Some Properties of a certain Set of Polynomials,

by

H. Bateman, Pasadena, Calif.

1. The generating function.

The polynomial \( F_n(z) \) is interesting analytically on account of
the two operational equations

\[
F_n(D) \operatorname{sech} x = \operatorname{sech} x \cdot P_n(\tanh x),
\]

\[
F_n(D) x \operatorname{sech} x = \operatorname{sech} x \cdot Q_n(\tanh x),
\]

in which \( D \) denotes the operator \( d/dx \) and \( P_n(t), Q_n(t) \) are the two
standard solutions of Legendre's differential equation. The chief
properties of the function may be studied systematically by starting
with a generating function and so the polynomial \( F_n(z) \) will be
defined at the outset by means of the expansion

\[
(1 - t)^α F\left(\frac{1+z}{2}, \frac{1+z}{2}; 1; t^2\right) = \sum_{n=0}^{\infty} t^n F_n(z),
\]

in which \( F(a, b; c; x) \) denotes the hypergeometric function and \(|t| < 1\).

Since

\[
F(a, b; c; t^2) = (1 - t^2)^{c-a-b} F(c-a, c-b; c; t^2),
\]

we have also

\[
(1+t)^{-z} F\left(\frac{1-z}{2}, \frac{1-z}{2}; 1; t^2\right) = \sum_{n=0}^{\infty} t^n F_n(z),
\]

consequently

\[
F_n(-z) = (-1)^n F_n(z).
\]

If \( m \) is a positive integer the last equation indicates that \( F_{2m-1}(0) = 0 \),
also the expansion

\[
F\left(\frac{1}{2}, \frac{1}{2}; 1; t^2\right) = 1 + \left(\frac{1}{2}\right)^2 t^2 + \left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right)t^4 + \ldots.
\]

indicates that

\[
F_{2m}(0) = \frac{1^2 3^2 \ldots (2m-1)^2}{2^2 4^2 \ldots (2m)^2}.
\]

2. Transformation of the generating function.

It may be shown that for sufficiently small values of \(|t|\)

\[
F\left(\frac{1+z}{2}, \frac{1+z}{2}; 1; t^2\right) = (1 - t)^{-z-1} F\left(\frac{1}{2}, \frac{1+z}{2}; 1; -\frac{4t}{(1-t)^2}\right).
\]
The relation is, in fact, a particular case of the following relation due to Gauss\(^{(1)}\)

\[
(1-t)^n F \left( \alpha, \alpha + \frac{1}{2} - \beta; \beta + \frac{1}{2}; t^2 \right) = F \left( \alpha, \beta; 2\beta; -\frac{4t}{(1-t)^2} \right).
\]


Starting with the equation

\[
\sum_{n=0}^{\infty} t^n F_n(z) = \frac{1}{1-t} F \left( \frac{1}{2}, \frac{1+z}{2}; 1; -\frac{4t}{(1-t)^n} \right)
\]

and expanding each term of type \(t^n(1-t)^{-2m-1}\) in the series on the right, it is readily seen that

\[
F_n(z) = 1 - \frac{n(n+1)}{1!1!} \frac{z+1}{2} + \frac{(n-1)n(n+1)(n+2)}{2!2!} \frac{(z+1)(z+3)}{2\cdot4} - \ldots
\]

With the following notation for the generalized hypergeometric series\(^{(2)}\)

\[
F'(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = 1 + \frac{a_1 a_2 \ldots a_p}{b_1 b_2 \ldots b_q} \frac{z}{1!} + \ldots,
\]

\[
F(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; 1) = F \left[ \frac{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \right],
\]

we may write

\[
F_n(z) = F \left( -n, n+1, \frac{1+z}{2}; 1, 1; 1 \right) = F \left[ \begin{array}{c} -n, n+1, \frac{1+z}{2} \\ 1, 1 \end{array} \right],
\]

and when the factorial series is written backwards it takes the form

\[
F \left[ \begin{array}{c} 2\alpha+n, -n, \beta \\ \alpha+\frac{1}{2}, 2\beta \end{array} \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \beta+\frac{1}{2} \right) \Gamma \left( \alpha+\frac{1}{2} \right) \Gamma \left( \beta-\alpha-\frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{1}{2} n \right) \Gamma \left( \alpha+\frac{1}{2} n+\frac{1}{2} \right) \Gamma \left( \beta+\frac{1}{2} n+\frac{1}{2} \right) \Gamma \left( \beta-\alpha-\frac{1}{2} n+\frac{1}{2} \right)}.
\]

\(^{(1)}\) For references see F. J. W. Whipple, “Some transformations of generalized hypergeometric series,” Proceedings of the London Mathematical Society, ser. 2, vol. 26, p. 257 (1927). It is interesting to notice that the value of \(F_{2m}(0)\) may be derived from our factorial series by using Whipple’s formula (8.2), viz.

\[
F \left[ \begin{array}{c} 2\alpha+n, -n, \beta \\ \alpha+\frac{1}{2}, 2\beta \end{array} \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \beta+\frac{1}{2} \right) \Gamma \left( \alpha+\frac{1}{2} \right) \Gamma \left( \beta-\alpha-\frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{1}{2} n \right) \Gamma \left( \alpha+\frac{1}{2} n+\frac{1}{2} \right) \Gamma \left( \beta+\frac{1}{2} n+\frac{1}{2} \right) \Gamma \left( \beta-\alpha-\frac{1}{2} n+\frac{1}{2} \right)}.
\]

