m M_n \) in question by the aid of the equation (3), utilizing \( m M_q \) \( (q \leq n-1) \) already obtained and \( m M_n' \) prepared above.

For instance let us find \( _2 M_3 \), the third moment coefficient of the
variance $\mu_2$ in sampling distribution.

Now, from the equation (3), we have

$$2M_3 = 2M_3' - 3 \times 2M_2 \times 2M_1' - (2M_3')^3$$

and

$$2M_2 = 2M_2' - (2M_1')^2.$$  \hspace{1cm} (24)

Therefore, for the deduction of $2M_3$, we have first to deduce $2M_1'$, $2M_2'$, and $2M_3'$.

Moreover, since

$$2M_1' = \text{mean} \left( \frac{1}{N} \left( s_2 - s_1^2 \right) \right) = \left( 1 - \frac{1}{N} \right) \bar{\mu}_2,$$  \hspace{1cm} (25)

$$2M_2' = \frac{1}{N^2} \left( \text{mean} s_2^2 - \frac{2}{N} \text{mean} s_2 s_1^2 + \frac{1}{N^2} \text{mean} s_1^5 \right),$$

$$2M_3' = \frac{1}{N^3} \left( \text{mean} s_2^3 - \frac{3}{N} \text{mean} s_2^2 s_1^2 + \frac{3}{N^2} \text{mean} s_2 s_1^4 - \frac{1}{N^3} \text{mean} s_1^6 \right),$$  \hspace{1cm} (26)

we have only to find here the following means

$$[s_2^2], [s_2 s_1^2], [s_1^4]; \quad [s_2^3], [s_2^2 s_1^2], [s_2 s_1^4], [s_1^6];$$

where the bracket $[ \ ]$ is used as a symbol for "mean". For example $[s_2^2]$ denotes mean $s_2^2$.

And consequently the following symbolical equation are what are necessary here,

$$\pi(2) = N[2] + N_2[1, 1],$$

$$\pi(3) = N[3] + 3N_2[2, 1] + N_3[1, 1, 1],$$

$$\pi(4) = N[4] + 4N_2[3, 1] + 3N_2[2, 2] + 6N_3[2, 1, 1] + N_4[1, 1, 1, 1],$$


$$+ 15N_3[2, 2, 1] + 10N_4[2, 1, 1, 1] + N_5[1, 1, 1, 1, 1],$$


$$+ 60N_3[3, 2, 1] + 20N_4[3, 1, 1, 1] + 15N_5[2, 2, 2]$$

$$+ 45N_4[2, 2, 1, 1] + 15N_6[2, 1, 1, 1, 1] + N_6[1, 1, \ldots, 1].$$  \hspace{1cm} (27)

For example, the symbolical equation $\pi = 6$ above is wanted now for the deduction of $[s_1^6]$ and all suffixes of $s$ in $s_1^6$ are 1, we have only to retain the following four terms, out of the eleven terms,


and we can easily show that

$$\text{mean} [6]; \quad s_1^6 = N \text{mean } s_1^5 = N \bar{\mu}_6.$$
mean [4, 2] ; $s_1^6 = 15 N \bar{\mu}_2 \text{mean}[4] ; s_1^4 = 15 N (N - 1) \bar{\mu}_4 \bar{\mu}_2$, mean [3, 3] ; $s_1^6 = \frac{N}{2} c_3 \bar{\mu}_3 \text{mean}[3] ; s_1^3 = 10 N (N - 1) \bar{\mu}_3^2$.

Similarly

mean [2, 2, 2] ; $s_1^6 = 15 N (N - 1) (N - 2) \bar{\mu}_4^3$.

