Spaces in which there exist Uncountable Convergent Sequences of Points,

by

C. W. Vickery, Austin, Texas, U.S.A.

Introduction.

In his thesis, Fréchet(*) studied spaces satisfying certain conditions with reference to the notions of point and the sequential limit point of a type $\omega$ sequence of points. Fréchet and others have since investigated these spaces further.

Our first object will be to study spaces, which we call spaces $A$, satisfying conditions similar to those for spaces $L$ of Fréchet except that the notion of the convergence of a sequence of points does not necessitate that the sequence be of type $\omega$. Since every well-ordered sequence of points is run through by a subsequence whose ordinal number is regular, our object is accomplished if we restrict our investigation to convergent sequences whose ordinal number is regular. In Part I is proved a certain theorem about transfinite numbers that is of use in the treatment that follows. In Part II spaces $A$ are studied.

In Part III we shall undertake to extend the notion of distance so that it may be applicable to spaces in which there exist convergent sequences of type greater than $\omega$. In order to do this we must restrict our attention to spaces such that in each of them all the convergent sequences have the same ordinal number. It is found that all spaces satisfying the generalized metric conditions and in which the convergent sequences are of type greater than $\omega$ are totally disconnected.

If, therefore, we wish to study properties of connected sets analogous to certain properties of connected sets in ordinary metric spaces, we must make use of a different set of conditions. In Part IV we study consequences of such a set of axioms. These axioms

C. W. VICKERY:

state certain conditions about point and region, which are regarded as undefined notions. Axiom 1($\aleph_0$), the most important axiom of that set, represents a modification of an axiom of R. L. Moore. It is found that Axioms 0, 1, and 2 do not necessitate that a space satisfying them be a space $A$.

In Part V the notion of $n$-dimensional Euclidean space is extended in a certain manner and properties of certain 1-dimensional and 2-dimensional spaces thus obtained are studied.

In Part VI certain systems of numbers are developed and a certain one of them is used to construct $n$-dimensional spaces. Properties of certain 1-dimensional and 2-dimensional spaces thus obtained are studied.

Certain spaces of Parts V and VI, and certain subsets of such spaces, are used as examples to show the consistency of certain sets of axioms of preceding parts and also as examples to show that certain propositions do not follow from certain conditions. In Part VI is given an example of a connected, locally connected space, no connected subset of which is metric, but which has many of the properties of metric spaces.

Many of the theorems of this treatment were suggested by analogous theorems for other spaces due to Fréchet, R. L. Moore, and others. In several cases where it seemed that the proof of a theorem could be obtained without difficulty by one familiar with the proof of its analogue in spaces previously studied, I have stated the theorems and omitted the proof.

Definitions of terms peculiar to the treatment are given in the text. Definitions of a number of terms in use in point-set theory and transfinite number theory are given, for the convenience of the reader, in foot-notes.

I wish to thank Professor R. L. Moore for attracting my attention to mathematics, for suggesting the present problem, and for many helpful suggestions in the course of its development.

**Part I. Transfinite Numbers.**

**Definition.** A cardinal number $\aleph_0$ is said to be *regular* provided $\omega$, the smallest ordinal number in number class $Z(\aleph_0)$, is regular(1).

---

(1) An ordinal number is said to be of the first kind provided it is the number 1 or there is an ordinal number immediately preceding it. All other ordinals
Theorem 1. Suppose that \( \aleph_a \) is a regular cardinal number and 
\( G \) is a collection of power less than \( \aleph_a \) of sets \( g \), each of power less 
than \( \aleph_a \). Suppose that \( G^* \) is the sum of all the elements of \( G \).
Then \( G^* \) is of power less than \( \aleph_a \).

It is easy to see that \( G^* \) is not of power greater than \( \aleph_a \). Suppose 
that \( G^* \) is of power \( \aleph_a \). Suppose \( G^* \) to be well-ordered in a 
sequence \( B \) of type \( \omega_\alpha \). Every set \( g \) of \( G \) forms a subsequence \( B_g \) 
of \( B \), which does not run through \( B \). For each such sequence \( B_g \) 
call \( b_g \) the first element of \( B \) following all the elements of \( B_g \). Call 
\( C \) the sequence of elements \( b_g \) preserving the order of \( B \). Then \( C \) 
runs through \( B \). But \( C \) is of type less than \( \omega_\alpha \), since \( G \) is of power 
less than \( \aleph_a \). Thus we have a contradiction. Hence \( G^* \) is of power 
less than \( \aleph_a \).

Part II. Spaces \( A \) and \( A_\mu \).

Fréchet has investigated spaces in which we have the notion 
of a type \( \omega \) sequence of points converging to a point. We shall 
try to extend that notion to include other types of sequences. 
Since it has been shown that every sequence whose ordinal number 
is of the second kind is run through by a subsequence whose or-
dinal number is regular, it is sufficient if we confine our attention 
to regular sequences of points in so far as the notion of convergence 
is concerned.

We shall consider the notion of point and the notion of a re-
gular sequence of points converging to a point as undefined. The 
set of spaces satisfying the following conditions will be called \( A \) and 
any particular space satisfying these conditions will be referred to 
as a space \( A \).

Axiom 0. If \( \mu \) is a regular ordinal number, \( P \) is a point, and 
\( M \) is a type \( \mu \) sequence of points, the statement \( M \) converges to \( P \).
has a definite meaning.

**Axiom 1.** If \( A \) is a well-ordered sequence of points, \( A \) does not converge to each of two distinct points.

**Axiom 2.** If \( P \) is a point, \( A \) is a convergent sequence of points, and \( B \) is a sequence of the same type as \( A \), each element of \( B \) being the point \( P \), then \( B \) converges to \( P \).

**Axiom 3.** If \( A \) is a sequence of points which converges to a point \( P \), then every subsequence of \( A \) that runs through \( A \) converges to \( P \).