\(^{(2)}\) In many papers \(F'\) is replaced by \(\nu F\), the suffixes \(p\) and \(q\) indicating the numbers of the two types of parameters. We prefer here to drop the suffixes as the numbers are sufficiently indicated by the position of the semicolons.
SOME PROPERTIES OF A CERTAIN SET OF POLYNOMIALS.

\[ F_n(z) = \frac{(2n)!}{n! \cdot n! \cdot n!} \frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{1-z-n}{2}\right)} \binom{-n, -n, -n}{-2n, \frac{1-z}{2}, -n} \]  

The equivalence of the two generalized hypergeometric functions is a simple consequence of Thomae's identity (1)

\[ F\left[\begin{array}{c} a, b, c \end{array}; \begin{array}{c} e, f \end{array}\right] = \frac{\Gamma(1-a) \Gamma(e) \Gamma(f) \Gamma(e-b) \Gamma(1+b-a) \Gamma(e)}{\Gamma(c-b) \Gamma(f-b) \Gamma(1+b-a) \Gamma(c)} F\left[\begin{array}{c} b, b-c+1, b-f+1 \end{array}; \begin{array}{c} 1+b-c, 1+b-a \end{array}\right]. \]

The first generating function gives the factorial series

\[ F_n(z) = (-1)^n \left[ \binom{z}{n} + \frac{(z+1)^2}{2^2} \binom{z}{n-2} + \frac{(z+1)^3}{2^3 \cdot 4^3} \binom{z}{n-4} + \ldots \right], \]

which may be expressed in the forms

\[ F_n(z) = (-1)^n \binom{z}{n} F\left[\begin{array}{c} \frac{n}{2}, \frac{1+n}{2}, \frac{1+z}{2} \end{array}; \begin{array}{c} \frac{z+1-n}{2}, \frac{z+2-n}{2} \end{array}\right] \]

\[ F_{im}(z) = F\left[\begin{array}{c} m+\frac{1}{2}, \frac{1+z}{2}, \frac{1-z}{2} \end{array}; \begin{array}{c} 1, 1, \frac{1}{2} \end{array}\right]. \]

The second expression may be derived from the first with the aid of Whipple's relation (2)

\[ F\left[\begin{array}{c} a, b, c, -m \end{array}; \begin{array}{c} u, v, w \end{array}\right] \]

\[ = F\left[\begin{array}{c} u-a, u-b, c, -m \end{array}; \begin{array}{c} 1-v+c-m, 1-w+c-m, u \end{array}\right] \frac{\Gamma(v) \Gamma(w) \Gamma(v-c+m) \Gamma(w-c+m)}{\Gamma(v-c) \Gamma(w-c) \Gamma(v+m) \Gamma(w+m)} \]

in which

\[ u+v+w-a-b-c+m=1. \]

4. Proof of the operational equation.

If \( D \) denotes the operator \( d/dx \) we have

\[ (D+2m-1)(D+2m-3) \ldots (D+1) \text{sech} x = m! (\text{sech} x)^{m+1} e^{-mx}. \]

Hence the factorial series gives the equation


\[ F_n(D) \operatorname{sech} x = \operatorname{sech} x \left[ 1 - \frac{n(n+1)}{1!1!} \left( \frac{1 - \tanh x}{2} \right) \right. \\
+ \frac{(n-1)n(n+1)(n+2)}{2!2!} \left( \frac{1 - \tanh x}{2} \right)^2 + \ldots \] = \operatorname{sech} x P_n(\tanh x).

It should be noticed that the factorial series also indicates that when \( m \) is zero or a positive integer
\[ F_n(-2m-1) = 1 + \binom{m}{1} \frac{n(n+1)}{1!1!} + \binom{m}{2} \frac{(n-1)n(n+1)(n+2)}{2!2!} + \ldots. \]

This expansion shows immediately that \( F_n(-2m-1) \) is the coefficient of \( x^m \) in
\[ (1 + x)^m P_n(1 + 2x) \]
and so
\[ F_n(-2m-1) = \frac{1}{m!} \left[ \frac{d^m}{dx^m} \left\{ (1 + x)^m P_n(1 + 2x) \right\} \right]_{x=0}. \]

Lagrange's theorem now indicates that for sufficiently small values of \( |t| \)
\[ \frac{1}{1-t} P_n \left[ 1 + \frac{2t}{1-t} \right] = \sum_{m=0}^{\infty} t^m F_n(-2m-1), \]
i.e.
\[ \frac{1}{1-t} P_n \left[ \frac{1+t}{1-t} \right] = \sum_{m=0}^{\infty} t^m F_n(-2m-1). \]

This equation should be compared with the equation
\[ (1-t)^n(1+t)^{-n-1} P_n \left( \frac{1+t^2}{1-t^2} \right) = \sum_{m=0}^{\infty} t^m F_n(2n+1), \]
which is obtained by transforming the generating function with the aid of the relation (1)
\[ P_n(\mu) = \left( \frac{2}{\mu+1} \right)^{n+1} F \left( n+1, n+1; 1; \frac{\mu-1}{\mu+1} \right). \]

