Historical Note of the Determination of all the Permutation Groups of Low Degrees,

by

G. A. MILLER, Urbana, Ill., U. S. A.

F. N. Cole inaugurated in America the determination of all the possible permutation groups of low degrees by completing in certain directions the determination of such groups published by A. Cayley in volume 25 of the Quarterly Journal of Mathematics, 1891. In particular, F. N. Cole seems to have been the first to complete the determination of all the possible permutation groups of degree 7 by adding to those previously published the intransitive group of order 24 which results from a (1,4) isomorphism between the symmetric groups of degrees 3 and 4 respectively. All the possible permutation groups whose degrees do not exceed 6 had been determined earlier by European mathematicians, so that the participation by American mathematicians in the determination of all the permutation groups of low degrees begins with those of degree 7.

The determination of the groups whose degrees do not exceed 6 is credited to A. L. Cauchy in the Encyclopédie des Sciences Mathématiques, tome 1, volume 1, page 563. This and various other statements found here and relating to the same subject are, however, not quite accurate. In fact, A. L. Cauchy and some of his followers, including E. Mathieu, confined their attention in these determinations to the enumeration of the possible orders of permutation groups and hence they did not find the number of the possible transitive or intransitive groups of the same order and also of the same degree. The honor of having been the first to determine all the possible permutation groups of certain low degrees seems to be due to J. A. Serret, the author of a well known higher algebra, who published a correct determination of all the 19 possible permutation groups (including the identity) whose degrees do not exceed 5.

The difficulties involved in determining all the possible permutation groups of a certain degree increase very rapidly with the in-

crease of the degree, but this increase is very irregular. While there are 7 such groups of degree 4 and 8 of degree 5 the number of these groups of degree 6 is 37. Hence it is natural that the determination of the total number of groups of the latter degree was accomplished somewhat slowly. The 16 possible transitive permutation groups of this degree were correctly determined for the first time by T. P. Kirkman in two articles which appeared in volumes 3 and 4 of the *Proceedings of the literary and philosophical society of Manchester*, 1864-5. The former of these articles contains also the first complete determination of the 7 transitive groups of degree 7, but the determinations of the transitive permutation groups of degrees 8, 9, and 10 published by T. P. Kirkman in these two articles are quite incomplete.

In 1883 G. Veronese published an article in the *Annali di Mathematica*, volume 11, page 93, in which he aimed, in particular, to give a geometric interpretation of the permutation groups whose degrees do not exceed 6 and hence he undertook to determine all these groups since no list of them had been previously published. In this respect he was, however, not entirely successful. Not only did he fail to find all of the permutation groups of degree 6 but he also listed the same groups twice and he missed two of the transitive permutation groups which appeared in the list of T. P. Kirkman, published about twenty years earlier as noted above. The geometric interpretations given by G. Veronese in this article involve, however, much that was then new and valuable but as an enumeration of all the groups concerned the article is not a creditable one.

About seven years after the appearance of this memoir by G. Veronese, E. H. Askwith published an article in volume 24 of the *Quarterly Journal of Mathematics*, 1890, in which he also undertook to determine all the possible permutation groups of degree 6. He succeeded in finding for the first time all the 21 possible intransitive groups of this degree but he failed to find three of the 16 possible transitive groups, although all of these had been published in his own country about a quarter of a century before, as was noted above. This oversight appeared also in the article by A. Cayley which was published in the following volume of the same periodical, but it was corrected by F. N. Cole in the article to which we referred above. Hence it results that while all the possible
permutation groups on six letters had been determined before F. N. Cole began to publish on this subject, he was the first to exhibit the lacking groups in a particular published list and hence to terminate the enumeration of these groups as an unsolved problem.

The history of the determination of all the possible permutation groups of degree 7 differs only slightly from that of the determination of all those of degree 6. In both cases the list of the possible transitive groups given by T. P. Kirkman in the articles cited above is complete, and in both cases F. N. Cole's corrections to the list given by A. Cayley completed the enumerations of these groups. The difference is that while both E. H. Askwith and A. Cayley found all the possible intransitive groups of degree 6 in the articles cited both of them missed therein one of these groups of degree 7. Hence F. N. Cole's addition thereto seems to include the first publication of this particular permutation group as was noted in the opening paragraph of this article, while all the other groups of degrees 6 and 7 noted by him in this addition had been published before but were overlooked by E. H. Askwith and A. Cayley.

There are only 40 permutation groups of degree 7 but there are 200 such groups of degree 8 and hence the determination of the latter was a much more difficult problem than the determination of the former. E. H. Askwith, A. Cayley and F. N. Cole endeavored successively to determine both the transitive and the intransitive groups of degree 8 in the volumes to which references have been made. Each of the last two aimed to complete the earlier work but failed to find all the possible groups and all failed to use the results obtained by T. P. Kirkman as noted above. Finally G. A. Miller added, in his first article on group theory, published in the Bulletin of the New York Mathematical Society, volume 3 (1894), page 168, the only group which had been overlooked by all these earlier writers, viz., the transitive group of degree 8 which is simply isomorphic with the symmetric group of degree 4, and he thus completed the enumeration of all the possible permutation groups whose degrees do not exceed 8. These were later listed and explained by him in the American Journal of Mathematics, volume 21 (1899), page 287, which seems to be complete as regards the existence of different groups which can be represented on eight letters or less.

