On a Generalized Lagrange's Formula of Interpolation and Some Class of Functions Defined by Dirichlet's Series,

by

Satoru Takenaka, Osaka.

1. Let \( \{z_n\} \) be a sequence of complex numbers with the property that

\[
(1.1) \quad z_0 \neq 0, \quad \left| \frac{z_{n+1}}{z_n} \right| > \lambda > 1, \quad (n = 0, 1, 2, \ldots),
\]

and \( \{A_n\} \) be another sequence of complex constants such that

\[
(1.2) \quad \sqrt[\lambda]{|A_n|} < \lambda^{\frac{1}{2}(n+1)-\varepsilon}, \quad (\varepsilon > 0, \ n \geq N)
\]

where \( N \) is an integer sufficiently large.

Then we have the following theorem:

**Theorem I.** If we put

\[
(1.3) \quad \phi(z) = \prod_{\nu=0}^{\infty} \left(1 - \frac{z}{z_{\nu}} \right),
\]

the function defined by

\[
(1.4) \quad f(z) = \sum_{n=0}^{\infty} \frac{\phi(z) \cdot A_n}{(1 - \frac{z}{z_n}) \phi'(z_n)}
\]

is an integral transcendental function of genus zero with the property that

\[
f(z_n) = A_n, \quad (n = 0, 1, 2, \ldots).
\]

**Proof.** From (1.1), we have

\[
|z_n| > \lambda |z_{n-1}| > \lambda^2 |z_{n-2}| > \ldots > \lambda^n |z_0|, \quad (n = 1, 2, \ldots),
\]

so that the function \( \phi(z) \) defined by (1.3) is an integral transcendental function of genus zero.

Putting

\[
M(z) = \prod_{\nu=0}^{\infty} \left(1 + \frac{|z|}{\lambda^\nu |z_0|} \right),
\]

we get

\[
\left| \frac{\phi(z)}{(1 - \frac{z}{z_n}) \phi'(z_n)} \right| < \frac{M(z)}{\phi'(z_n)}.\]
On the other hand, we have
\[ \frac{1}{|\phi'(z_n)|} = \frac{1}{\prod_{\nu=n}^{\infty} \left(1 - \frac{z_n}{z_\nu}\right)} \]
\[ \leq \frac{|z_0 z_1 \ldots z_{n-1}|}{|z_n|^n \left|\left(\frac{z_n - 1}{z_n}\right) \ldots \left(\frac{z_{n-1} - 1}{z_n}\right)\right| \prod_{\nu=n+1}^{\infty} \left(1 - \frac{z_n}{z_{\nu+1}}\right) \prod_{\nu=n+2}^{\infty} \left(1 - \frac{z_n}{z_{\nu+2}}\right)} \]
\[ \leq \frac{1}{\lambda^n \lambda^{n-1} \ldots \lambda \prod_{\nu=1}^{n} \left(1 - \frac{1}{\lambda^\nu}\right)^{\infty}} \left(1 - \frac{1}{\lambda^\nu}\right) \]

or
\[ \frac{1}{|\phi'(z_n)|} \leq \{\pi(\lambda)\}^\frac{n(n+1)}{2} \]
where
\[ \pi(\lambda) = \prod_{\nu=1}^{n} \left(1 - \frac{1}{\lambda^\nu}\right)^{-1} \]

Therefore we get
\[ |\frac{\phi(z)}{\left(1 - \frac{z}{z_n}\right) \phi'(z_n)}| = \{\pi(\lambda)\}^\frac{n(n+1)}{2} M(z) \lambda^{-\frac{n(n+1)}{2}} \]

From this inequality together with (1.2), it follows that
\[ |\frac{\phi(z) \cdot A_n}{\left(1 - \frac{z}{z_n}\right) \phi'(z_n)}| \leq |\pi(\lambda)|^2 M(z) \lambda^{-n+1} \]

so that, for any finite value of \(z\),
\[ \lim_{n \to \infty} \left|\frac{\phi(z) \cdot A_n}{\left(1 - \frac{z}{z_n}\right) \phi'(z_n)}\right| \leq \lambda^{-\varepsilon} < 1, \]
from which we can conclude that the function \(f(z)\) defined by (1.4) is an integral transcendental function having the property that
\[ f(z_n) = A_n, \quad (n = 0, 1, 2, \ldots). \]

It is remarkable that the series on the right hand side of (1.4) is convergent absolutely and uniformly for \(|z| \leq R\), \(R\) being positive however large it may be.