Therefore from the equation (28), we obtain the following result,

$$[s_1^6] = 15 N^3 \bar{\mu}_4^3 + N^2 (15 \bar{\mu}_3 \bar{\mu}_4 + 10 \bar{\mu}_3^2 - 45 \bar{\mu}_4^3)$$

$$+ N (\bar{\mu}_6 - 15 \bar{\mu}_4 \bar{\mu}_2 - 10 \bar{\mu}_3^2 + 30 \bar{\mu}_4^3). \quad (29a)$$

Similarly

$$[s_2^6] = \text{mean} s_2^6 = N^3 \bar{\mu}_4^3 + N (\bar{\mu}_4 - \bar{\mu}_3^2),$$

$$[s_2 s_1^6] = N^2 \bar{\mu}_4^2 + N (\bar{\mu}_3 - 3 \bar{\mu}_3^2),$$

$$[s_1^4] = N^2 \bar{\mu}_4^2 + N (\bar{\mu}_3^2 - 3 \bar{\mu}_3^3);$$

$$[s_2^3] = N^3 \bar{\mu}_4^3 + N^2 (3 \bar{\mu}_4 \bar{\mu}_2 - 3 \bar{\mu}_2^3) + N (\bar{\mu}_6 - 3 \bar{\mu}_4 \bar{\mu}_2 + 2 \bar{\mu}_3^3),$$

$$[s_2^2 s_1^4] = N^3 \bar{\mu}_4^3 + N^2 (3 \bar{\mu}_4 \bar{\mu}_2 + 2 \bar{\mu}_3^2 - 3 \bar{\mu}_3^3)$$

$$+ N (\bar{\mu}_6 - 3 \bar{\mu}_4 \bar{\mu}_2 - 2 \bar{\mu}_3^2 + 2 \bar{\mu}_2^3),$$

$$[s_2 s_1^3] = 3 N^3 \bar{\mu}_4^3 + N^2 (7 \bar{\mu}_4 \bar{\mu}_2 + 4 \bar{\mu}_3^2 - 9 \bar{\mu}_2^3)$$

$$+ N (\bar{\mu}_6 - 7 \bar{\mu}_4 \bar{\mu}_2 - 4 \bar{\mu}_3^2 + 6 \bar{\mu}_2^3). \quad (29b)$$

Consequently, from the equation (26) and (28), we have

$$2 M_4' = \bar{\mu}_4^2 + \frac{1}{N} (\bar{\mu}_4 - 3 \bar{\mu}_3^2) + \frac{1}{N^3} (-2 \bar{\mu}_4 + 5 \bar{\mu}_3^2) + \frac{1}{N^3} (\bar{\mu}_4 - 3 \bar{\mu}_3^2), \quad (30a)$$

$$2 M_3' = \bar{\mu}_3^3 + \frac{1}{N} (3 \bar{\mu}_4 \bar{\mu}_2 - 6 \bar{\mu}_2^3) + \frac{1}{N^3} (\bar{\mu}_6 - 12 \bar{\mu}_4 \bar{\mu}_2 - 6 \bar{\mu}_3^2 + 20 \bar{\mu}_2^3)$$

$$+ \frac{1}{N^3} (-3 \bar{\mu}_6 + 30 \bar{\mu}_4 \bar{\mu}_2 + 18 \bar{\mu}_3^2 - 48 \bar{\mu}_2^3)$$

$$+ \frac{1}{N^4} (3 \bar{\mu}_6 - 36 \bar{\mu}_4 \bar{\mu}_2 - 22 \bar{\mu}_3^2 + 63 \bar{\mu}_2^3)$$

$$+ \frac{1}{N^6} (-\bar{\mu}_6 + 15 \bar{\mu}_4 \bar{\mu}_2 + 10 \bar{\mu}_3^2 - 30 \bar{\mu}_2^3). \quad (30b)$$

Finally, from the equations (24), (25) and (30), we obtain

$$2 M_3 = \frac{1}{N^2} (\bar{\mu}_6 - 3 \bar{\mu}_4 \bar{\mu}_2 - 6 \bar{\mu}_3^2 + 2 \bar{\mu}_2^3)$$

$$- \frac{1}{N^3} (3 \bar{\mu}_6 - 21 \bar{\mu}_4 \bar{\mu}_2 - 18 \bar{\mu}_3^2 + 26 \bar{\mu}_2^3)$$
which was deduced first by Tchouproff and checked afterward by Church and which I have now verified again.