**Definition.** A point \( P \) is said to be a limit point of a point set \( M \) provided there exists a regular sequence of distinct points of \( M \) converging to \( P \).

**Definition.** If \( \mu \) is an ordinal number, a point \( P \) is said to be a type \( \mu \) limit point of a point set \( M \) provided there exists a type \( \mu \) sequence of distinct points of \( M \) which converges to \( P \).

It will be noted that a given point \( P \) may be a type \( \mu \) limit point of a point set \( M \) for more than one value of \( \mu \).

**Definition.** A space \( A \) in which the derived \( ^2 \) set of every point set is closed \( ^2 \) is called a space \( \Sigma \).

**Theorem 1.** In a space \( \Sigma \) in order that a point \( P \) should be a limit point of a point set \( M \) it is necessary and sufficient that every domain \( ^2 \) containing \( P \) should contain a point of \( M \) distinct from \( P \).

The proof is not essentially different from that in the case of a space \( S \) of Fréchet.

**Definition \(^a\).** In a space \( \Sigma \) a point \( P \) is said to be a maximal limit point of a point set \( M \) if and only if every domain containing \( P \) contains a subset of \( M \) of the same power as \( M \).

\(^2\) By the derived set of a point set is meant the set of all its limit points. A point set is said to be closed provided it contains all its limit points. A point set is said to be a domain provided no point of it is a limit point of a set of points not belonging to it. Two point sets are said to be mutually separated provided neither contains a point or a limit point of the other. A point set is said to be connected if it is not the sum of two mutually separated subsets. A point set is said to be totally disconnected provided it contains no connected subset of more than one point.

Definition $(3a)$. If $\kappa_a$ is a transfinite cardinal number, a point set $M$ in a space $\Sigma$ is said to be $\aleph_a$ compact provided every subset of $M$ of power $\kappa_a$ has a maximal limit point.

Definition $(3)$. A point set $M$, in a space $\Sigma$, is said to be perfectly compact provided every subset of $M$ has a maximal limit point.

Theorem 2. Suppose that $S$ is a space $\Sigma$, $\aleph_a$ is a cardinal number, and $H$ is a monotonic descending type $\omega_a$ sequence of distinct closed point sets each of which is $\aleph_a$-compact. Then there exists a point common to all the elements of $H$.

This theorem offers no particular difficulties not involved in the ordinary case.

Definition. Suppose that $M$ is a point set and $\aleph_a$ is a cardinal number. $M$ is said to have the $\aleph_a$-Borel property provided it is true that if $G$ is a collection of domains of power $\aleph_a$ covering $M$, then there exists a subcollection $H$ of $G$, of power less than $\aleph_a$, covering $M$.

Theorem 3. Suppose that $S$ is a space $\Sigma$, $\aleph_a$ is a transfinite cardinal number, and $M$ is a closed and $\aleph_a$-compact point set. Then $M$ has the $\aleph_a$-Borel property.

Theorem 4. In any space $\Sigma$, if $G$ is a monotonic collection $(4)$ of closed and perfectly compact point sets, then there exists a point common to all the elements of $G$.

Theorem 5. Suppose that $M$ is a closed and perfectly compact point set in a space $\Sigma$. Then $M$ has the Borel-Lebesgue property $(5)$.

Suppose that $G$ is a collection of domains covering $M$ such that no finite subcollection of $G$ covers $M$. Let $\kappa_a$ denote the smallest cardinal number which is the cardinal number of a subcollection $H$ of $G$ covering $M$. Since $M$ is perfectly compact it is $\aleph_a$-compact. Hence, by Theorem 3, there exists a subcollection of $H$ of power less than $\aleph_a$ covering $M$, which is a contradiction. Hence there

$(3a)$ Since the original preparation of this paper there has come to my attention a very interesting paper by Keitaro Haratomi, "Über höherstufige Separabilität und Kompacktheit, Jap. Jour. Math., vol. 8, 1931, pp. 113-141, vol. 9, pp.1-18, 1932.

$(4)$ A collection $G$ of point sets is said to be monotonic if, for every two point sets of $G$, one of them is a subset of the other. Moore, R. L.: "On the Most General Class $L$ of Fréchet in which the Heine-Borel-Lebesgue Theorem holds true," Proc. Nat. Acad., Vol. 1919, pp. 206-210, 337.


Kuratowski, C. and Sierpiński, W.: Ibid.
exists a finite subcollection of $G$ covering $M$.

**Theorem 6.** Suppose that $\aleph_a$ is a regular cardinal, $G$ is a collection of power less than $\aleph_a$ of point sets in a space $A$, and $P$ is a type $\omega_a$ limit point of $G^*$, the sum of the elements of collection $G$. Then $P$ is a type $\omega_a$ limit point of an element of $G$.

Since $P$ is a type $\omega_a$ limit point of $G^*$, there exists a sequence $A$ of type $\omega_a$ of distinct points of $G^*$ which converges to $P$. Call $H$ the set of all the points of sequence $A$. The point set $H$ is of power $\aleph_a$. Since $G$ is of power less than $\aleph_a$, it follows from Theorem 1, Part I, that there exists an element $g$ of $G$ such that $g H$ is of power $\aleph_a$. Thus it is false that there exists an element of sequence $A$ such that all the elements of $g H$ precede that element. Hence there exists a subsequence $B$ of $A$ running through $A$ such that all of the elements of $B$ belong to $g$. The sequence $B$ converges to $P$. Thus $P$ is a type $\omega_a$ limit point of $g$.

**Definition.** Suppose that $\mu$ is a regular ordinal number. A space $A$ in which every convergent sequence is of type $\mu$ is called a space $A_{\mu}$. A space $\Sigma$ in which every convergent sequence is of type $\mu$ is called a space $\Sigma_{\mu}$.

It will be noted that spaces $A_{\omega_a}$ and $\Sigma_{\omega_a}$ are spaces $L$ and $S$ of Fréchet respectively.

