Note on the Geometry of the Triangle,

BY

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1. Let $P_0, P_1, P_2$ and $P_3$ be the points of contact at which the nine-point circle of the fundamental triangle $ABC$ is touched by the in-circle and the ex-circles, and the tangents to the nine-point circle at $P_0, P_1, P_2$ and $P_3$ be denoted by $p_0, p_1, p_2$ and $p_3$; then we have

$$p_0 \equiv x/(b-c) + y/(c-a) + z/(a-b) = 0,$$

$$p_1 \equiv x/(b-c) + y/(c+a) - z/(a+b) = 0,$$

$$p_2 \equiv -x/(b+c) + y/(c-a) + z/(a+b) = 0,$$

$$p_3 \equiv x/(b+c) - y/(c+a) + z/(a-b) = 0.$$

Hence, if $P_{mn}$ denote the intersection of $p_m$ and $p_n$, we get

$$P_{01}: (b-c)^2: (c^2-a^2) : -(a^2-b^2),$$

$$P_{01}: -(b^2-c^2): (a-c)^2 : (a^2-b^2),$$

$$P_{03}: (b^2-c^2): -(c^2-a^2) : (a-b)^2,$$

$$P_{23}: (b+c)^2 : (c^2-a^2) : -(a^2-b^2),$$

$$P_{31}: -(b^2-c^2): (c+a)^2 : (a-b^2),$$

$$P_{13}: (b^2-c^2): -(c^2-a^2) : (a+b)^2.$$

2. The equations to the lines $AP_{01}, BP_{01}$ and $CP_{03}$ are

$$0 + (a^2-b^2)y + (c^2-a^2)z = 0,$$

$$(a^2-b^2)x + 0 + (b^2-c^2)z = 0,$$

$$(c^2-a^2)x + (b^2-c^2)y + 0 = 0.$$

We see at once that these are also the equations to the lines $AP_{23}, BP_{31}$ and $CP_{12}$ respectively; therefore

$$A, \ P_{01}, \ P_{23}$$

$$B, \ P_{01}, \ P_{31}$$

$$C, \ P_{03}, \ P_{12}$$

are three systems of collinear points.

(1) The coordinates should always be understood to be barycentric, if it be not otherwise expressly stated.
3. The coordinates of \( P_a, P_b, P_c \) and \( P_s \) are

\[
\begin{align*}
(s-a)(b-c)^2 : & \quad (s-b)(c-a)^2 : \quad (s-c)(a-b)^2, \\
-s(b-c)^2 : & \quad (s-c)(c+a)^2 : \quad (s-b)(a+b)^2, \\
(s-c)(b+c)^2 : & \quad -s(c-a)^2 : \quad (s-a)(a+b)^2, \\
(s-b)(b+c)^2 : & \quad (s-a)(a+c)^2 : \quad -s(a-b)^2,
\end{align*}
\]

where \( 2s \) stands for \( a+b+c \) as usual, or in normal coordinates

\[
\begin{align*}
\sin^2 \frac{1}{2} (B-C) : & \quad \sin^2 \frac{1}{2} (C-A) : \quad \sin^2 \frac{1}{2} (A-B), \\
-s \sin^2 \frac{1}{2} (B-C) : & \quad \cos^2 \frac{1}{2} (C-A) : \quad \cos^2 \frac{1}{2} (A-B), \\
\cos^2 \frac{1}{2} (B-C) : & \quad -\sin^2 \frac{1}{2} (C-A) : \quad \cos^2 \frac{1}{2} (A-B), \\
\cos^2 \frac{1}{2} (B-C) : & \quad \cos^2 \frac{1}{2} (C-A) : \quad -\sin^2 \frac{1}{2} (A-B).
\end{align*}
\]