Similarly another moments $mM_n$, for any $m,n$, provided $m \leq 4$, can be deduced by the same method of deduction.

**14. Special Features and Merits of New Method of Deduction.**

We may enumerate the following points as the special features and merits of this new method of deduction.

(i) **All processes are purely analytic and quite mechanical.** In this method inspection is little used and it can be used by all with equal success. This is a feature, which is not found in Fisher’s method.

(ii) **The processes of summation $\sum$ in the reduction formulae are very simple in almost all cases and are quite practicable to any user.**

This is another feature of this method which is not found in Craig’s method.

(iii) **The process of deduction is quite the same for any product $\pi(A)$ or for any moment of moment $mM_n$.**

(iv) **We can check the results at each step of deduction of the mean $[\alpha_1, \ldots, \alpha_\lambda]; s_{11}^{s_{12}} s_{22}^{s_{23}} \ldots s_{\lambda}^{s_{\lambda \lambda}}$ where operation becomes most complicated in the course of our deduction.**

(v) **There are some special devices and arrangements, or preliminary preparations which make the deduction considerably simple.**

By these points of excellence, the new method is, we can say, made pretty useful.

**15. Devices and Arrangements for Simplification.**

For instance, if the parent distribution is normal, we have not only $\mu_1=0$, but also $\mu_{2m-1}=0$ for any integer $m$.

Thus, in this case, the mean of any term, which contains some odd power of $x$’s, becomes zero and here we can expect some extension of the theorem VII which will be very useful for simplification like the theorem VII and which I hope to develope in a later paper.

Now, from the point of statistical order of mean $[\alpha_1, \alpha_2, \ldots, \alpha_\lambda]$ as a term of $mM_n'$, we can obtain also some theorems, which are
very useful for simplification and which I wish to discuss in the next chapter.

Of the special arrangements, I want first to mention the generality of application of symbolical equations.

The necessary symbolical equation for the mean \((s_{m1}^1 \ldots s_{mk}^k)\) is determined by the equation \(\tau = l_1 + l_2 + \ldots + l_k\).

But this symbolical equation is also applicable to any other product

\[
s_{m1}^{m1} s_{m2}^{m2} \ldots s_{mk}^{mk} \text{ where } m_1 + m_2 + \ldots + m_k = \tau.
\]

Hence to record symbolical equations once found is a cautious preparation or arrangement for future purposes and if the following forty-nine equations be once found

\[\tau = 2, 3, 4, 5, \ldots, 49, 50.\]

Then, in usual cases, we have no symbolical equations to find.

The second important arrangement or preparation is to make use of the known

\[
\text{mean } [a_1, \ldots, a_h]; \quad s_{a1}^{a1} \ldots s_{ah}^{ah}.
\]

For example, let us find

\[
\text{mean } [3, 2, 2, 1]; \quad s_3^3 s_2^2.
\]

From the reduction rule (I), we have

\[
\text{mean } [3, 2, 2, 1]; \quad s_3^3 s_2^2 = 6N\bar{m}_2 \text{ mean } [3, 2, 2]; \quad s_2^2 s_1^2.
\]

Therefore, if the mean \([3, 2, 2]; \quad s_2^2 s_1^2\) be known as follows

\[
\text{mean } [3, 2, 2]; \quad s_2^2 s_1^2 = 5N_3 (2\bar{m}_0 \bar{m}_4 \bar{m}_2 + 4\bar{m}_0 \bar{m}_3^3 + 12\bar{m}_0 \bar{m}_4 \bar{m}_3 + 3\bar{m}_4^3),
\]

then we can obtain at once the following result:

\[
\text{mean } [3, 2, 2, 1]; \quad s_2^2 s_1^2
\]