To prove the relation
\[ F_n(D) x \operatorname{sech} x = \operatorname{sech} x. Q_n(\tanh x) \]
we use Christoffel's equation
\[ Q_n(\tanh x) = x P_n(\tanh x) - \frac{2n-1}{1(n)} P_{n-1}(\tanh x) \]
\[ - \frac{2n-5}{3(n-1)} P_{n-3}(\tanh x) - \ldots, \]

(1) E. W. Hobson, Spherical and ellipsoidal harmonics, p. 100.
which indicates that we must establish the formula

$$F_n'(D) = -\frac{2n-1}{1(n)} F_{n-1}(D) - \frac{2n-5}{3(n-1)} F_{n-3}(D)$$

$$- \frac{2n-9}{5(n-2)} F_{n-5}(D) - \ldots.$$  

This can be done by induction with the aid of the first recurrence relation of § 5. This proof of the operational equation is then completed with the aid of the formula

$$F_n(D) \text{sech } x = x F_n(D) \text{sech } x + F_n'(D) \text{sech } x.$$  

The truth of the formula

$$F_{2m}(z) = 1 - \binom{m}{1} \frac{2m+1}{1! 2!} \frac{1^2 - z^2}{2}$$

$$+ \binom{m}{2} (2m+1)(2m+3) \frac{(1^2 - z^2)(3^2 - z^2)}{2! 4!} - \ldots.$$

follows immediately from the fact that

$$(1^2 - D^2)(3^2 - D^2) \ldots [(2m-1)^2 - D^2] \text{sech } x = (2m)! \text{sech } x^{2m+1},$$

$$P_{2m} \text{sech } x = 1 - \binom{m}{1} \frac{2m+1}{2} \text{sech } x$$

$$+ \binom{m}{2} \frac{(2m+1)(2m+3)}{2 \cdot 4} \text{sech } x - \ldots.$$

$$= F\left(-m, m + \frac{1}{2}; 1; \text{sech } x\right).$$

There is a related formula

$$z F_{2m+1}(z) = -1 - \sum_{s=1}^{m+1} \frac{m!}{s! (m+1-s)!} \frac{(2m+3) \cdots (2m+2s-1)}{s! (2s)!} M_s,$$

where

$$M_s = [(m+1)(2m+1) + s(4m^2 + 6m + 3)] \frac{(z^2 - 1^2) \cdots (z^2 - (2s-1)^2)}{2 \cdot 4 \cdots (2s)}.$$  

The formula for $F_{2m}(z)$ may be obtained by using Sheppard's modification of Bessel's formula of interpolation¹. Since $F_{2m}(z)$ is an even function this formula is

$$F_{2m}(z) = F_{2m}(1) + \frac{z^2 - 1^2}{2^2 2!} [F_{2m}(3) - F_{2m}(1)]$$

$$+ \frac{(z^2 - 1^2)(z^2 - 3^2)}{2^4 4!} [F_{2m}(5) - 3F_{2m}(3) + 3F_{2m}(1) - F_{2m}(-1)].$$

28 H. BATEMAN:  

\[ \frac{(z^2-1^3)(z^2-3^3)(z^2-5^3)}{2^6 \cdot 6!} [F_{2m}(7) - 5F_{2m}(5) + 10F_{2m}(3) - 10F_{2m}(1) \]
\[ + 5F_{2m}(-1) - F_{2m}(-3)] \ldots . \]

The formula indicates, moreover, that if \( s \leq m \)

\[ F_{2m}(2s+1) - \binom{2s-1}{1} F_{2m}(2s-1) + \binom{2s-1}{2} F_{2m}(2s-3) - \ldots . \]

\[ = \binom{m}{s} \binom{2s}{s} (2m+1)(2m+3) \ldots (2m+2s-1). \]

On the other hand, the series for \( \frac{1}{1-t} \) indicates that when \( s \geq p > n \)

\[ F_n(2s+1) - \binom{p}{1} F_n(2s-1) + \binom{p}{2} F_n(2s-3) - \ldots . \]

\[ + (-1)^p F_n(2s-2p+1) = 0. \]

Testing this result with the aid of the series for \( F_n(-2m-1) \) we see that it is a consequence of the identity

\[ 0 = \binom{s}{r} - \binom{p}{1} \binom{s-1}{r} + \binom{p}{2} \binom{s-2}{r} - \ldots . \]

\[ \ldots + (-1)^p \binom{s-p}{r}, \quad s \geq p > r, \]

which, when it is written in the form \( F(-p, r-s; -s; 1) = 0, \) is seen to be a consequence of the formula given by Gauss for the sum of a hypergeometric series with argument unity.