Again it is evident that
\[ |f(z)| \leq C \cdot M(R), \quad (|z| \leq R), \]
where $C$ is a positive constant depending only on $\lambda$ and
\[
M(R) = \prod_{n=0}^{\infty} \left( 1 + \frac{R}{\lambda^n |z_0|} \right),
\]
so that the genus of $f(z)$ must be zero. Q. E. D.

2. Now let $\{\sigma_n\}$ be a sequence of positive numbers such that
\[
\sigma_{n+1} - \sigma_n \geq \delta > 0, \quad (n = 0, 1, 2, \ldots)
\]
and
\[
\lim_{n \to \infty} \frac{\sigma_n}{n^2} = 0,
\]
and define a sequence $\{z_n\}$ by
\[
z_n = \lambda_n = \sigma_n a^{\sigma_n}, \quad (\alpha > 0, \ n = 0, 1, 2, \ldots).
\]
Then it follows that
\[
\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{\lambda_{n+1}}{\lambda_n} \right| = \left| \frac{\sigma_{n+1}}{\sigma_n} e^{(\sigma_{n+1} - \sigma_n) \alpha} \right| \geq \left( 1 + \frac{\delta}{\delta_n} \right) e^{\delta \alpha} = \lambda > 1.
\]
Since
\[
\lim_{n \to \infty} \frac{\sigma_n}{n^2} = 0 \text{ and hence } \lim_{n \to \infty} \frac{1}{n} \log \sigma_n = 0,
\]
putting
\[
\lambda_n = \frac{z_n}{\lambda_n} = \frac{z_n}{m}, \quad (n = 0, 1, 2, \ldots)
\]
in which $m$ is an integer $\geq 0$, we obtain
\[
\sqrt{\lambda_n} = \sigma_n e^{\sigma_n a} = \exp \left( \frac{\sigma_n}{\lambda_n^2} n \alpha + \frac{1}{n} \log \sigma_n \right) < \exp \left( \frac{n+1}{2} \delta \alpha - \varepsilon \delta \alpha \right) = \lambda^{\frac{n+1}{2} - \varepsilon} \quad (\varepsilon > 0, \ n \geq N)
\]
$N$ being an integer sufficiently large.

Therefore, from Theorem I, the series of functions
\[
g(z) = \sum_{n=0}^{\infty} \frac{\lambda_n \phi(z)}{(1 - \frac{z}{\lambda_n}) \phi'(\lambda_n)}, \quad \phi(z) = \prod_{n=0}^{\infty} \left( 1 - \frac{z}{\lambda_n} \right)
\]
is convergent absolutely and uniformly for $|z| \leq R$.

On the other hand, from Lagrange’s formula of interpolation, we have
\[
(2.1) \quad z^m = \sum_{n=0}^{\infty} \frac{\lambda_n \phi_n(z)}{(1 - \frac{z}{\lambda_n}) \phi_n'(\lambda_n)}, \quad \phi_n(z) = \prod_{n=0}^{\infty} \left( 1 - \frac{z}{\lambda_n} \right)
\]
which holds good for any integer $n > m$. 
Putting \( z = x + iy \), we obtain
\[
g(x) - x^m = \sum_{n=0}^{\infty} \frac{\lambda_n^m \phi_n(x)}{\left(1 - \frac{x}{\lambda_n}\right) \phi_n'(\lambda_n)} \left(\prod_{\mu=n+1}^{\infty} \frac{1 - \frac{x}{\lambda_\mu}}{1 - \frac{x}{\lambda_\mu}} - 1\right) + R_n,
\]
where
\[
R_n = \sum_{\nu=n+1}^{\infty} \frac{\lambda_\nu^m \phi(x)}{\left(1 - \frac{x}{\lambda_\nu}\right) \phi'(\lambda_\nu)}.
\]