\[
= 6N\bar{m}_2 \times 5N_3(2\bar{m}_0 \bar{m}_4 \bar{m}_2 + 4\bar{m}_0 \bar{m}_3^3 + 12\bar{m}_0 \bar{m}_4 \bar{m}_3 + 3\bar{m}_4^3)
\]

\[
= 30N(N-1)(N-2)(N-3)\bar{m}_2(2\bar{m}_0 \bar{m}_4 \bar{m}_2
\]

\[
+ 4\bar{m}_0 \bar{m}_3^3 + 12\bar{m}_0 \bar{m}_4 \bar{m}_3 + 3\bar{m}_4^3).
\]

But it is not so simple a problem to those who do not know the relation (32) and they have to deduce the mean \([3, 2, 2]; \quad s_2^2 s_1^2\) by themselves.

V. **Statistical Order of mean** \([a_1, a_2, \ldots, a_h]\), as a **Term of** \(\mu M_{n'}\).

16. **Statistical Order of Mean** \([a_1, a_2, \ldots, a_h]\), as a term of \(\mu M_{n'}\).
A NEW METHOD OF FINDING MOMENTS OF MOMENTS.

Let us first consider the moment \( zM'_n \), moments of the sampling distribution of variances and examine the "statistical order", the order in sample size \( N \), of the mean \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]\), as a term of \( zM'_n \). Now

\[
zM'_n = \text{mean} \left\{ \frac{s_2^2}{N^2} \right\}^n
\]

or

\[
zM'_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{N^{n+i}} \text{mean} s_2^{n-i} s_1^{2i}
\]

\[
= \sum_{i=0}^{n} (-1)^i \frac{n!}{N^{n+i}} \left( \sum_{\lambda=1}^{n+i} \text{mean} [\alpha_1, \alpha_2, \ldots, \alpha_\lambda]; s_2^{n-i} s_1^{2i} \right), \quad (34)
\]

where \( \alpha_1 + \alpha_2 + \ldots + \alpha_\lambda = n + i \), and, in \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]\), for each term, there are \( N_\lambda \) like terms.

Moreover, since

\[
N_\lambda = N(N-1)(N-2)\ldots(N-\lambda+1)
\]

\[
= N^{\lambda-1} \frac{N(N-1)}{2} \ldots ,
\]

we can easily ascertain that the highest order term of the mean \( \frac{1}{N^{n+i}} \)

\([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]; s_2^{n-i} s_1^{2i} \) is of order \( \lambda - (n + i) \) in \( N \).

Therefore, if we have only to find some approximate expressions for \( zM'_n \), say up to the order \( N^{-m} \), we can neglect those types \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]\), of which

\[
n+i-\lambda > m \quad \text{or} \quad \lambda < n+i-m, \quad \text{Criterion (J)}
\]

for the order of terms of such a mean \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]; s_2^{n-i} s_1^{2i} \) are all less than \(-m\).

Thus we obtain the following theorem:

**Theorem IX.** If we have to find an approximate expression for \( zM'_n \) only up to the order \( N^{-m} \), we can neglect those types \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]\), of which

\[
\lambda < n+i-m
\]

for \( s_2^{n-i} s_1^{2i} \) as a term of \( zM'_n \).

For instance, if \( m = 4 \), then the criterion (J) becomes

\[(1) \text{ In my study of the sampling distribution of } \sigma (\text{Biometrika, Vol. XXII, 1930, London}), I deduced required moments up to the order } N^4 \text{ and I found that the results thus obtained, are pretty accurate and especially when the parent distribution is normal or almost normal, they are accurate enough as far as the fourth decimal place.}\]
and we get the following table

<table>
<thead>
<tr>
<th>n+i((s_{i}^{n-1}s_{1}^{2}))</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>......</th>
</tr>
</thead>
<tbody>
<tr>
<td>negligible order ((\lambda))</td>
<td>(\lambda = 1)</td>
<td>(\lambda \leq 2)</td>
<td>(\lambda \leq 3)</td>
<td>(\lambda \leq 4)</td>
<td>(\lambda \leq 5)</td>
<td>(\lambda \leq 6)</td>
<td>(\lambda \leq 7)</td>
<td>......</td>
</tr>
</tbody>
</table>

**Ex. (1).** What terms of the symbolical equation \(\tau = 9\) must be retained for the mean \(s_{i}^{2}\) as a term of \(M_{n}'\) provided that an approximate expression up to the order \(N^{-4}\) is accurate enough?