5. Recurrence relations.

With the aid of the expansion in series of factorials it may be shown that the polynomial \( F_n(z) \) satisfies the following difference equations or recurrence relations

\[ (n+1)^2 F_{n+1}(z) = n^2 F_{n-1}(z) - (2n+1)z F_n(z), \]

\[ (z+1)^2 \left[ F_n(z+2) - F_n(z) \right] + (z-1)^2 \left[ F_n(z-2) - F_n(z) \right] = 4n(n+1) F_n(z), \]

\[ 2z^2 F_n(z-1) = [z^2 + (z-2n)^2] F_n(z+1) + 4n^2 F_{n-1}(z+1), \]

\[ 2z^2 F_{n-1}(z-1) = 4n^2 F_n(z+1) + [z^2 + (z+2n)^2] F_{n-1}(z+1), \]

\[ 2z^2 F_n(z+1) = [z^2 + (z+2n)^2] F_n(z-1) - 4n^2 F_{n-1}(z-1), \]

\[ 2z^2 F_{n-1}(z+1) = [z^2 + (z-2n)^2] F_{n-1}(z-1) - 4n^2 F_n(z-1). \]

It may also be shown that

\[ F_n(z-2) - F_n(z) = \frac{n(n+1)}{1! \cdot 1!} - \frac{(n-1)n(n+1)(n+2)}{2! \cdot 2!} \frac{z+1}{2} \]
SOME PROPERTIES OF A CERTAIN SET OF POLYNOMIALS. 29

\[ \frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{3 \cdot 4} \]

and that
\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} [F_n(z-2) - F_n(z)] = \frac{2t}{(1-t)^2} F\left( \frac{3}{2}, \frac{z+1}{2}; \frac{2}{2} \; ; \; \frac{-4t}{(1-t)^2} \right). \]

6. **Definite integrals for** \( F_n(z) \).

The well known definite integral for the hypergeometric series gives the formula (in which \( t > \sqrt{8} - 3 \))
\[ \frac{\pi}{1+t} F\left( \frac{1}{2}, \frac{1+z}{2}; 1; \frac{4t}{(1+t)^2} \right) \sec \frac{\pi z}{2} = \int_{0}^{1} u^{\frac{z-1}{2}} (1-u)^{-\frac{z+1}{2}} [(1+t)^2 - 4tu]^{-\frac{1}{2}} du \]
and by equating coefficients of \( t^n \) we obtain the equation
\[ F_n(z) \sec \left( \frac{\pi z}{2} \right) = \frac{1}{\pi} \int_{0}^{1} u^{\frac{z-1}{2}} (1-u)^{-\frac{z+1}{2}} P_n(1-2u) \; du, \quad (-1 < z < 1). \]

A related integral is obtained by starting with the formula
\[ \text{sech } x = \frac{1}{2} \int_{-\infty}^{\infty} e^{t x} \text{sech} \left( \frac{\pi z}{2} \right) \; dz, \]
in which we shall suppose \( z \) to be real. Operating on both sides with \( F_n\left( \frac{d}{dx} \right) \) we find that
\[ \text{sech } x P_n(\tanh x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{t x} \text{sech} \left( \frac{\pi z}{2} \right) F_n(iz) \; dz. \]

Inverting the integral by Fourier's theorem we obtain the formula (1)
\[ F_n(iz) \text{sech} \left( \frac{\pi z}{2} \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-t x} \text{sech } x \; P_n(\tanh x) \; dx. \]

By a change of variable the operational equation may be written in the form
\[ F_n\left( i \frac{d}{ds} \right) \cosec s = \cosec \text{ } s \; P_n(i \cot s). \]

Combining this with Euler's formula
\[ \cosec s = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{t z}}{1 + e^z} \; dt = \int_{-\infty}^{\infty} \frac{e^{st}}{1 + e^{s t}} \; dt \quad (0 < s < \pi) \]
we obtain

(1) This formula may be verified by noting that \( e^{-t x} F_n\left( \frac{d}{dx} \right) \text{sech } x - \text{sech } x \times F_n\left( -\frac{d}{dx} \right) e^{-t x} \) is the exact differential of a function which vanishes at both limits.
Another definite integral for $F_n(z)$ may be derived from the factorial series of section 3 by making use of Cauchy's integral

$$
\frac{\Gamma\left(\frac{z+1}{2}+n\right)}{\Gamma(n+1)\Gamma\left(\frac{z+1}{2}\right)} = \frac{2}{\pi} \int_0^\pi (2\cos \theta)^{n+\frac{z-1}{2}} \cos \left(\left(n+1-\frac{z}{2}\right)\theta\right) d\theta,
$$

which is valid for $z > -1$. The result is that for these values of $z$

$$
F_n(z) = (-1)^n \frac{2}{\pi} R \int_0^\pi (1+e^{-i\theta})^{\frac{z+1}{2}} P_n(1+2e^{i\theta}) d\theta,
$$

where the symbol $R$ is used to denote the real part of the expression on the right. The result may be rewritten by writing $-x$ in place of $z$ and putting $\phi = 2\theta$. Since $F_n(-x) = (-1)^n F_n(x)$, we have the formula

$$
F_n(x) = \frac{1}{\pi} R \int_0^\pi (1+e^{-i\phi})^{-\frac{x+1}{2}} P_n(1+2e^{i\phi}) d\phi,
$$

$x < 1$.