**Theorem 7.** Suppose that in a space $A_{\omega_a}$ $G$ is a collection of power less than $\aleph_a$ of point sets and $P$ is a limit point of $G^*$, the sum of the elements of collection $G$. Then $P$ is a limit point of at least one element of $G$.

This theorem follows immediately from Theorem 6. It is noted that in case $\alpha=0$ this theorem states that if $G$ is a finite collection of point sets and $P$ is a limit point of their sum, then $P$ is a limit point of at least one element of $G$.

**Definition.** A space $A$ is said to be regular provided it is true that if $D$ is a domain and $P$ is a point of $D$, then there exists a domain $E$ which contains $P$ and such that $E$ is a subset of $D$.

**Theorem 8.** If $M$ is a connected point set containing more than one point in a regular space $\Sigma$, then the set of type $\omega$ limit points of $M$ belonging to $M$ is everywhere dense in $M$.

Let $S$ denote the space considered. Suppose that $P$ is a point of $M$ and $D$ is a domain containing $P$. It will be shown that $D$ contains a point $Q$ of $M$ which is a type $\omega$ limit point of $M$. Let $A$ denote a point of $M$ distinct from $P$. Since $S$ is regular, there exists a
domain $D_1$ containing $P$ and such that $\overline{D_1}$ is a subset of $D - A$, and there exists a domain $D_2$ containing $P$ and such that $\overline{D_2}$ is a subset of $D_1$. This process may be continued. Thus there exists a type $\omega$ sequence of domains $D_1, D_2, D_3, \ldots$ such that for each $n$, $D_n$ contains $P$ and $\overline{D_{n+1}}$ and such that $\overline{D_1}$ is a subset of $D - A$. For each $n$ let $H_n$ denote $S - D_n$. Let $T$ denote common part of the domain of the sequence $\sum_{i=1}^{\omega} D_i$. Then $T = S - \sum_{i=1}^{\omega} H_i$. The point set $T$ is closed and $M$ is connected. Hence $T.M$ contains a limit point $Q$ of $(M - T.M)$. Then $Q$ is a limit point of $\sum_{i=1}^{\omega} M.H_i$, but of no set of the sequence $\sum_{i=1}^{\omega} M.H_i$. Hence, by Theorem 6, there exists a type $\omega$ sequence of points of $(M - T.M)$ converging to $Q$.

Theorem 9. If $\mu$ is a regular ordinal greater than $\omega$, then every regular space $\sum_{\mu}$ is totally disconnected.

Theorem 9 is a corollary to Theorem 8.

Theorem 10. Suppose that in a space $\sum_{\kappa}$, $M$ is a point set and $C$ is a collection of domains which properly covers(\textsuperscript{a}) $M$. Then there exists a subset $K$ of $M$ such that no domain of $G$ contains two distinct points of $K$ and such that if $P$ is a point of $M$ then there exists a domain of $G$ which contains $P$ and a point of $K$.

Let $S$ denote the particular space considered, $\overline{M}$ the cardinal number of $M$ and $\mu$ the smallest ordinal corresponding to $\overline{M}$. Let $A$ denote a type $\mu$ sequence whose terms are the elements of $M$. Suppose that the theorem is false. I shall now define a subsequence $B$ of $A$. The first element of $B$ is the first element of $A$. The second element of $B$ is the first element of $A$ which is not contained in any domain of $G$ containing the first element of $B$. There exists such an element, otherwise the theorem would be true. Suppose that $B_v$ is any order-preserving subsequence of $A$ such that if $b$ is any element of $B_v$, then $b$ is the first element of $A$ not contained in any domain of $G$ containing the first element of $B$. There exists such an element, otherwise the theorem would be true. Suppose that $B_v$ is any order-preserving subsequence of $A$ such that if $b$ is any element of $B_v$, then $b$ is the first element of $A$ not contained in any domain of $G$ containing an element of $B$ that precedes $b$. The first element of $B$ following all the elements of $B_v$ is the first element of $A$ following all the elements of $B_v$ and having the property that no domain of $G$ contains it and a point of $B_v$. By hypothesis there is such an element, otherwise the set of

\textsuperscript{a} A collection $G$ of domains is said to properly cover a point set $M$ provided every domain of $G$ contains a point of $M$ and provided every point of $M$ belongs to a domain of $G$. Moore, R. L.: Foundations of Point-set Theory, p. 59, New York, 1932.
points of $B_v$ would suffice for the point set $K$ and the theorem would be true. Let $K$ denote the set of all points of the sequence $B$ thus defined. Suppose there exists an element of $A$ such that there does not exist a domain of $G$ which contains that element and a point of $K$. Let $c$ denote the first such point of $A$. The point $c$ is not an element of $B$. In the sequence $A$, $c$ is not followed by any element of $B$. For if it were, then, by the definition of $B$, if $d$ denotes the first element of $B$ following $c$, $c$ is the first element of $B$ following all the elements of $B$ that precede $d$, which is a contradiction. Furthermore, $c$ does not follow all the elements of $B$ for then by the definition of $B$, $c$ would be an element of $B$. Hence the supposition that there exists a point $P$ of $M$ such that there does not exist a domain of $G$ connecting $P$ and a point of $K$ leads to a contradiction. Furthermore, it follows immediately from the definition of $B$ that no domain of $G$ contains two points of $K$.

**Theorem 11**: Suppose that in a space $\Sigma$, $M$ is a point set and $G$ is a collection of domains which properly covers $M$. Suppose that there exists a cardinal number $\aleph_\alpha$ such that every subset of $M$ of power greater than $\aleph_\alpha$ has a limit point. Then there exists a subset $K$ of $M$ such that $K$ is of power less than or equal to $\aleph_\alpha$, such that no domain of $G$ contains two distinct point of $M$, and such that if $P$ is a point of $M$ then there exists a domain of $G$ which contains $P$ and a point of $K$.