(Sol.) The symbolical equation \(\tau = 9\) or the equation (21) for the mean \(s_{i}^{2}\) was once examined in Ex. (14), Art. (12) and it was shown that only fifteen terms, as shown in the equation (22), must be retained.

Now as we have \(M_{n}'\) to deduce, only up to the order \(N^{-4}\) and, since \(N = 5\), \(m = 4\) and \(i = \frac{\delta}{2} = 4\), we can neglect more types, of which

\[\lambda < n + i - m = 5\]

and we have only one type \([2, 2, 2, 2, 1]\) to consider, the others being all negligible.

Thus, in this special case, we have

\[\pi(9) = 945N_5[2, 2, 2, 2, 1]\]

only one type out of the thirty possible types.

**17. Statistical Order of mean \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]\) (continued).**

The highest order of the terms in mean \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda] ; s_{1}^{1} s_{2}^{2}\) depends, also, upon the suffix \(m\) of \(M_{n}'\) in question and the criterion (J) applies only to the deduction of \(M_{n}'\).

But we can deduce similar criteria for \(M_{n}', \mathcal{M}_{n}'\) and \(M_{n}'\) by the same method of reasoning, without much difficulty, from the equation (2) and the theorem I.

The results, I obtained, are as follows.

**Theorem X.** If we have to find some approximate expressions for \(M_{n}', (l = 1, 3 \text{ or } 4)\) only up to the order \(N^{-m}\), we can neglect those types \([\alpha_1, \alpha_2, \ldots, \alpha_\lambda]\) in the symbolical equations, of which

(i) \(\lambda < n - m\) for \(s_{i}^{n}\) as a term of \(M_{n}'\);

(ii) \(\lambda < 3n - 2i - j - m\) for \(s_{i}^{1} s_{i}^{2} s_{i}^{3n-3i-2j}\), as a term of \(M_{n}'\);

(iii) \(\lambda < 4n - 3i - 2j - k - m\) for \(s_{i}^{1} s_{i}^{2} s_{i}^{3n-4i-3j-2k}\), as a term of \(M_{n}'\). Criteria (K).
A NEW METHOD OF FINDING MOMENTS OF MOMENTS.

18. Approximate Expressions for $mM_n$.

As an example of the application of the theory in Art. (16) and (17), let us find an approximate expression for $zM_5$, up to the order $N^{-4}$.

From the equation ( ), to obtain $zM_5$, we have first to find $zM_1'$, $zM_2'$, $zM_3$, $zM_4$ and $zM_5'$, of which the first four moments were found by Tchouproff or Church, but the last moment is not yet found.

Now, to obtain $zM_5'$, we have first to find the following means

$$\frac{1}{N^5}[s_5^5], \frac{5}{N^6}[s_5^4s_1^1], \frac{10}{N^7}[s_5^3s_1^2], \frac{10}{N^8}[s_5^2s_1^3], \frac{5}{N^9}[s_2^5], \frac{1}{N^{10}}[s_1^5];$$

as is shown by the equation (34), $n$ being five.