The equations to \( AP_1, BP_2 \) and \( CP_3 \) are, therefore, respectively

\[
\begin{align*}
0 + (s-b)(a+b)^2 y - (s-c)(c+a)^2 z = & 0, \\
-(s-a)(a+b)^2 x + 0 + (s-c)(b+c)^2 z = & 0, \\
(s-a)(c+a)^2 x - (s-b)(b+c)^2 y + 0 = & 0.
\end{align*}
\]

The determinant formed with the coefficients of these three linear equations is evidently a skew symmetric determinant of odd order, which is always equal to zero; thus \( AP_1, BP_2 \) and \( CP_3 \) are concurrent. If \( P \) be the point of concurrence, its coordinates are

\[
\frac{(b+c)^2}{s-a} : \frac{(c+a)^2}{s-b} : \frac{(a+b)^2}{s-c},
\]

or in normal coordinates

\[
\cos^2 \frac{1}{2} (B-C) : \cos^2 \frac{1}{2} (C-A) : \cos^2 \frac{1}{2} (A-B).
\]

4. Let the points \((p_1, x), (p_2, y)\) and \((p_3, z)\) be denoted by \( Q_1, Q_2 \) and \( Q_3 \), then we get

\[
\begin{align*}
Q_1 : & \quad 0 : c+a : a+b, \\
Q_2 : & \quad b+c : 0 : a+b, \\
Q_3 : & \quad b+c : c+a : 0.
\end{align*}
\]

Hence the equations to the lines \( AQ_1, BQ_2 \) and \( CQ_3 \) are

\[
\begin{align*}
0 + (a+b)y - (c+a)z = & 0, \\
-(a+b)x + 0 + (b+c)z = & 0, \\
(c+a)x - (b+c)y + 0 = & 0.
\end{align*}
\]
These three straight lines, as it will easily be seen, meet in the point
\( b + c : c + a : a + b, \)
or in normal coordinates
\[
(b + c)/a : (c + a)/b : (a + b)/c,
\]
which is the centre of the conic \( \Sigma(b + c)y z = 0. \)

5. The point \( P \) lies on the join of the in-centre and the centre of
the nine-point circle.

For,
\[
\begin{vmatrix}
\cos^2 \frac{1}{2} (B - C) & \cos^2 \frac{1}{2} (C - A) & \cos^2 \frac{1}{2} (A - B) \\
\cos (B - C) & \cos (C - A) & \cos (A - B) \\
1 & 1 & 1
\end{vmatrix} = 0,
\]
since we have
\[
\cos 2 \theta = \cos^2 \theta + 1.
\]

6. The six points \( A, B, C, P_a, P_b, P_c \) lie on a conic. Let
\( K_0 \) be a conic on which \( A, B, C, P_a \) and \( P_b \) lie; then \( K_0 \) must be of
the form
\[
p/x + q/y + r/z = 0.
\]
Since this conic passes through \( P_a \) and \( P_b \), we have
\[
p/(b + c)^2 + q/(c^2 - a^2) - r/(a^2 - b^2) = 0,
-2p/(b^2 - c^2) + q/(c + a)^2 + r/(a^2 - b^2) = 0;
\]
whence follows that
\[
p : q : r = (b - c) (b + c)^2 : (c - a) (c + a)^2 : (a - b) (a + b)^2.
\]
Therefore \( K_0 \equiv (b - c) (b + c)^2/x + (c - a) (c + a)^2/y + (a - b) (a + b)^2/z = 0. \)
The symmetry of this equation shows that \( P_1 \) is also on \( K_0. \)

Similarly we see that \( (A, B, C, P_a, P_b, P_c) \) respectively lie on conics \( K_1, K_2 \) and \( K_3, \)
and that
\[
K_1 \equiv (b - c) (b + c)^2/x + (c - a) (c + a)^2/y - (a + b) (a - b)^2/z = 0,
K_2 \equiv -(b + c) (b - c)^2/x + (c - a) (c + a)^2/y + (a + b) (a - b)^2/z = 0,
K_3 \equiv (b + c) (b - c)^2/x - (c + a) (c - a)^2/y + (a - b) (a + b)^2/z = 0.
\]