Since the series which defines \( g(z) \) converges absolutely and uniformly for \( |z| \leq R \), we have
\[
|R_n| < \varepsilon_n,
\]
uniformly for \( |x| \leq R \), in which \( \{\varepsilon_n\} \) is a sequence of positive constants such that
\[
\lim_{n \to \infty} \varepsilon_n = 0.
\]

Further, for \( -R \leq x < 0 \), we have
\[
0 \leq \prod_{\mu=n+1}^{\infty} \frac{1 - \frac{x}{\lambda_\mu}}{1 - \frac{x}{\lambda_\mu}} - 1 \leq \prod_{\mu=n+1}^{\infty} \frac{1 + |x|}{1 - \frac{x}{\lambda_\mu}} - 1 < \varepsilon_n',
\]
\( \{\varepsilon_n'\} \) being another sequence of positive constants such that
\[
\lim_{n \to \infty} \varepsilon_n' = 0.
\]

Consequently, by the use of the inequality which may be obtained by a similar method employed for (1.5), we get
\[
|g(x) - x^m| < \varepsilon_n' \sum_{\nu=0}^{\infty} \left|\frac{\lambda_\nu^m \phi_n(x)}{\left(1 - \frac{x}{\lambda_\nu}\right) \phi_n'(\lambda_\nu)}\right| + \varepsilon_n
\]
\[
< \{\pi(\lambda)\}^2 M(R)\varepsilon_n' + \varepsilon_n, \quad ( -R \leq x < 0),
\]
where
\[
\pi(\lambda) = \prod_{\nu=1}^{\infty} (1 - e^{-\nu \lambda_0})^{-1} \quad \text{and} \quad M(R) = \prod_{\nu=0}^{\infty} \left(1 + \frac{R}{\lambda_0} e^{-\nu \lambda_0}\right),
\]
so that
\[
\lim_{n \to \infty} \sum_{\nu=0}^{n} \frac{\lambda_\nu^m \phi_n(x)}{\left(1 - \frac{x}{\lambda_\nu}\right) \phi_n'(\lambda_\nu)} = \sum_{\nu=0}^{\infty} \frac{\lambda_\nu^m \phi_n(x)}{\left(1 - \frac{x}{\lambda_\nu}\right) \phi_n'(\lambda_\nu)}.
\]
which holds good uniformly for \(-R \leq x < 0\), or

\[ g(x) = x^m, \quad (-R \leq x < 0). \]

Since \(g(z)\) is regular for all values of \(z\), we finally obtain

\[
(2.2) \quad z^m = \sum_{n=0}^{\infty} \frac{\lambda_n^m \phi(z)}{(1 - \frac{z}{\lambda_n}) \phi'(\lambda_n)}.
\]

Again let us define a sequence \( \{C_n\} \) of constants such that

\[
\lim_{n \to \infty} \sqrt[n]{|C_n|} = R_0,
\]

and let \(\chi(z)\) be a function defined by

\[
(2.3) \quad \chi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n C_n}{z^{n+1}}
\]

which is shown to be regular for \(|z| > R_0\).