When $x < -1$ the binomial series converges absolutely and we can write

$$
F_n(x) = \frac{1}{2\pi} \int_0^{2\pi} (1+e^{-i\phi})^{-\frac{x+1}{2}} P_n(1+2e^{i\phi}) d\phi.
$$

Since

$$
x \sech x = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{\partial}{\partial z} \left(e^{iz} \sech \left(\frac{\pi z}{2}\right)\right) dz
$$

$$
= \frac{\pi}{4i} \int_{-\infty}^{\infty} e^{iz} \sech \left(\frac{\pi z}{2}\right) \tanh \left(\frac{\pi z}{2}\right) dz,
$$

we have the equation

$$
\sech x \cdot Q_n(\tanh x) = \frac{\pi}{4i} \int_{-\infty}^{\infty} e^{iz} F_n(iz) \sech \left(\frac{\pi z}{2}\right) \tanh \left(\frac{\pi z}{2}\right) dz,
$$

which, on inversion, gives rise to the equation

$$
F_n(iz) \sech \left(\frac{\pi z}{2}\right) \tanh \left(\frac{\pi z}{2}\right) = \frac{2i}{\pi^2} \int_{-\infty}^{\infty} e^{-iz} Q_n(\tanh x) \sech x dx,
$$

which may be verified in a device similar to that used in verifying the formula for $F_n(iz) \sech \left(\frac{\pi z}{2}\right)$. There is a corresponding equation
which may be established with the aid of the recurrence relation satisfied by $F_n(z)$.

7. The sign of $F_n(z)$ when $z$ is real.

The first recurrence relation may be used to prove step by step that when $z$ is negative $F_n(z)$ is positive for all values of $n$ and that when $z$ is positive $F_n(z)$ is positive when $n$ is even and negative when $n$ is odd. These statements are evidently true when $n=0$ and when $n=1$, for $F_0(z)=1$ and $F_1(z)=-z$; they are thus true for $n=2$, then for $n=3$ and so on.

This recurrence relation gives additional information when $|z|<1$, for then $|F_0(z)|=1$, $|F_1(z)|<1$ and we find by induction that

$$|F_n(z)| \leq 1.$$ 

With the aid of the inequality and the fifth recurrence relation an inequality may be obtained for the case in which $z$ lies in the interval $1<z<3$, this inequality may be used to find one for the case in which $z$ lies in the interval $3<z<5$ and so on.

The analysis in Section 10 indicates that the coefficients in the power series for $F_{2n}(z)$ and $-F_{2n+1}(z)$ are all positive. We may infer, then that when $|z|<1$ we have the inequality

$$|F_n(iz)| \leq 1.$$ 

8. Expansion of the exponential function.

When $a>0$ and $-1<z<1$ we may write

$$\sum_{n=0}^{\infty} (2n+1) \coth a \cdot Q_n(\coth a) F_n(z) \sec \left( \frac{\pi z}{2} \right)$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{2n+1} {\pi} \coth a \cdot Q_n(\coth a) \int_0^1 u^{\frac{z-1}{2}} (1-u)^{-\frac{z+1}{2}} Q_n(1-2u) \, du$$

$$= \frac{1}{\pi} \int_0^1 u^{\frac{z-1}{2}} (1-u)^{-\frac{z+1}{2}} \frac{du}{e^{-a} + 2u \sinh a}$$

on account of Heine's expansion. The integral is known to have the value $e^{-az} \sec \left( \frac{\pi z}{2} \right)$ and so

$$e^{-az} = \sum_{n=0}^{\infty} (2n+1) \coth a \cdot Q_n(\coth a) F_n(z).$$

Changing the sign of $z$ and making use of the fact that $Q_n(x)$ is an
odd function of \( z \) when \( n \) is even and an even function of \( z \) when \( n \) is odd, we see that the foregoing equation holds also when \( a \) is negative. When \( z = \pm 1 \) the equation still holds because, by Heine's expansion, if \( a > 0 \)

\[
e^{-a} = \sum_{n=0}^{\infty} (-1)^n (2n + 1) \text{cosech} \ a \cdot Q_n(\coth a),
\]

\[
e^a = \sum_{n=0}^{\infty} (2n + 1) \text{cosech} \ a \cdot Q_n(\coth a).
\]

The absolute convergence of the series for \( e^{-az} \) when \(-1 < z < 1\) may be deduced from the last result by using the inequalities

\[Q_n(\coth a) > 0, \quad |F_n(z)| < 1.\]

On account of these results the validity of the expansion for \( e^{-az} \) can, when \( a > 0 \), be extended to odd integral values of \( z \) by making use of the fact that the function \( f = e^{-az} \) satisfies the equation

\[4 \sinh^2 a \left( \frac{\partial^2 f}{\partial a^2} + f \right) + 8 \sinh a \cosh a \frac{\partial f}{\partial a} = (z + 1)^q [f(a, z + 2) - f(a, z)]
\]
\[+ (z - 1)^q [f(a, z - 2) - f(a, z)].\]

When \( z < -1 \) the series for the exponential function may be studied with the aid of the formula at the end of Section 6. When \( |a| \) is sufficiently large the point \( 1 + 2i \phi \) lies within the ellipse with \(-1, 1\) as foci which passes through the point \( a \), consequently Heine's series converges uniformly in \( \phi \) and we may write

\[
\sum (2n + 1) \text{cosech} \ a \cdot Q_n(\coth a) F_n(z)
\]
\[= \frac{1}{2\pi} \sum_{n=0}^{\infty} (2n + 1) \text{cosech} \ a \cdot Q_n(\coth a) \int_{0}^{2\pi} (1 + e^{-i\phi})^{-\frac{z+1}{2}} P_n(1 + 2e^{i\phi}) \, d\phi
\]
\[= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1 + e^{-i\phi})^{-\frac{z+1}{2}}}{\cosh a - \sinh a (1 + 2e^{i\phi})} \, d\phi
\]
\[= e^{-az}.
\]

By changing the signs of \( z \) and \( a \) this result may be extended to the case in which \( z > 1 \).