By the preceding theorem there exists a subset $K$ of $M$ such that no domain of $G$ contains two distinct points of $K$ and such that if $P$ is a point of $M$ then there exists a domain of $G$ which contains $P$ and a point of $K$. Suppose that $K$ is of power greater than $\aleph_\alpha$. Then $K$ has a limit point $P$. The point $P$ belongs to $M$ and therefore to a domain of $G$ and this domain contains infinitely many points of $K$. This contradicts the statement that no domain of $G$ contains two points of $K$.

**Part III. Spaces $A_\mu$.**

We shall now undertake to extend the notion of distance in such a manner that it will be applicable to spaces in which there exist convergent sequences of type greater than $\omega$.

Suppose that $\mu$ is a regular ordinal number. Let $W$ denote a

---

(7) A generalization of Theorem 17, p. 12, of R. L. Moore's *Foundations of Point-set Theory.*
type \( \mu \) sequence such that (1) in order that \( x \) should be an element of \( W \) it is necessary and sufficient that \( x \) should be either 0 or a sensed pair \((1, \nu)\), where \( \nu \) denotes any ordinal number less than \( \mu \), (2) an element \( a_1 \) of \( W \) precedes an element \( a_2 \) of \( W \) distinct from \( a_1 \) provided either \( a_2 = 0 \) or \( a_1 = (1, \nu) \) and \( a_2 = (1, \nu_2) \), where \( \nu_1 < \nu_2 \). An element \( a_1 \) of \( W \) will be said to be greater than an element \( a_2 \) of \( W \) provided \( a_1 \) precedes \( a_2 \).

Suppose that \( B \) is a sequence of ordinal numbers all less than \( \mu \) such that if \( x \) is any ordinal number less than \( \mu \) then there exists an element \( y \) of \( B \) such that every element that follows \( y \) in \( B \) is greater than \( x \). Suppose that for each element \( \nu \) of \( B \), \( a_\nu \) is an element of \( W \). Then

\[
\lim_{\nu \to \mu} a_\nu = 0
\]

will be understood to mean that if \( e \) is an element of sequence \( W \) different from 0, then there exists an ordinal number \( \delta_\nu \) less than \( \mu \) such that if \( \nu \) is any element of \( B \) greater than \( \delta_\nu \) then \( a_\nu < e \). The notation

\[
\lim_{\nu \to \mu} a_\nu = 0
\]

will be understood to mean

\[
\lim_{\nu \to \mu} a_\nu = 0
\]

where \( M \) denotes the sequence of all ordinal numbers less than \( \mu \).

A space \( A_\mu \) will be called a space \( \Delta_\mu \) provided it satisfies the following conditions:

1. Associated with every pair of points \((A, B)\) of the space considered there is an element of sequence \( W \), \( d(A, B) \), called the distance \( A \) to \( B \), such that \( d(A, B) = d(B, A) \).

2. \( d(A, B) = 0 \) if and only if \( A = B \).

3. A type \( \mu \) sequence of points \( \sum_{v=1}^{n} P_v \) converges to a point \( P \) if and only if \( \lim d(P_v, P) = 0 \).

4. Suppose that \( \sum_{v=1}^{\mu} A_v \), \( \sum_{v=1}^{\mu} B_v \), and \( \sum_{v=1}^{\mu} C_v \) are three type \( \mu \) sequences of points. If \( \lim d(A_v, B_v) = 0 \) and \( \lim d(B_v, C_v) = 0 \) then \( \lim d(A_v, C_v) = 0 \).

It will be noted that if we consider each pair \((1, n)\) (where \( n \) is a natural number greater than 1) to be the rational number \( 1/n \), a space \( \Delta_n \) is, by a theorem of Chittenden's(8), a space \( D \) of Fré-
chet provided we re-define distance in the proper manner. The following three theorems can be proved by arguments similar to those used in proving the corresponding theorems for space ($\mathcal{F}$) of Fréchet in which the ecart is regular.

**Theorem 1.** Every space $\Delta_\mu$ is a space $\Sigma_\mu$.

*Notation.* If $P$ is a point and $\epsilon$ is an element of $W$, the symbol $R_{pa}$ denotes the set of all points $X$ such that $d(XP) < \epsilon$.

**Theorem 2.** If $P$ is a point and $\epsilon$ is an element of $W$, there exists a domain $D$ containing $P$ such that $\overline{D}$ is a subset of $R_{pa}$.

**Theorem 3.** Every space $\Delta_\mu$ is regular.

*Note.* If $P$ is a point and $\epsilon$ is an element of $W$, it is not necessarily true that $R_{pa}$ is a domain. Consider the space whose points are the number 1 and the elements of $W$. Let $A$ and $B$ denote the points 1 and 0 respectively and let $P$, denote $(1, \nu)$. For each $v[1 < \nu < \mu]$ let $d(B, P_r) = (1, \nu)$, let $d(A, P_r) = (1, 2)$, and let $d(P_r, P_t) = (1, \nu) [\nu < \xi < \mu]$. Let $d(A, B) = (1, 8)$. Let $e_1 = (1, 4)$. $R_{A, e_1}$ contains $B$ but it contains no point of sequence $\sum_{\nu=2}^\mu P_{\nu}$.

**Theorem 4.** If $\mu$ is a regular ordinal greater than $\omega$, then every space $\Delta_\mu$ is totally disconnected.

This theorem follows immediately from Theorems 1 and 3 of this Part and Theorem 9 of Part II.

**Theorem 5.** Suppose that $\sum_{\nu=1}^\mu R_{\nu}$ and $\sum_{\nu=1}^\mu T_{\nu}$ are two type $\mu$ sequences such that for each $\nu < \mu$, $R_{\nu}$ and $T_{\nu}$ are intersecting spheroids of radii $r_{\nu}$ and $t_{\nu}$ respectively. Suppose that for each $\nu$, $A_{\nu}$ and $B_{\nu}$ are two points belonging to $(R_{\nu} + T_{\nu})$. Suppose that $\lim r_{\nu} = 0$ and $\lim t_{\nu} = 0$. Then $\lim d(A_{\nu}, B_{\nu}) = 0$.

