On the Expansion of $\theta$ in the Mean-Value Theorem of the Differential Calculus (2nd paper),

by

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1. In the first paper (1) I gave the general term in the expansion of $\theta$ in the mean-value theorem

$$f(x+h)-f(x) = hf'(x+\theta h)$$

in two cases

(i) when $f''(x) \neq 0$,

(ii) when $f''(x) = 0$, $f'''(x) \neq 0$.

The object of the present paper is to study the general case i.e., when $z=x$ is a multiple root of order $m$ of the equation $f'(z)-f'(x) = 0$. It is found that the expressions for the $m$ values of $\theta$ depend on the solution of a rational and integral equation of $m$th degree, the coefficients of which may be obtained from a set of $m$ known symmetric functions of the $\theta'$s.

2. We shall make use of the following inversion theorem:

Let $\phi(z)$ and $\chi(z)$ be two functions analytic on and inside a closed curve $C$ containing a point $z=x$, such that the function $\phi(z)$ takes on the value $\phi(z)=\beta$ (i.e., $\phi(x)$) $m$ times at the point. Suppose that a variable parameter $\alpha$ is so determined that at every point on the perimeter of the curve

$$|\alpha \chi(z)| < |\phi(z)-\beta|,$$

then the equation

$$F(z) = \phi(z) - \beta - \alpha \chi(z) = 0 \quad (2)$$

has $m$ roots in the interior of $C$. Denoting these roots by $\zeta_1, \zeta_2, \ldots, \zeta_m$, it follows that

$$\Psi(\zeta_1) + \Psi(\zeta_2) + \ldots + \Psi(\zeta_m)
= m \left[ \psi(x) + \sum_{n=1}^{\infty} \frac{z^{mn-1} f'(\zeta) \chi'(\zeta)^n}{n! \phi_n(\zeta)^n} \right], \quad z = \lambda (3)$$


(2) Corresponding to an equation of the equivalent form $f(z)=c$, this formula is hinted in a paper by A. C. Dixon (Proc. Lond. Math. Soc., 34, p. 153). For the sake of completeness, we give here a method of obtaining (3).
where
\[(z-x)^m \phi_m(z) = \psi(z) - \beta, \quad (4)\]
\(\psi(z)\) being an analytic function in \(C\).

To prove this, let us consider the integral
\[
\frac{1}{2\pi i} \int_C \frac{\pi(z)}{F(z)} \, dz,
\]
where \(\pi(z)\) is an analytic function in \(C\) and \(F(z)\) as defined in (2).

The function \(\frac{\pi(z)}{F(z)}\) has \(m\) poles in the interior of \(C\) at the points \(\xi_1, \xi_2, \ldots, \xi_m\). We have then
\[
\frac{1}{2\pi i} \int_C \frac{\pi(z)}{F(z)} \, dz = \frac{\pi(\xi_1)}{F'(\xi_1)} + \frac{\pi(\xi_2)}{F'(\xi_2)} + \cdots + \frac{\pi(\xi_m)}{F'(\xi_m)}. \quad (5)
\]
Developing the integral on the left in powers of \(\alpha\), we get also
\[
\frac{1}{2\pi i} \int_C \frac{\pi(z)}{F(z)} \, dz = H_0 + H_1 \alpha + H_2 \alpha^2 + \cdots + H_n \alpha^n + R_{n+1}, \quad (6)
\]

where
\[
H_0 = \frac{1}{2\pi i} \int_C \frac{\pi(z)}{\phi(z) - \beta} \, dz,
\]
\[
H_n = \frac{1}{2\pi i} \int_C \frac{\pi(z) \left\{ \chi(z) \right\}^n}{\left( \phi(z) - \beta \right)^{n+1}} \, dz,
\]
\[
R_{n+1} = \frac{1}{2\pi i} \int C \frac{\pi(z)}{\phi(z) - \beta - \alpha \chi(z)} \left[ \frac{\alpha \chi(z)}{\phi(z) - \beta} \right]^{n+1} \, dz.
\]

Writing \((z-x)^m \phi_m(z)\) for \(\phi(z) - \beta\) in the above, we find the values\(^1\) of
\[
H_0 = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \pi(z) \phi_m(z),
\]
\[
H_n = \frac{1}{(nm+m-1)!} \frac{d^{mn+m-1}}{dz^{mn+m-1}} \frac{\pi(z) \left\{ \chi(z) \right\}^n}{\phi_m(z)^{n+1}},
\]
where in each we replace \(z\) by \(x\) after performing the indicated differentiations.

It may be shown by virtue of the inequality (1) that the integral
\[
(1) \text{ In the particular case, when } m=1, \text{ each of these may be exhibited in an alternative form. For instance}
\]
\[
H_n = \frac{1}{n!} \left( \frac{1}{\phi'(x)} \frac{d}{dx} \right)^n \pi(x) \left\{ \chi(x) \right\}^n.
\]
$R_{n+1}$ approaches zero as $n$ increases indefinitely and the convergence of the series (6) is absolute.