Other expansions for the exponential function may be derived from Fourier integrals of Section 6. The results are

\[e^{-iax} = \sum_{n=0}^{\infty} (2n + 1) \text{cosech} \ a \cdot Q_n(\coth a) F_n(ix),
\]
\[e^{-izx} \cosh x \cosh \left( \frac{\pi z}{2} \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} (2n + 1) F_n(iz) P_n(\tanh x).\]
When $z=0$ the last series takes a well known form (1)
\[
\cosh x = \frac{\pi}{2} \sum_{m=0}^{\infty} \frac{(4m+1) \frac{1^2 \cdot 3^2 \cdots \cdot (2m-1)^2}{2^2 \cdot 4^2 \cdots \cdot (2m)^2} P_{2m}(\tanh x)}{2m+1}.
\]
The first definite integral for $F_n(z) \sec \left( \frac{\pi z}{2} \right)$ gives the related expansion
\[
e^{-z} \cosh x \cos \left( \frac{\pi x}{2} \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1) F_n(z) P_n(\tanh x)
\]
when $-1 < z < 1$, $-\infty < x < \infty$.


When $m$ is a positive integer we have the expansion
\[
(1-x)^m = \sum_{n=0}^{m} (-1)^n \frac{(2n+1) (m!)^2 2^m}{(m-n)! (m+n+1)!} P_n(x).
\]
Writing $x = 1 + 2\epsilon^x$, multiplying by $(1+\epsilon^{-1}x) \frac{e^{x}}{2}$ and integrating with respect to $\phi$ between 0 and $2\pi$ we obtain the expansion
\[
\frac{(z+1)(z+3) \cdots (z+2m-1)}{1 \cdot 2 \cdots m} = \sum_{n=0}^{m} (-1)^n \frac{(2n+1) (m!)^2 2^m}{(m-n)! (m+n+1)!} F_n(z)
\]
\[
= \sum_{n=0}^{m} c_m^{(n)} F_n(z), \text{ say.}
\]
If, then, a function $f(z)$ can be expanded in the series
\[
f(z) = a_0 + a_1 (z+1) + a_2 (z+1)(z+3) + \ldots.
\]
we shall also have a formal expansion
\[
f(z) = \sum_{n=1}^{\infty} b_n F_n(z),
\]
in which
\[
b_n = \sum_{m=n}^{\infty} m! a_m c_m^{(n)}.
\]
In particular, since when $w > z$
\[
\frac{1}{w-z} = \frac{1}{w+1} + \frac{z+1}{(w+1)(w+3)} + \frac{(z+1)(z+3)}{(w+1)(w+3)(w+5)} + \ldots,
\]
we may write
\[
\frac{1}{w-z} = \sum_{n=0}^{\infty} b_n(w) F_n(z),
\]

(1) E. Heine, Handbuch der Kugelfunctionen, Bd. 1, p. 85.
where

\[ B_n(w) = \sum_{m=0}^{\infty} \frac{m! \ c_m^{(n)}}{(w+1)(w+3) \ldots (w+2m+1)}. \]

A series for \( \frac{1}{w+z} \) may also be obtained by multiplying our expansion for \( e^{-az^2} \) by \( e^{-aw} \) and integrating between 0 and \( \infty \). A comparison of the two expansions gives the formula

\[ \int_0^\infty e^{-aw} \cosech a \cdot Q_n(\coth a) \, da = \sum_{m=0}^{\infty} \frac{(m!)^2 2^m}{(m-n)! (m+n+1)! (w+1)(w+3) \ldots (w+2m+1)} \]

This relation and the associated relation

\[ \int_0^\infty e^{-aw} \cosech a \cdot \coth a \cdot Q_{n+1}(\coth a) \, da = \sum_{m=0}^{\infty} \frac{(m!)^2 2^m}{(m-n)! (m+n+3)! (w+1)(w+3) \ldots (w+2m+1)} \]

may be established by induction, use being made of the particular relations in which \( n \) has the values 0 and 1 and in which

\[ Q_0(\coth a) = a, \quad Q_1(\coth a) = a \coth a - 1. \]

When use is made of Heine's expansion the first relation gives (1)

\[ \int_0^\infty e^{-aw} \cosh a - \mu \sinh a \, da = \sum_{m=0}^{\infty} \frac{m! (1+\mu)^n}{(w+1)(w+3) \ldots (w+2m+1)}. \]

Equating coefficients of \( \mu^n \) on the two sides of this equation we obtain the relation

\[ \int_0^\infty e^{-aw} \sech a \cdot (\tanh a)^n \, da = \sum_{m=0}^{\infty} \frac{m! m!}{n! (m-n)! (w+1)(w+3) \ldots (w+2m+1)}. \]

There is an associated equation

\[ \int_0^\infty e^{-aw} (\tanh a)^n \, da = \sum_{m=0}^{\infty} \frac{m! m! (w+m+1)}{n! (m-n)! w(w+2) \ldots (w+2m+2)}. \]

If we denote the last integral by \( T_n(w) \) and the previous one by

\[ \begin{align*}
(1) \text{ The integral is a particular case of one considered by T. J. Stieltjes, Quarterly Journal of Mathematics, vol. 24, p. 374 (1890). One of his formulas gives the expansion} \\
\frac{1}{w-\mu} = \frac{1(1-\mu^2)}{w-\mu} + \frac{2(1-\mu^2)}{w-3\mu} + \frac{3(1-\mu^2)}{w-5\mu} + \ldots \quad \text{for} \ w > 0, \ -1 \leq \mu \leq 0.
\end{align*} \]
SOME PROPERTIES OF A CERTAIN SET OF POLYNOMIALS.