Putting \(\xi\) instead of \(z\) and multiplying both sides of (2.2) by

\[
\frac{1}{2\pi i} \chi(\xi)
\]

and integrating term by term about the circle \(|\xi| = R_0 + \varepsilon\), we get

\[
(2.4) \quad C_m = \sum_{n=0}^{\infty} (-1)^n \lambda_n^m \frac{1}{\phi'(\lambda_n)} \frac{1}{2\pi i} \int_{|\xi| = R_0 + \varepsilon} \frac{\chi(\xi) \phi(\xi)}{(1 - \frac{\xi}{\lambda_n})} d\xi.
\]

We now consider a series of functions defined by

\[
(2.5) \quad \sum_{n=0}^{\infty} a_n e^{-\lambda_n z}, \quad a_n = \frac{1}{2\pi i} \int_{|\xi| = R_0 + \varepsilon} \frac{\chi(\xi) \phi(\xi)}{(1 - \frac{\xi}{\lambda_n})} d\xi.
\]

Since

\[
\left| \frac{1}{\phi'(\lambda_n)} \right| < \left\{ \pi(\lambda) \right\}^2 \lambda^{-\frac{n(n+1)}{2}} = \left\{ \pi(\lambda) \right\}^2 e^{-\frac{\nu(n+1)}{2} \delta a}.
\]

putting \(z = x + iy\), we obtain

\[
|a_n e^{-\lambda_n z}| < \left\{ \pi(\lambda) \right\}^2 e^{-\frac{\nu(n+1)}{2} \delta a} e^{-\lambda_n x} \left| \frac{1}{2\pi i} \int_{|\xi| = R_0 + \varepsilon} \frac{\chi(\xi) \phi(\xi)}{(1 - \frac{\xi}{\lambda_n})} d\xi \right|,
\]

which shows that (2.5) is convergent absolutely and uniformly for \(z\) whose real part \(x \geq p > 0\).

Denote the function defined by (2.5) by \(F(z)\). Then, as the series

\[
\sum_{n=0}^{\infty} a_n (-\lambda_n)^m e^{-\lambda_n z}
\]

may be shown to be convergent absolutely and uniformly for \(z = x + iy\),
$x \geq p > 0$, as well as (2.5), we get

$$F^{(m)}(z) = \sum_{n=0}^{\infty} \sigma_n (-\lambda_n)^m e^{-\lambda_n z}, \quad (z = x + iy, \ x > 0).$$

On the other hand, since

$$\frac{1}{|\phi'(\lambda_n)|} \geq 1 \geq \frac{1}{n! \prod_{p=0}^{n-1} \frac{1}{1 + \frac{\lambda_n}{n}} \prod_{p=0}^{\infty} \left(1 + \frac{\lambda_n}{p n} \right)} \geq e^{-n \log \lambda_n} \prod_{p=0}^{\infty} \left(1 + \frac{1}{\lambda_p} \right)^{\frac{2}{\lambda_p}}, \quad (\lambda = e^{\delta n}),$$

it follows that

$$|\sigma_n e^{-\lambda_n z}| > h e^{-n \log \lambda_n} e^{-\lambda_n x},$$

in which $h$ is a positive constant independent of $z$ and $n$.

Therefore the series (2.5) which defines $F(z)$ does not converge for $x < 0$. Moreover, as

$$\lambda_{n+1} - \lambda_n = \sigma_n e^{\sigma_n a} \left(\frac{\sigma_n+1}{\sigma_n} e^{(\sigma_n+1-\sigma_n) a} - 1\right) > \lambda_n e^{a n} - 1,$$

the imaginary axis must be the natural cut for the series (2.5).

Above results can be stated as follows:

**Theorem II.** The function $F(z)$ defined by (2.5) is regular and analytic for $z$ whose real part is positive and it has the property that

$$F^{(m)}(+0) = C_m, \quad (m = 0, 1, 2, \ldots)$$

where

$$\lim_{n \to \infty} \mathcal{N} |C_n| = R_0 < +\infty.$$

Moreover the imaginary axis is the natural cut for the series which defines $F(z)$.