Putting

$$\pi(z) = \Psi(z) F'(z),$$

i.e.,

$$\Psi(z)\{\phi'(z) - \alpha \chi'(z)\}$$

we obtain from (5) and (6) the development of

$$\Psi(\zeta_1) + \Psi(\zeta_2) + \ldots + \Psi(\zeta_m)$$

in an absolutely convergent series

$$K_m,0 + K_m,1 \alpha_1 + \ldots + K_m,n \alpha^n + \ldots,$$

where

$$K_m,0 = \frac{1}{(m-1)!} \frac{d^{m-1} \Psi(z) \phi'(z)}{\phi_m(z)}, \quad z = x,$$

$$K_m,n = \frac{1}{(mn + m-1)!} \frac{d^{mn+m-1} \Psi(z) \phi'(z) \chi(z)^n}{\phi_m(z)^{n+1}}, \quad z = x.$$

For $\phi'(z)$ let us write in the above

$$m(z-x)^{m-1} \phi_m(z) + (z-x)^n \phi_m'(z)$$

as derived from (4); then it is easily seen that

$$K_m,0 = m \Psi(x),$$

$$K_m,n = \frac{m}{(mn)!} \frac{d^{mn-1} \Psi(z) \chi(z)^n}{\phi_m(z)^n}, \quad z = x.$$

Hence the formula (3) follows.

3. Let us now revert to the equation

$$f(x+h) - f(x) = hf'(x + \theta h),$$

where $f(z)$ is analytic in the neighbourhood of $z = x$. Expanding by means of Taylor's series and remembering that $f''(x)$, $f'''(x)$, ..., $f^{(m)}(x)$ all vanish we may write the above in the form

$$f'(x + \theta h) = f'(x) + \lambda_m,$$

where

$$\lambda_m = h^m \frac{f^{(m+1)}(x)}{(m+1)!} + h^{m+1} \frac{f^{(m+2)}(x)}{(m+2)!} + \ldots.$$ 

Let the $m$ values of $\theta$ be denoted by $\theta_1, \theta_2, \ldots, \theta_m$. Comparing (7) with (2) we take $\alpha = \lambda_m$, $\phi(z) = f'(z)$ and $\chi(z) = 1$. The conditions
for (8) being assumed to be satisfied with regard to the equation (7), we get

\[ \psi(x + \theta_1 h) + \psi(x + \theta_2 h) + \ldots + \psi(x + \theta_m h) \]

\[ = m \left[ \psi(x) + \sum_{n=1}^{\infty} \frac{\lambda_m}{(mn)!} \frac{d^{mn-1}}{dz^{mn-1}} \left\{ g_m(z) \right\}^n \right], \quad z = x, \quad (8) \]

where

\[ (z-x)^m g_m(z) = \frac{f^m(z) - f^m(x)}{m!} \]

or

\[ g_m(z) = \frac{f^{m+1}(x)}{m!} + (z-x)^2 \frac{f^{m+2}(x)}{(m+1)!} + \ldots. \]

Let \( m_{\lambda_1, \ldots, \lambda_m} \) denote the coefficient of \( (q+1) \)th term in the expansion of

\[ \lambda_m = \left( \frac{h^m f^{(m+1)}(x)}{(m+1)!} + h^{m+1} \frac{f^{(m+2)}(x)}{(m+2)!} + \ldots \right)^n \]

in ascending powers of \( h \). Substituting then the development of \( \lambda_m^n \) in (8) and collecting coefficients of similar powers of \( h \) we may express (8) in the form

\[ \psi(x + \theta_1 h) + \psi(x + \theta_2 h) + \ldots + \psi(x + \theta_m h) \]

\[ = m \left[ \psi(x) + \sum_{p=1}^{\infty} C_{pm+t} h^{pm+t} \right], \quad (9) \]

where

\[ C_{pm+t} = \sum_{n=1}^{\infty} m_{\lambda_1, \ldots, \lambda_m} \frac{d^{mn-1}}{dz^{mn-1}} \left\{ g_m(z) \right\}^n, \quad z = x, \quad t \]

admitting the values 0, 1, 2, \ldots, \( m-1 \).

Putting in succession

\[ \psi(z) = z - x, \quad (z-x)^2, \quad \ldots, \quad (z-x)^m \]

we deduce from (9) a set of \( m \) symmetric functions involving \( \theta \)'s. Thus the explicit expressions for each of the \( \theta \)'s depends on the solution of a rational and integral equation of \( m \)th degree.

4. In particular, let us put in (9) \( m=2 \) and take \( \psi(z)=z-x, \quad (z-x)^2 \) successively. We then have

\[ (\theta_1 + \theta_2) h = 2 \sum_{p=1}^{\infty} h^{2p+t} \sum_{n=1}^{p} \frac{z A_{\lambda_1, \lambda_2, \ldots, \lambda_m}^{(2p+t-2z)}}{(2n)!} \frac{d^{2n-1}}{dz^{2n-1}} \left\{ g(z) \right\}^n, \quad z = x, \]

\[ (\theta_1^2 + \theta_2^2) h^2 = 2 \sum_{p=1}^{\infty} h^{2p+t} \sum_{n=1}^{p} \frac{z A_{\lambda_1, \lambda_2, \ldots, \lambda_m}^{(2p+t-2n)}}{(2n-2)!} \frac{d^{2n-2}}{dz^{2n-2}} \left\{ g(z) \right\}^n, \quad z = x, \]

where \( t \) is either 0 or 1.
The above formulae are, in fact, less complicated than the corresponding ones of my last paper. It may be noticed that the formula (D) used therein was not in its simplest form.

If, again, we put \( m = 1 \) and take \( \psi(z) = z - x \) we get from (9)

\[
\theta = \sum_{\nu=1}^{\infty} h_{\nu-1} \sum_{n=1}^{\nu} \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \frac{1}{\{g_{\nu}(z)\}^n}, \quad z = x,
\]

which is an alternative form of expressing \((C)\) of my last paper.