We shall now extend an axiom of R. L. Moore's to read as follows, $\mu$ being a regular ordinal number:

**Axiom 1'($\mu$).** There exists a type $\mu$ sequence such that (1) for each $\nu < \mu$, $G_{\nu}$ is a collection of regions covering the space, (2) if $R$ is any region and $A$ and $B$ are any two points of $R$, then there exists an ordinal number $\delta < \mu$ such that if $\nu$ is any ordinal number such that $\delta < \nu < \mu$ and $g$ and $h$ are any two regions of $G_{\nu}$ having a point in common and such that $g$ contains $A$, then $(g + h)$ is a subset of $(R - B) + A$.

Note. Every space satisfying Axiom 1'(μ) is obviously a regular space $\sum_\mu$ and hence, if $\mu > \omega$, is totally disconnected.

**Theorem 6.** Every space $\Delta_\mu$ satisfies Axiom 1'(μ), provided domains are called regions.

For each ordinal $\nu < \mu$, let $G_\nu$ be the set of all domains $D$ such that $D$ is a subset of some spheroid of radius $(1, \nu)$. It follows easily, with the help of Theorems 2 and 5, that if every domain is called a region then the sequence $\sum_{\nu=1}^\mu G_\nu$ satisfies the conditions of Axiom 1'(μ).

**Definition.** Suppose that $\aleph_\alpha$ is a cardinal number. A point set $M$ is said to be $\aleph_\alpha$-separable provided there exists a subset $K$ of $M$ of power $\aleph_\alpha$ such that every point of $M$ is either a point of $K$ or a limit point of $K$.

**Definition.** Suppose that $\aleph_\alpha$ is a cardinal number. A point set $M$ is said to be $\aleph_\alpha$-completely-separable provided there exists a collection $G$ of power $\aleph_\alpha$ of open subsets of $M$ such that if $P$ is a point and $D$ is any open subset of $M$, there exists an element of $G$ which contains $P$ and which is a subset of $D$.

**Theorem 7.** In order that a space $\Delta_\omega$ of power $\aleph_\alpha$ or greater should be $\aleph_\alpha$-separable it is necessary and sufficient that every point set of power greater than $\aleph_\alpha$ should have a limit point.

The sufficiency of the condition is shown as follows: If domains are called regions, Axiom 1'(ω) is satisfied. By Theorem 11 of Part II for each $\nu < \omega_\alpha$ there exists a point set $K_\nu$ of power $\leq \aleph_\alpha$ such that no region of $G_\nu$ contains two points of $K_\nu$, and such that if $P$ is a point there exists a region of $G_\nu$ which contains $P$ and a point of $K_\nu$. Call $K$ the point set $\sum_{\nu=1}^\omega K_\nu$. $K$ is of power less than or equal to $\aleph_\alpha$. If $P$ is a point not belonging to $K$ and $D$ is a domain containing $P$ then it is easy to see that $D$ contains a point of $K$ and hence that $P$ is a limit point of $K(\alpha)$.

Suppose now that is a separable space $\Delta_\omega$. It will be shown in the proof of Theorem 8 that $S$ is $\aleph_\alpha$-completely-separable. From this it can be shown that every point set of power greater than $\aleph_\alpha$ has a limit point.

**Theorem 8.** In order that a space $\Delta_\omega$ should be $\aleph_\alpha$-completely-separable it is necessary and sufficient that it be $\aleph_\alpha$-separable.

---

(9) This is an extension of an argument in R. L. Moore's *Foundations of Point-set Theory*, Chap. I.
This condition is clearly necessary. It will be shown that it is sufficient. The space considered satisfies Axiom $1'(\omega_a)$ provided domains are called regions. By hypothesis there exists a point set $K$ of power $\aleph_a$ such that every point is either a point or a limit point of $K$. For each point $P$ of $K$ and each ordinal number $\nu<\mu$ let $H_{\nu,P}$ denote the set of all regions of $G_{\nu}$ containing $P$. Let $D_{\nu,P}$ denote the sum of the domains of $H_{\nu,P}$. Let $C$ denote the collection of all domains $D_{\nu,P}$. The collection $C$ is of power $\aleph_a$. It is easy to see that if $D$ is any domain and $P$ is a point of $D$ then there exists a domain of collection $C$ containing $P$ and lying in $D$.

Suppose that $\omega_a$ is any regular ordinal. We shall now give an example of an $\aleph_a$-compact, regular space $\Sigma_{\omega_a}$ which is not a space $\Delta_{\omega_a}$.

The points of the space will be the ordinal numbers less than $\omega_{a+1}$. Let $L$ denote this sequence of ordinal numbers taken in natural order. If $H$ is any type $\omega_a$ subsequence of $L$ preserving the order of $L$, it will be said to converge to the first element of $L$ that comes after all its elements; and there will be such an element since $\omega_{a+1}$ is regular ($^{10}$). If $G$ is a point and $\sum_{\nu=1}^{\omega_a} P_{\nu}$ is any type $\omega_a$ sequence such that for each $\nu<\omega_a$, $P_{\nu}=P$, then this sequence will be said to converge to $P$. The space thus defined is a space $\Sigma_{\omega_a}$. It is easy to see that it is $\aleph_a$-compact. The space is regular, since if $P$ is any point and $D$ is any domain containing $P$, there exists an interval of the space which is a subset of $D$ and which contains $P$ as an interior point. The space is not a space $\Delta_{\omega_a}$, since, being $\aleph_a$-compact it would be $\aleph_a$-separable. Suppose that the space is $\aleph_a$-separable. Then there exists a subset $K$ of power $\aleph_a$ such every point of $L$ belongs to $K$ or is a limit point of $K$. But since $\omega_{a+1}$ is regular, there exists a point $A$ which comes after all the elements of $K$ in $L$ and a point $B$ which follows $A$. $B$ does not belong to $K$ and is not a limit point of $K$. Hence the space is not $\aleph_a$-separable, and hence not a space $\Delta_{\omega_a}$.