7. The equation to the line joining \( P_3 \) and \( Q_1 \) is

\[ (1) \] See Vol. 4, p. 29 of this Journal.
\[
\begin{vmatrix}
 x & (b+c)^2 & 0 \\
y & -c^2 - a^2 & c-a \\
z & -(c^2 - b^2) & a+b \\
\end{vmatrix} = 0, \\
i.e. \\
(b-c)(c+a)(a+b)x + (a+b)(b+c)(b+c)y - (c+a)(a+b)^2z = 0. \\
\]

Similarly, for the lines \( P_{31}Q_{2}, P_{12}Q_{3} \) we get

\[-(c+a)^2(a+b)x + (b+c)(c-a)(a+b)y + (b+c)(c+a)^2z = 0, \]

\[(c+a)(a+b)^2x - (a+b)^2(b+c)y + (b+c)(c+a)(a-b)z = 0. \]

These three straight lines meet in the point

\[(b+c)^2 : (c+a)^2 : (a+b)^2. \]

8. It is already known, as e.g. in Koehler's Exercices de géométrie analytique p. 176, that \( \Sigma x^2 - 2 \Sigma xy = 0 \), a conic touching the three sides of the fundamental triangle at their mid points, touches \( p_{a}, p_{b}, p_{c} \) and \( p_{a} \). If \( R_{a}, R_{b}, R_{2} \) and \( R_{a} \) be the points of contact, then we get

\[ R_{a} : (b-c)^2 : (c-a)^2 : (a-b)^2, \]

\[ R_{b} : (b-c)^2 : (c+a)^2 : (a+b)^2, \]

\[ R_{2} : (b+c)^2 : (c-a)^2 : (a+b)^2, \]

\[ R_{3} : (b+c)^2 : (c+a)^2 : (a-b)^2. \]

Hence the equations to the lines \( AR_{1}, BR_{2} \) and \( CR_{3} \) are

\[ 0 + (a+b)^2y - (c+a)^2z = 0, \]

\[-(a+b)^2x + 0 + (b+c)^2z = 0, \]

\[(c+a)^2x - (b+c)^2y + 0 = 0. \]

These also meet in the point

\[(b+c)^2 : (c+a)^2 : (a+b)^2. \]

9. We can easily see that \( R_{1}, R_{2} \) and \( R_{3} \) lie on \( P_{31}Q_{1}, P_{31}Q_{2}, P_{21}Q_{3} \) respectively.

10. Let the points \( (BC, p_{a}), (BC, p_{b}), (CA, p_{a}), (CA, p_{b}), (AB, p_{a}), (AB, p_{b}) \) be denoted by \( A_{1}, A_{2}, B_{1}, B_{2}, C_{1}, C_{2} \) respectively; then \( A_{1}A_{2}C_{1}C_{2}B_{1}B_{2} \) is a hexagon circumscribed about the conic \( \Sigma x^2 - 2 \Sigma yz = 0 \). Hence, by Brianchon's theorem, \( A_{1}C_{1}, B_{1}A_{2}, \) and \( C_{1}B_{2} \) meet in a point, and it will at once be found that its coordinates are

\[ a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2). \]
or in normal coordinates

\[ b^3 - c^3 : c^3 - a^3 : a^3 - b^3. \]

This point evidently lies on the trilinear polars of \((b+c, c+a, a+b)\), \((a \overline{b+c}, b \overline{c+a}, c \overline{a+b}), \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)\) and the in-centre \((a, b, c)\).

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**ERRATA**

in my two notes in this Journal, Vols. 3 and 4.

Vol. 3, p. 91, l. 22; read "conyclic" for "collinear."

Vol. 4, p. 29; omit the sentence in lines 9, 10 and 11 from below.

Soon after having written the latter notes, I found the \(\Sigma X(Y^2 - Z^2) = 0\) is not a proper cubic, but

\[ (Y-Z)(Z-X)(X-Y) = 0. \]

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