The determination of the possible permutation groups of degrees 9 and 10 was completed shortly after all of those of degree 8 had
been determined. After E. H. Askwith had made a very unsuccessful attempt to determine all of the groups of degree 9, F. N. Cole studied the problem anew and with much greater success. In fact, he missed only two of the possible 258 groups and these were noted in G. A. Miller's second article on the theory of groups which was published in the same volume as his first article noted above. Hence in his first two articles on group theory, published in 1894, G. A. Miller completed the determination of all the possible permutation groups of degrees 8 and 9 respectively. In his next two articles on this subject which were published during the same year he completed the determination of the 1039 possible permutation groups of degree 10(1). At least no such additional groups have been published as yet.

An interesting element of the history of the determination of the permutation groups of low degrees is that three prizes, offered by three different academies stimulated work along this line. The first of these prizes was offered in 1858 by the Paris Academy of Sciences which announced for its "Grand Prix des Mathématiques" in 1860 the following subject: "Quels peuvent être les nombres de valeurs des fonctions bien définies qui contiennent un nombre donné de lettres, et comment peut-on former les fonctions pour lesquelles il existe un nombre donné de valeurs?" The enumeration of transitive groups by T. P. Kirkman noted above was a part of a memoir submitted by him in competing for this prize but the prize was not awarded and it was later withdrawn by the Academy.

The second of these prizes was offered by the Belgian Royal Academy, which announced for 1881 the following subject: "Étudre, autant que possible, les théories des points et des droites de Steiner, Kirkman, Cayley, Salmon, Hesse, Bauer aux propriétés qui sont pour les courbes supérieures, pour les surfaces et pour les courbes gauches les analogues des théorèmes de Pascal et de Brianchon." The memoir by G. Veronese noted above was submitted in competition for this prize but failed to be adjudged as meeting the requirements. Both T. P. Kirkman and G. Veronese expressed in connection with the publication of the results contained therein their disagreement with the verdict rendered with respect to these memoirs.

(1) The intransitive groups of degree 10 were listed in the Quart. Jour. of Math., vol. 27 (1894), p. 99. The listing of the transitive groups of this degree was completed in the Bull. of the Amer. Math. Soc., vol. 1 (1894), p. 67.
The third of the three prizes under consideration was awarded to G. A. Miller in 1900 by the Cracow Academy of Sciences for his work relating to the following subject: "To find all the groups of a system of ten letters, or, at least, to increase the number of known groups of this degree." In this case the prize was offered by mistake after the groups in question had been published but additional information relating to these groups was submitted by the recipient of this prize while competing for the same. In 1900 the Paris Academy of Sciences discerned to E. Maillet its "Prix Francoeur" for his contributions towards the development of the theory of permutation groups.

C. Jordan was the first to publish, in the Paris Comptes Rendus, volume 75 (1872), page 1754, a complete enumeration of the 8 possible transitive groups of degree 11. These groups were later re-determined by F. N. Cole who gave considerable additional information relating thereto but did not refer to the earlier enumeration by C. Jordan. The 1492 intransitive groups of this degree were published jointly by G. A. Miller and G. H. Ling in the Quarterly Journal of Mathematics, volume 32 (1901), page 342. The intransitive groups of any of the higher degrees have not yet been published. It is obvious that the number of such groups of a given degree $n > 3$ must always be larger than the total number of the possible permutation groups of degree $n-2$, but it has not yet been proved whether the total number of the permutation groups of degree $n$ is always larger than the number of these groups of degree $n-1$. When $n$ does not exceed 12 it is easy to see that this condition is satisfied.

A. Cayley's main contribution to the enumeration of all the permutation groups of low degrees is his notation for intransitive groups. This was adopted with slight modifications by the later writers on this subject. It does not always characterize uniquely the various possible intransitive groups but it exhibits the most fundamental ideas involved, viz., the isomorphisms between the various transitive constituents. This widely used notation is not mentioned in Cajori's History of Mathematical Notations (1928-9), which also omits the important notation for the commutator of two operators of a group as well as that of the normal form of a permutation. The notation employed by T. P. Kirkman to distinguish the various transitive groups exhibits the number of similar
permutations involved therein and the number of the various cycles contained in such a permutation together with the number of the conjugates of this transitive group under the symmetric group of the same degree. This notation does not always enable us to distinguish between two distinct transitive groups and it has not been widely adopted by others. In fact, there is now no widely adopted notation which distinguishes the different transitive permutation groups of the same degree and of the same order. By means of generating permutations we can secure this distinction after some effort, which in some cases is quite laborious.

From the properties of a given finite abstract group we can deduce directly all its possible representations as a transitive permutation group and hence every possible transitive representation of such a group exhibits abstract group properties. This is, however, not true as regards the possible intransitive representations as results directly from the fact that the number of such representations is always infinite while a finite abstract group can have only a finite number of properties. The determination of all the possible intransitive groups of a given degree exhibits abstract properties of the constituent groups of lower degrees as regards isomorphisms but not necessarily of the group itself. From what precedes it results that the next step in the complete determination of all the permutation groups of low degrees is the enumeration of the possible intransitive groups of degree 12. This step presents no unusual difficulties but the number of cases to be examined is very much larger than it is for the lower degrees, and hence the notation becomes either more complex or less definite.