$S_n(w)$, it is readily seen that we have the relations

\[ wT_0(w) = 1, \quad wT_n(w) = n [ T_{n-1}(w) - T_{n+1}(w) ], \quad n > 0 \]
\[ S_n(w) = (n - w - 1) S_{n-1}(w) - (n - 1) S_{n+1}(w), \quad n > 1 \]
\[ S_1(w) = 1 - wS_0(w), \]
\[ S_n(w + 1) = T_n(w) - T_{n+1}(w), \]
\[ S_n(w - 1) = T_n(w) + T_{n+1}(w), \]
\[ T_n(w) = \sum_{m=0}^{\infty} \frac{m! (m-1)!}{(n-1)! (m-n)!} \frac{1}{w(w+2) \ldots (w+2m+1)}, \quad n \geq 1 \]
\[ S_0(w) = \frac{1}{w} + \frac{1}{w} + \frac{2}{w} + \frac{3}{w} + \ldots. \]

The definite integral

\[ U_n(w) = \int_0^\infty e^{-aw} \text{sech} \, a \, P_n(\tanh a) \, da, \]

which is equal to $S_n(w)$ when $n = 0$, may be calculated by expanding $P_n(\tanh a)$ in powers of tanh $a$. It is thus found that

\[ U_n(w) = \sum_{m=0}^{\infty} \frac{m! \alpha_n^m}{(w+1)(w+3) \ldots (w+2m+1)}, \]

where $\alpha_n^m$ is the coefficient of $\mu^m$ in $(1 + \mu)^m P_n(\mu)$.

The function $U_n(w)$ satisfies the recurrence relation

\[ (n+1)^2 U_{n+1}(w) - n^2 U_{n-1}(w) + (2n+1)w U_n(w) = (2n+1) P_n(0), \]

from which it is seen that \(^{(1)}\)

\[ U_n(w) = F_n(w) S_0(w) - V_{n-1}(w), \]

where $V_{n-1}(w)$ is a polynomial of degree $n - 1$; when

\[ F_n(w) \sum_{m=0}^{n-1} \frac{m!}{(w+1)(w+3) \ldots (w+2m+1)} \]

is expressed in partial fractions, $V_{n-1}(w)$ represents the remainder.

10. The power for $F_n(z)$.

The coefficients in the power series for $F_n(z)$ may be determined

\(^{(1)}\) This result may also be obtained by noticing that

\[ e^{-aw} \text{sech} \, a \, F_n(w) = \text{sech} \, a \, F_n \left( -\frac{d}{da} \right) e^{-aw} \]
\[ = e^{-aw} F_n \left( \frac{d}{da} \right) \text{sech} \, a + \frac{d}{da} R_{n-1} \{ e^{-aw}, \text{sech} \, a \}, \]

where $R_{n-1} \{ y, z \}$ involves $y, z$ and their derivatives up to order $n - 1$. 

by means of the difference equation of § 5. Writing

\[ F_{2m}(z) = \frac{1^2.3^2 \cdots (2m-1)^2}{2^2.4^2 \cdots (2m)^2} G_{2m}(z), \]

\[ F_{2m+1}(z) = -\frac{2^2.4^2 \cdots (2m)^2}{1^2.3^2 \cdots (2m+1)^2} G_{2m+1}(z), \]

we have the difference equations

\[ G_{2m+1}(z) - G_{2m-1}(z) = (4m+1)z \cdot \frac{1^4.3^4 \cdots (2m-1)^4}{2^4.4^4 \cdots (2m)^4} G_{2m}(z), \]

\[ G_{2m+2}(z) - G_{2m}(z) = (4m+3)z \cdot \frac{2^4.4^4 \cdots (2m)^4}{1^4.3^4 \cdots (2m+1)^4} G_{2m+1}(z), \]

which give

\[ G'_{2m+1}(0) = 1 + 5 \frac{1^4}{2^4} + 9 \frac{1^4.3^4}{2^4.4^4} + \cdots + (4m+1) \frac{1^4.3^4 \cdots (2m-1)^4}{2^4.4^4 \cdots (2m)^4}, \]

\[ G'_{2m}(0) = 2 \left[ 3 + 7 \left( 1 + 5 \frac{1^4}{2^4} \right) \frac{2^4}{1^4.3^4} + \cdots \right. \]

\[ + (4m-1) \frac{2^4 \cdots (2m-2)^4}{1^4 \cdots (2m-1)^4} \left( 1 + 5 \frac{1^4}{2^4} + \cdots \right) \]

and so on. The value of \( G'_{2m+1}(0) \) may also be calculated by means of the Fourier integral of § 6. This leads to the formula