The deduction of the exact expressions for these means is by no means easy for the terms of the necessary symbolical equations for these means are numerous, but if we apply the theorem IX, used together with the theorem VII, we can neglect very many terms among them and the following results are obtained without much difficulty.

$$\frac{1}{N_5^5}[s_5^5] = \bar{\mu}_5^5 + \frac{1}{N}(10\bar{\mu}_4\bar{\mu}_2^3 - 10\bar{\mu}_5^5)$$

$$+ \frac{1}{N^2}(10\bar{\mu}_6\bar{\mu}_2^3 + 15\bar{\mu}_4^2\bar{\mu}_2 - 60\bar{\mu}_4\bar{\mu}_2^3 + 35\bar{\mu}_5^5)$$

$$+ \frac{1}{N^3}(5\bar{\mu}_8\bar{\mu}_2 + 10\bar{\mu}_6\bar{\mu}_2 - 30\bar{\mu}_6\bar{\mu}_2^3 - 45\bar{\mu}_4^2\bar{\mu}_2 + 10\bar{\mu}_4\bar{\mu}_2^3 - 50\bar{\mu}_5^5)$$

$$+ \frac{1}{N^4}(\bar{\mu}_{10} - 5\bar{\mu}_8\bar{\mu}_2 - 10\bar{\mu}_6\bar{\mu}_2 + 20\bar{\mu}_6\bar{\mu}_2^3 + 30\bar{\mu}_4^2\bar{\mu}_2 - 60\bar{\mu}_4\bar{\mu}_2^3 + 24\bar{\mu}_5^5),$$

$$\frac{1}{N^6}[s_5^4s_1^1] = \frac{1}{N}\bar{\mu}_2^5 + \frac{1}{N^2}(10\bar{\mu}_4\bar{\mu}_2^3 + 12\bar{\mu}_5^2\bar{\mu}_2^2 - 10\bar{\mu}_5^5)$$

$$+ \frac{1}{N^3}(10\bar{\mu}_6\bar{\mu}_2^3 + 24\bar{\mu}_6\bar{\mu}_2^3 + 15\bar{\mu}_4^2\bar{\mu}_2 + 12\bar{\mu}_4\bar{\mu}_2^3 - 60\bar{\mu}_4\bar{\mu}_2^3 - 72\bar{\mu}_5^2\bar{\mu}_2^2 + 35\bar{\mu}_5^5)$$

$$+ \frac{1}{N^4}(5\bar{\mu}_8\bar{\mu}_2 + 8\bar{\mu}_6\bar{\mu}_2 + 10\bar{\mu}_6\bar{\mu}_2 + 6\bar{\mu}_4^2\bar{\mu}_2 - 30\bar{\mu}_4\bar{\mu}_2^3 - 72\bar{\mu}_6\bar{\mu}_2^3 - 35\bar{\mu}_5^5)$$

$$- 45\bar{\mu}_4^2\bar{\mu}_2 + 36\bar{\mu}_4\bar{\mu}_2^3 + 110\bar{\mu}_4\bar{\mu}_2^3 + 132\bar{\mu}_5^2\bar{\mu}_2^3 - 50\bar{\mu}_5^5),$$

$$\frac{1}{N^7}[s_5^3s_1^2] = \frac{3}{N^2}\bar{\mu}_2^5 + \frac{1}{N^3}(28\bar{\mu}_4\bar{\mu}_2^3 + 48\bar{\mu}_5^2\bar{\mu}_2^2 - 30\bar{\mu}_5^5)$$
Substituting these results into the equation (34) and after simplification, we have

$$2M_5' = \bar{\mu}^5 + \frac{1}{N} (10 \bar{\mu}_3 \bar{\mu}_2 - 15 \bar{\mu}_2^5)$$

$$+ \frac{1}{N^2} (10 \bar{\mu}_6 \bar{\mu}_2^2 + 15 \bar{\mu}_4 \bar{\mu}_2 - 110 \bar{\mu}_2 \bar{\mu}_4 + 60 \bar{\mu}_3 \bar{\mu}_2^2 + 115 \bar{\mu}_2^5)$$

$$+ \frac{1}{N^3} (5 \bar{\mu}_8 \bar{\mu}_2 + 10 \bar{\mu}_6 \bar{\mu}_2 - 80 \bar{\mu}_6 \bar{\mu}_2 - 120 \bar{\mu}_6 \bar{\mu}_3 \bar{\mu}_2$$