Part IV. Spaces $\Gamma(\aleph_a)$.

Axiom $0$. Every region is a set of points.

Axiom $1(\aleph_a)$. There exists a family $F(\aleph_a)$ of power $\aleph_a$ such that (1) every element of $F(\aleph_a)$ is a collection of regions covering

---

the space, (2) if $R$ is a region and $A$ and $B$ are two points of $R$, then there exists a collection $G$ of $F(\mathcal{N}_a)$ such that if $g$ and $h$ are any two regions of $G$ having a point in common and such that $g$ contains $A$ then $g+h$ is a subset of $(R-B)+A$.

**Axiom 2.** If $H$ and $K$ are two regions having a point $P$ in common, then there exists a region $R$ containing $P$ and such that $R$ is a subset both of $H$ and of $K$.

**Axiom 3.** If $P$ is a point there exists a well-ordered sequence of regions closing down on $P$.

**Axiom 4.** If $H$ is a well-ordered sequence of regions such that each region of $H$ contains those that come after it together with their boundaries, then there exists a point $P$ common to all the regions of $H$.

**Definition.** A point $P$ is said to be a limit point of a point set $M$ if and only if every region containing $P$ contains a point of $M$ distinct from $P$.

**Definition.** If $M$ is a point set, and $\mathcal{N}_a$ is a cardinal number, the point $P$ is said to be an $\mathcal{N}_a$-limit-point of $M$ provided there exists a region containing $P$ and not more than $\mathcal{N}_a$ points of $M$ and every region containing $P$ contains at least $\mathcal{N}_a$ points of $M$.

**Definition.** If $M$ is a point set of power $\mathcal{N}_a$, the point $P$ is said to be a maximal limit point of $M$ provided it is an $\mathcal{N}_a$-limit-point of $M$.

**Definition.** If $\mathcal{N}_a$ is a cardinal number, a point set $M$ is said to be $\mathcal{N}_a$-compact provided every subset of $M$ of power $\mathcal{N}_a$ has a limit point.

**Definition.** A point set $M$ is said to be perfectly compact provided every subset of $M$ has a maximal limit point.

**Definition.** If $S$ is a set of points and $\mathcal{N}_a$ is a cardinal number, $S$ is said to be a space $\Gamma(\mathcal{N}_a)$ provided it satisfies Axioms 0 and 1 and provided $\mathcal{N}_a$ is the smallest cardinal number for which

---

(11) This is a modification of an axiom of R. L. Moore's.

(12) A well-ordered sequence of regions $H$ is said to close down on a point $P$ provided (1) every region of $H$ contains $P$, (2) every region of $H$ contains all the regions of $H$ that follow it together with their boundaries, (3) $P$ is the only point common to all the regions of $H$, (4) if $R$ is a region containing $P$, there exists region of $H$ that is a subset of $R$.

this is true. A space $\Gamma(\mathcal{K}_a)$ that satisfies Axiom 2 is called a **distributive** space $\Gamma(\mathcal{K}_a)$.

**Definition.** If $S$ is a space $\Gamma(\mathcal{K}_a)$, a subset $M$ of $S$ is said to be **compact** provided every subset of $M$ of power less than or equal to $\mathcal{K}_a$ has a maximal limit point.

**Theorem 1.** In order that a space $\Gamma(\mathcal{K}_a)$ should be $\mathcal{K}_a$-separable it is necessary and sufficient that it be $\mathcal{K}_{a+1}$-compact.

The proof is similar to that of Theorem 7, Part III.

**Theorem 2.** In order that a space $\Gamma(\mathcal{K}_a)$ should be $\mathcal{K}_a$-completely-separable it is necessary and sufficient that it be $\mathcal{K}_a$-separable.

The proof is similar to that of Theorem 8, Part III.

**Note.** It will thus be observed that every compact space $\Gamma(\mathcal{K}_a)$ is a separable space (D) of Fréchet.

**Definition.** If $\mathcal{K}_a$ is a cardinal number and $M$ is a point set, $M$ is said to have the $\mathcal{K}_a$-**Lindelöf property** provided it is true that if $G$ is any collection of power greater than $\mathcal{K}_a$ of regions covering $M$ there exists a subcollection of $G$ of power $\mathcal{K}_a$ which also covers $M$.

**Theorem 3.** In order that a space $\Gamma(\mathcal{K}_a)$ have the $\mathcal{K}_a$-Lindelöf property it is necessary and sufficient that it be $\mathcal{K}_a$-completely separable.

**Theorem 4.** In a separable space $\Gamma(\mathcal{K}_a)$ every monotonic descending sequence of distinct closed point sets runs through by a subsequence of type less than or equal to $\omega_a$.

**Theorem 5.** In a space $\Gamma(\mathcal{K}_a)$ every monotonic descending sequence of type less than or equal to $\omega_a$ of closed and compact point sets has a common point.

The proof is similar to that of Theorem 2, Part II.

**Theorem 6.** In a space $\Gamma(\mathcal{K}_a)$ every monotonic collection of closed and compact point sets has a common point.

**Theorem 7.** In a space $\Gamma(\mathcal{K}_a)$ every closed and compact point set has the Borel-Lebesgue property.

This follows without difficulty with the help of Theorem 6.

I have verified that Theorems 31 to 35 inclusive of Chapter I of R. L. Moore's *Foundations of Point-set Theory* hold true for every distributive space $\Gamma(\mathcal{K}_a)$, provided the term "compact" is interpreted in accordance with the definition given in this Part:

+ **(13a)** Axiom 2 is equivalent to the proposition that if $H$ and $K$ are two point sets, then $(H+K)'=H'+K'$. The operations of derivations are distributive and hence the space is called a distributive space.
**Definition.** A space \( \Gamma(\mathcal{N}_a) \) which satisfies Axiom 3 is called a **uniform space** \( \Gamma(\mathcal{N}_a) \).