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} x \text{sech} x P_{2m+1}(\tanh x) \, dx \]

\[ = \frac{2^2.4^2 \cdots (2m)^2}{1^2.3^2 \cdots (2m+1)^2} \left[ 1 + 5 \frac{1^4}{2^4} + \cdots + (4m+1) \frac{1^4.3^4 \cdots (2m-1)^4}{2^4.4^4 \cdots (2m)^4} \right]. \]

If, on the other hand, we calculate \( F'_{n}(0) \) from the expansion

\[ F_{n}(z) = (-1)^{n} \left[ \binom{z}{n} + \frac{(z+1)^2}{2^2} \binom{z}{n-2} + \frac{(z+1)^2(z+3)^2}{2^2.4^2} \binom{z}{n-4} + \cdots \right], \]

we find that(1)

\[ G'_{2m+1}(0) = \frac{1^2.3^2 \cdots (2m+1)^2}{2^2.4^2 \cdots (2m)^2} \left[ \frac{1}{2m+1} + \frac{1^2}{2^2} \frac{1}{2m-1} \right. \]

\[ + \frac{1^2.3^2}{2^2.4^2} \frac{1}{2m-3} + \cdots + \frac{1^2.3^2 \cdots (2m-1)^2}{2^2.4^2 \cdots (2m)^2} \left. \right], \]

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} x \text{sech} x P_{2m+1}(\tanh x) \, dx = \frac{1}{2m+1} + \frac{1^2}{2^2} \frac{1}{2m-1}. \]

(1) Still another expression for \( G'_{2m+1}(0) \) may be obtained from the series for \( F'_{n}(z) \) given in § 4.
SOME PROPERTIES OF A CERTAIN SET OF POLYNOMIALS.

To determine the power series for the function \( F_n(iz) \text{ sech} \left( \frac{\pi z}{2} \right) \) from the Fourier integral we need the value of the integral

\[
C_n = \int_{-\infty}^{\infty} x^n \text{ sech} x (\tanh x)^n \, dx,
\]

which may be calculated step by step with the aid of the reduction formula

\[
(p + 1) \ C_p = (q + 1) \ C_{p+1}^{q+1} - q \ C_p^q,
\]

\[
C_0^0 = 0, \quad C_{2m+1}^0 = 0,
\]

\[
C_{2m}^0 = \frac{4}{2m+1} \left[ 1 - \left( \frac{1}{3} \right)^{2m+1} + \left( \frac{1}{5} \right)^{2m+1} - \cdots \right].
\]

It is thus found that

\[
q! \ C_p^q = p(p-1) \ldots (p-q+1) \ C_{p-q}^0
\]

\[
+ N_{q(1)} p(p-1) \ldots (p-q+3) \ C_{p-q+2}^0
\]

\[
+ N_{q(2)} p(p-1) \ldots (p-q+5) \ C_{p-q+4}^0
\]

\[
+ \cdots,
\]

where the coefficients \( N_{q}^{(a)} \) satisfy the relations

\[
q^2 N_{q-1}^{(a)} = N_{q+1}^{(a+1)} - N_{q+1}^{(a+1)}, \quad N_{q}^{(0)} = 1,
\]

which give

\[
N_{q}^{(1)} = 1^2 + 2^2 + \cdots + (q-1)^2,
\]

\[
N_{q}^{(2)} = (q-1)^2 [1^2 + 2^2 + \cdots + (q-3)^2] + (q-2)^2 [1^2 + 2^2 + \cdots + (q-4)^2] + \cdots
\]

and so on. Hence if the summation with respect to \( q \) is over values for which \( n-q \) is zero or a positive integer we have

\[
F_n(iz) \text{ sech} \left( \frac{\pi z}{2} \right)
\]

\[
= \frac{1}{\pi} \frac{2n!}{2^n \cdot n!} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{n-q} \frac{(-iz)^n}{n!} \frac{(p-q)_{n-q}}{2.4 \ldots (n-q)(2n-1)(2n-3) \ldots (n+q+1)} \ f(p,q)
\]

where

\[
f(p,q) = \frac{C_{p-q}^0}{(p-q)!} + N_{q}^{(1)} \frac{C_{p-q+2}^0}{(p-q+2)!} + N_{q}^{(2)} \frac{C_{p-q+4}^0}{(p-q+4)!} + \cdots.
\]

But

\[
\frac{1}{\pi} \sum_{s=0}^{\infty} \frac{(-iz)^s}{s!} C_s = \text{ sech} \left( \frac{\pi z}{2} \right).
\]
Therefore

\[ F(z) = \sum_{q} \frac{(-iz)^q}{2.4.\ldots\ldots(n-q) q! q!} \left[ (\sum_{q} (-1)^{n-q} \frac{(n-q-1)!}{(n-q)!} \right] \]

Hence finally

\[ F_n(x) = \sum_{q} (-1)^{\frac{n+q}{2} x^q} \frac{1.3\ldots\ldots(n+q-1)(n+q-3)}{2.4.\ldots\ldots(n-q) q! q!} \left[ \frac{1}{q! q!} \right] \]

where \(2.4\ldots\ldots(n-q)\) is to be interpreted as \(2^{\frac{n-q}{2}}\left(\frac{n-q}{2}\right)!\) when \(q = n\).