$$- 120 \bar{\mu}_4 \bar{\mu}_2 - 60 \bar{\mu}_4 \bar{\mu}_3 + 690 \bar{\mu}_4 \bar{\mu}_2 + 840 \bar{\mu}_3 \bar{\mu}_2^2 - 675 \bar{\mu}_2^5)$$

$$+ \frac{1}{N^4} (\bar{\mu}_10 - 30 \bar{\mu}_8 \bar{\mu}_2 - 40 \bar{\mu}_7 \bar{\mu}_4 - 60 \bar{\mu}_6 \bar{\mu}_4 + 410 \bar{\mu}_6 \bar{\mu}_2$$

$$- 30 \bar{\mu}_5^2 + 1080 \bar{\mu}_8 \bar{\mu}_3 \bar{\mu}_2 + 645 \bar{\mu}_4 \bar{\mu}_2 + 660 \bar{\mu}_4 \bar{\mu}_3$$

$$- 3490 \bar{\mu}_4 \bar{\mu}_2^3 - 5740 \bar{\mu}_3 \bar{\mu}_2^3 + 3349 \bar{\mu}_2^5)$$

Finally, from the equations (3), (35) and Tchouproff's formulae, we obtain the following result,

$$2M_5 = \frac{1}{N^3} (10 \bar{\mu}_6 \bar{\mu}_2 - 10 \bar{\mu}_6 \bar{\mu}_2 - 30 \bar{\mu}_4 \bar{\mu}_2$$

$$+ 50 \bar{\mu}_3 \bar{\mu}_2^3 - 60 \bar{\mu}_3 \bar{\mu}_3 \bar{\mu}_2^2 + 60 \bar{\mu}_5 \bar{\mu}_2^5 - 20 \bar{\mu}_2^5)$$

$$+ \frac{1}{N^4} (\bar{\mu}_10 - 5 \bar{\mu}_8 \bar{\mu}_2 - 40 \bar{\mu}_7 \bar{\mu}_3 - 60 \bar{\mu}_6 \bar{\mu}_4 + 90 \bar{\mu}_6 \bar{\mu}_2 - 30 \bar{\mu}_5^2 + 480 \bar{\mu}_5 \bar{\mu}_3 \bar{\mu}_2$$

$$+ 300 \bar{\mu}_4 \bar{\mu}_2 + 660 \bar{\mu}_4 \bar{\mu}_2^2 - 690 \bar{\mu}_4 \bar{\mu}_3^2 - 1980 \bar{\mu}_3 \bar{\mu}_2^3 + 364 \bar{\mu}_2^5)$$

which is nothing but the required expression (1).

In my study of the sampling distribution of $\sigma^2$, I had to find $2M_n$ ($n = 5, 6, 7$ or 8) as a preparation for my study at that time and I deduced the following expression for $2M_5$

(1) Mr. A. Fisher deduced the so-called $K(2^d)$-function (Dec. 1928), a function which corresponds to $2M_5$ in his theory and, from some point of view, we may say the moment $M_5$ became known from that time. But no one has directly deduced the moments $2M_5'$ or $2M_5$.

A NEW METHOD OF FINDING MOMENTS OF MOMENTS.

\[ zM_6 = 10 \times zM_2 \times zM_3 + \frac{4U}{N^5} \mu_2^5, \]

where

\[ U = \frac{1}{4} [\bar{\beta}_n - 5\bar{\beta}_n - 10\bar{\beta}_3 (\bar{\beta}_1 - 2) - 30\bar{\beta}_3 (\bar{\beta}_1 - 16) + 30\bar{\beta}_2 (\bar{\beta}_1 + 12\bar{\beta}_1 - 2) - 1500\bar{\beta}_1 + 24]. \] (37)

Here \( \bar{\beta}'s \) are the constants \( \beta' \)s for the parent distribution and defined by the equations

\[ \bar{\beta}_{2r-2} = \frac{\mu_{2r}}{(\bar{\mu}_{2})^r}, \quad \bar{\beta}_{2r-1} = \frac{\mu_{2r} \mu_{2r+1}}{(\mu_{2})^{r+\frac{1}{2}}}. \]

I deduced this expression, at that time, by the aid of Fisher's formula for \( K (2^\circ) \) and of the theory of semi-invariants.