**Theorem 8.** If \( P \) is a point in a uniform space \( \Gamma(\mathcal{N}_a) \) then there exists a regular sequence \( I_P \) of type less than or equal to \( \omega_a \) of regions closing down on \( P \).

Let \( H \) denote a collection of regions obtained by selecting from each collection of family \( F(\mathcal{N}_a) \), whose existence is postulated by Axiom 1(\( \mathcal{N}_a \)), a region containing \( P \). Suppose the elements of \( H \) to be arranged in a well-ordered sequence \( K \) of type \( \omega_a \). Suppose that \( P \) is a point and \( \beta \) is the smallest ordinal such that there exists a sequence of regions of that type closing down on \( P \). Then \( \beta \) is regular. Suppose that \( \beta \) is greater than \( \omega_a \). Let \( B \) denote a type \( \beta \) sequence of regions closing down on \( P \). For each ordinal \( \nu < \omega_a \) let \( b'_{\kappa \nu} \) denote the first element of \( B \) that is contained in element \( k_{\nu} \) of sequence \( K \); it is easy to see that there exists such an element. Call \( B' \) the sequence of elements \( b'_{\kappa \nu} \) thus obtained, preserving the order of \( B \). Sequence \( B' \) runs through sequence \( B \). For suppose that sequence \( B' \) does not run through \( B \) then there exists an element \( c \) of \( B \) which comes after all the elements of \( B' \). By Axiom 1(\( \mathcal{N}_a \)) there exists an element \( k_{\xi} \) of \( K \) which is contained in \( c \) and there is an element \( b'_{\kappa \xi} \) of \( B' \) corresponding to it, which is a contradiction. Furthermore, \( B' \) is run through by a subsequence \( B'' \) of type \( \leq \omega_a \), since \( K \) is of type \( \omega_a \), and \( B'' \) runs through \( B \) which contradicts the hypothesis that the smallest ordinal number of a sequence of regions closing down on \( P \) is greater than \( \omega_a \).

**Theorem 9.** Every uniform space \( \Gamma(\mathcal{N}_a) \) is a space \( \Lambda \) in which every convergent sequence of points is of type less than or equal to \( \omega_a \).

If \( A \) is a regular sequence of points of type less than or equal to \( \omega_a \), we shall say that \( A \) converges to a point \( P \) provided every region containing \( P \) contains a residue of \( A \). It is easy to see that with respect to this definition of convergence the set of points of the original space is a space \( \Lambda \). We wish to show that it preserves the topological proportion of the original space. We can do this by showing that in order that the point \( P \) should be a limit point of a point set \( M \) it is necessary and sufficient that there should exist a sequence of distinct points of \( M \) converging to \( P \). It is obvious that this condition is sufficient. It is also necessary. For suppose that \( P \) is a limit point of a point set \( M \). By Theorem 8 there
exists a regular sequence $B$ of type $\beta$ less than or equal to $\omega_n$ of regions closing down on $P$. Each region of $B$ contains a point of $M$ distinct from $P$; for each ordinal $\nu<\beta$ let $P_\nu$ denote a point of $M$ distinct from $P$ belonging to region $b_\nu$ of sequence $B$. Then every region containing $P$ contains a residue of $B$ and hence $B$ converges to $P$. Furthermore, there exists a subsequence $B'$ of $B$ running through $B$ all of whose elements are distinct and $B'$ converges to $P$.

Part V. Spaces $U_{\mu, \xi}$ and $U_{\mu, \xi}$.

Let $\xi$ denote a transfinite ordinal number and $\xi^*$ the inverse type\(^{(14)}\). Let $I_\xi$ denote the set of all ordinals $\nu\leq \xi$, all inverses $\nu^*$ of such ordinals, all negative integers, and 0. If $a, b, c, \text{ and } d$ are elements of $I_\xi$ such that $a$ is the inverse of an ordinal, $b$ is a negative integer, $c$ is 0, $d$ is an ordinal number, we shall say that $a<b<c<d$. If $\alpha^*$ and $\beta^*$ are two inverse ordinals we shall say that $\alpha^*<\beta^*$ provided $\alpha>\beta$. We shall adopt the natural order of magnitude for the negative integers and the ordinal numbers. Now let $K$ denote the set of all sensed pairs $(a, b)$ such that (1) $a$ is an element of $I_\xi$, (2) $b$ is a real number such that $-1 < b < +1$, (3) if $a=\xi$ or $a=\xi^*$ then $b=0$; if $0 < a < \xi$ then $b$ is positive or 0; if $\xi^* < a < 0$ then $b$ is negative or 0. If $(a, b)$ and $(c, d)$ are two distinct pairs then $(a, b) < (c, d)$ provided either (1) $a < c$ or (2) $a = c$ and $b < d$.

We shall now consider the elements of $K$ as a space which we shall call $\overline{U}_{1, \xi}$. If the pairs $(\xi^*, 0)$ and $(\xi, 0)$ are omitted we shall call the space $U_{1, \xi}$. In this notation the first subscript denotes the number of dimensions of the space. A point $(a, b)$ is said to precede a point $(c, d)$ provided $(a, b) < (c, d)$. If $A$ and $B$ are two distinct points of the space, by segment $AB$ is meant the set of all points that lie between $A$ and $B$. A point $P$ is said to be a limit point of a point set $M$ if and only if every segment that contains $P$ contains a point of $M$ distinct from $P$. If $H$ is a regular sequence of points of type less than or equal to $\xi$, a point $P$ is said to be the sequential limit point of $H$ provided every segment containing $P$ contains a residue of $H$. It is easy to see that space $U_{1, \xi}$ is a regular, connected, locally connected space $A$.

We shall now consider spaces $U_{1,\omega}$ and $\bar{U}_{1,\omega}$ in particular. We shall show that every segment of space $U_{1,\omega}$ is homeomorphic with the segment of real numbers $(0, 1)$. As a means to doing this, we shall first prove the following theorem.