Furthermore it is evident that the series

$$\sum_{n=0}^{\infty} \sigma_n (-\lambda_n)^m e^{-\lambda_n z}$$

is absolutely convergent for any real value of $y$, and so is it with the series

$$\sum_{n=0}^{\infty} \sigma_n (-p_n)^m e^{-p_n y}, \quad a_{\nu_n} = \frac{1}{2\pi i} \int_{|\xi| = R_4 + \epsilon} \frac{\chi(\xi) \phi_0(\xi)}{(1 - \frac{\xi}{\nu_n}) \phi_0'(\nu_n)} d\xi,$$

where

$$\phi_0(\xi) = \prod_{p=0}^{\infty} \left(1 - \frac{\xi}{\nu_p} \right)$$
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and \( \{ p_n \} \) is a sequence of positive integers defined by

\[
p_n = \lfloor \sigma_n e^{\gamma_n} \rfloor, \quad (n = 0, 1, 2, \ldots)
\]

in which \( \lfloor a \rfloor \) denotes the integral part of \( a \).

It may also be shown that

\[
F_0(z) = \sum_{n=0}^{\infty} a_{pn} e^{-pnz}
\]

is regular and analytic for \( z \) whose real part is positive and that

\[
\frac{d^m}{dz^m} F_0(iz) = i^m \sum_{n=0}^{\infty} a_{pn} (-p_n)^m e^{-p_n z}, \quad (m = 0, 1, 2, \ldots)
\]

for any real value of \( y \), the series on the right hand side being convergent absolutely, so that

\[
\left[ \frac{d^m}{dy^m} F_0(iz) \right]_{y=0} = i^m C_m, \quad (m = 0, 1, 2, \ldots).
\]

As before, we can show that

\[
h_1 e^{-n \log p_n} < |a_{pn}| < h_2 e^{\frac{n(n+1)}{2}}
\]

in which \( h_1 \) and \( h_2 \) are positive constants independent of \( n \).

Accordingly we get

\[
\log \frac{1}{|a_{pn}|} = O(n \log p_n) = O((\log p_n)^2)
\]

as \( n \) tends to infinity.

For example, putting

\[
p_n = \lfloor \sigma_n e^{\gamma_n} \rfloor = \lambda^n, \quad (n = 0, 1, 2, \ldots)
\]

where \( \lambda \) is an integer \( \geq 2 \), we get

\[
\log \frac{1}{|a_N|} = O((\log N)^2), \quad N = p_n.
\]

Whence \( F_0(iz) \) is a periodic function of a real variable \( y \) which belongs to a class considered by de la Vallée Poussin relating to quasi-analytic functions(1).

N. B. Putting

\[
e^{-x} = u = r(\cos \theta + i \sin \theta),
\]

the function \( f(u) \) defined by

\[
f(u) = \sum_{n=0}^{\infty} a_{pn} u^{p_n} = \sum_{n=0}^{\infty} a_{pn} r^{p_n}(\cos p_n \theta + i \sin p_n \theta)
\]

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is regular and analytic for $|u|<1$ and has the circle $|u|=1$ as its natural cut. This function can be indefinitely differentiable with respect to $\theta$ even when $r=1$, and

$$\left[\frac{d^m}{d\theta^m}f(u)\right]_{\theta=0} = i^m C_m, \quad (|u|=1, \ m=0, 1, 2, \ldots).$$

Further it has the property that

$$\lim_{u=1^{-0}} \frac{d^m f(u)}{d (\log \frac{1}{u})^m} = \lim_{u=1^{-0}} \left(-u \frac{d}{du}\right)^m f(u) = C_m, \quad (m=0, 1, 2, \ldots).$$

Writing $z=r(\cos \theta + i \sin \theta)$ in the place of $u$, we can state above results as follows:

**Theorem III.** There exists a class $\{f(z)\}$ of functions defined by the power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

having the following natures:

(i) $f(z)$ is regular and analytic for $|z|<1$;

(ii) it has the circle $|z|=1$ as its natural cut;

(iii) it can be indefinitely differentiable with respect to $\theta$ even when $|z|=1$;