The expression (37) for \( zM_6 \) can easily be transformed into the form (36) and now I have checked the relation, which I used before in my paper.

If we express \( zM_6 \) in terms of \( \bar{\beta}'s \), we obtain, from the equation (36), the following result:

\[ zM_6 = \frac{1}{N^3} \bar{\sigma}^{10} [3(\bar{\beta}_3 - 1)^3 + \frac{1}{N} \{3\bar{\beta}_6 (\bar{\beta}_1 - 1) - 2\bar{\beta}_4 (\bar{\beta}_1 - 10) + 72\bar{\beta}_3 (\bar{\beta}_1 - 3) - 3\bar{\beta}_2 (\bar{\beta}_3 - 3) - 30\bar{\beta}_3 (\bar{\beta}_1 - 16) + 30\bar{\beta}_3 (\bar{\beta}_1 + 22\bar{\beta}_1 - 23) - 1980\bar{\beta}_1 + 364}], \] (38)

where

\[ \bar{\sigma} = \sqrt{\bar{\mu}_2}. \]

Similarly, as the application of the theory in this chapter, I have deduced the following formulae

\[ zM_6 = \frac{5}{N^3} \bar{\sigma}^{10} \{3(\bar{\beta}_2 - 1)^3 + \frac{1}{N} \{3\bar{\beta}_6 (\bar{\beta}_1 - 1) - 2\bar{\beta}_4 (\bar{\beta}_1 - 10) + 72\bar{\beta}_3 (\bar{\beta}_1 - 3) - 3\bar{\beta}_2 (\bar{\beta}_3 - 3) - 30\bar{\beta}_3 (\bar{\beta}_1 - 16) + 30\bar{\beta}_3 (\bar{\beta}_1 + 22\bar{\beta}_1 - 23) - 1980\bar{\beta}_1 + 364] \}, \] (39)

\[ zM_7 = \frac{105}{N^4} \bar{\sigma}^{14} \{\bar{\beta}_4 (\bar{\beta}_2^2 - 2\bar{\beta}_2 + 1) - \bar{\beta}_3 (3\bar{\beta}_2^2 - 8\bar{\beta}_2 + 6\bar{\beta}_1 - 12\bar{\beta}_1 + 7) - 6\bar{\beta}_1 + 2 \}, \] (40)

\[ zM_8 = \frac{105}{N^4} \bar{\sigma}^{16} (\bar{\beta}_2 - 1)^4, \] (41)

which themselves are useful and are interesting to me, not only as some applications of my theory, but also as the verification of my
own results, used before.

Especially, when the parent distribution is normal,

\[ \bar{\beta}_{2r-1} = 0 \]

for any integer \( r \), and

\[ \bar{\beta}_2 = 3, \quad \bar{\beta}_4 = 15, \quad \bar{\beta}_6 = 105, \quad \bar{\beta}_8 = 945. \]

Therefore, from the equations (38), (39), (40) and (41), we have

\[ 2M_5 = \frac{1}{N^3} \left( 160 + \frac{64}{N} \right) \bar{\sigma}^{10}, \]

\[ 2M_8 = \frac{1}{N^3} \left( 120 + \frac{1720}{N} \right) \bar{\sigma}^{12}, \]

\[ 2M_7 = 3360 \frac{\bar{\sigma}^{11}}{N^4}, \]

and

\[ 2M_9 = 1680 \frac{\bar{\sigma}^{12}}{N^4}, \] (42)

for the case of normal parent distribution.

December, 1930.