(iv) its successive derivatives with respect to $\theta$ take respectively the given values $C_0, iC_1, i^2 C_2, \ldots$ for $|z|=1$ and $\theta=0$, where $\{C_n\}$ is a sequence of constants such that

$$\lim_{n \to \infty} \sqrt[n]{|C_n|} = R_0 < +\infty;$$

(v) moreover, if we denote the operation $\left(-z \frac{d}{dz}\right)$ by $D_z$,

$$\lim_{z=1^{-0}} D_z^m f(z) = C_m, \quad (m=0, 1, 2, \ldots).$$

3. If we replace $\lambda_n$ by its value $\sigma_n e^{\sigma_n a}$ in (2.3), we have

$$(-1)^m C_m = \sum_{n=0}^{\infty} a_n \sigma_n^m e^{\sigma_n a},$$

in which

$$a_n = \frac{1}{2\pi i} \int_{|\xi|=\rho_0+\xi} \frac{\chi(\xi) \phi(\xi)}{1 - \frac{\xi}{\lambda_n}} d\xi, \quad \lambda_n = e^{\sigma_n a}, \quad (n=0, 1, 2, \ldots).$$

By the use of the inequality for $|\phi'(\lambda_n)|^{-1}$ obtained in the last section and by the definition of $\{\sigma_n\}$, we can easily show that
is convergent absolutely and uniformly for $z$ in any finite domain, so that, if we denote the function defined by (3.2) by $\Phi(z)$, it must be an integral transcendental function having the property that

$$\Phi^{(m)}(m\alpha) = (-1)^m C_m, \quad (m = 0, 1, 2, \ldots).$$

In particular, if we put

$$\sigma_n = n + 1, \quad (n = 0, 1, 2, \ldots),$$

and hence

$$\lambda_n = (n + 1)e^{(n+1)\alpha}, \quad (\alpha > 0, \quad n = 0, 1, 2, \ldots),$$

the function

$$\Phi_0(z) = \sum_{n=0}^{\infty} a_n e^{(n+1)z}$$

is an integral transcendental function with the property that

$$\Phi_0^{(m)}(m\alpha) = (-1)^m C_m, \quad (m = 0, 1, 2, \ldots).$$

In this case, as $\sigma_{n+1} - \sigma_n = 1$, it can be shown that

$$|a_n e^{(n+1)z}| \leq h e^{(n+1)|z| - \frac{\alpha}{2}(n+1)},$$

where $h$ is a positive constant.

For a given $\epsilon > 0$, let us take an integer $N$ so large that

$$|z| - \frac{\alpha}{2} n < -\epsilon |z|, \quad (n \equiv N)$$

Then

$$|\Phi_0(z)| < h \sum_{n=0}^{N-1} e^{(n+1)(|z| - \frac{\alpha}{2} n)} + h \sum_{n=N}^{\infty} e^{-\epsilon |z|(n+1)}$$

$$< h' \sum_{n=0}^{N-1} e^{(n+1)(|z| - \frac{\alpha}{2} n)}$$

in which $h'$ is also a positive constant.

For a given $z$, we can easily show that the largest value of

$$(n + 1) \left( |z| - \frac{\alpha}{2} n \right)$$

may be given by

$$n = \left\lfloor \frac{|z| - \frac{1}{2}}{\alpha} \right\rfloor < N.$$

Therefore
so that, for a sufficiently large value of $r$, we have

$$|\Phi_0(z)| < e^{2\alpha(1+\varepsilon')|z|^2}, \quad (\varepsilon' > 0, \ |z| > r),$$

which shows that the order of $\Phi_0(z)$ is at most 2.

**Theorem IV.** The function $\Phi(z)$ or $\Phi_0(z)$ defined by (3.2) or (3.3) is an integral transcendental function having the property that

$$\Phi^{(m)}(m\alpha) = (-1)^m C_m, \quad (m = 0, 1, 2, \ldots)$$

where $\{C_m\}$ is a sequence of constants such that

$$\lim_{n \to \infty} \sqrt[n]{|C_n|} = R_0 < +\infty.$$

Particularly the order of $\Phi_0(z)$ is at most 2.

Dec. 1932, Shiomi Institute, Osaka.