**Theorem 1.** If $\xi$ is an ordinal number of the first kind belonging to the second number class $Z(\aleph_\alpha)$, then there exists a type $\xi$ sequence of real numbers between 0 and 1 preserving the natural ascending order, and such that the set of elements of this sequence is closed.

We shall show first that if $\alpha$ is any ordinal number of $Z(\aleph_\alpha)$, then there exists a type $\alpha$ sequence of real numbers between 0 and 1 preserving the ascending order. It is clear that this is true if $\alpha=\omega$. Suppose the theorem is false. Let $a$ denote the smallest number of $Z(\aleph_\alpha)$ such that there does not exist a sequence of numbers between 0 and 1 of that type preserving the ascending order. Then $\alpha>\omega$. $\alpha$ is clearly not of the first kind; for suppose that it were and call $\beta$ the ordinal next preceding it. Then there exists on the real number segment $(0, \frac{1}{2})$, which is homeomorphic with the segment $(0, 1)$, a type $\beta$ sequence $B$ of real numbers preserving the ascending order. By adding to $B$ the number $\frac{1}{2}$, which will be said to follow all the elements of $B$, a sequence is obtained which is of type $\alpha$ and which preserves the ascending order. Hence, $\alpha$ is of the second kind. Then there exists a type $\omega$ sequence of ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$, such that for each natural number $n$, $\alpha_n<\alpha_{n+1}<\alpha$ and, such that $\lim_{n \to \infty} \alpha_n=\alpha$. Let $C$ be a type $\omega$ sequence of real numbers such that for each natural number $n$, $0<\alpha_n<\alpha_{n+1}<\frac{1}{2}$. For each $n$, there exists on the real segment $(0, 1)$ an ascending sequence of type $\alpha_n$ and hence there exists a sequence of type $\rho_n$, where for $n=1$, $\rho_1=\alpha_n$ and for $n>1$, $\alpha_{n-1}+\rho_n=\alpha_n$. For each $n$, the real number segment $(c_n, c_{n+1})$ is homeomorphic with the real number segment $(0, 1)$. Hence for each $n$, there exists on segment $(c_n, c_{n+1})$ an ascending sequence $E_n$ of type $\rho_n$. Let $E$ denote the sequence $\sum_{n=1}^{\xi} E_n$. Since $\sum_{n=1}^{\xi} \rho_n=\alpha$, $E$ is of type $\alpha$, which is a contradiction.

Suppose now that $\xi$ is an ordinal of the first kind belonging to $Z(\aleph_\alpha)$. From what we have just shown there exists a type $\xi$ ascending sequence $\sum_{n=1}^{\xi} f_n$ of real numbers between 0 and $\frac{1}{2}$. Suppose that
\( \nu \) is an ordinal of the second kind less than \( \xi \). In sequence \( \sum_{\nu=1}^{\xi} f_\nu \), let \( f_\nu \) be replaced by \( f'_\nu \), the smallest real number greater than all the numbers of sequence \( \sum_{\nu=1}^{\xi} f_\nu \). Call the new sequence thus obtained \( \sum_{\nu=1}^{\xi} h_\nu \); this sequence is an increasing sequence of type \( \xi \) and the set of all its elements is closed.

**Theorem 2.** Every segment of space \( U_{1,\omega_1} \) is homeomorphic with the real number segment \((0, 1)\).

This theorem follows without difficulty with the help of Theorem 1.

**Corollary.** Every segment of space \( \overline{U}_{1,\omega_1} \) neither of whose end points is \((\omega_1^*, 0)\) or \((\omega_1, 0)\) is homeomorphic with the real number segment \((0, 1)\).

**Theorem 3.** Space \( U_{1,\omega_1} \) does not satisfy Axiom 1\((\aleph_0)\).

Suppose that there exists a family \( F \), of power \( \aleph_0 \), of collections of domains satisfying the conditions of Axiom 1\((\aleph_0)\). Suppose that \( G \) is any collection of domains covering the space; it will be shown that there exists a point \( P \) such that if \( Q \) is any point following \( P \), \( P \) and \( Q \) both belong to a domain of \( G \). Suppose that this is not true. Then if \( A \) is any point, there exists a point \( B \) following \( A \) such that no domain of \( G \) contains both \( A \) and \( B \). Hence there exists a sequence \( \sum_{n=1}^{\omega} A_n \) of points such that for each \( n, A_n \) precedes \( A_{n+1} \) and such that no domain of \( G \) contains both \( A_n \) and \( A_{n+1} \). There exists a point following all the points of this sequence since no sequence of type \( \omega \) runs through by a sequence of type \( \omega \); let \( A \) denote the first such point. Then if \( D \) is any domain of \( C \) containing \( A \) there exists a natural number \( i \) such that \( \sum_{n=i}^{\omega} A_n \) belongs to \( D \), which is a contradiction. Let \( G_1, G_2, \ldots \) denote the collections of family \( F \). For each \( n \), there exists a point \( P_n \) such that \( P_n \) precedes \( P_{n+1} \) and such that if \( Q \) is any point following \( P_n \), \( Q \) and \( P_n \) belong to a domain of \( G_n \). Let \( P \) denote the first point following all the points of \( \sum_{n=1}^{\omega} P_n \). Let \( T \) denote any point following \( P \). Then for each \( n \), \( P \) and \( T \) belong to some domain of \( G_n \), which contradicts Axiom 1\((\aleph_0)\).

It is easy to see that space \( \overline{U}_{1,\omega_1} \) also fails to satisfy Axiom 1\((\aleph_0)\).

**Theorem 4.** Space \( \overline{U}_{1,\omega_1} \) satisfies Axiom 1\((\aleph_1)\) provided domains are taken for regions.
SPACES WITH UNCOUNT. CONVERG. SEQUENCES OF POINTS.

For each ordinal \( \nu \) such that \( \nu < \omega_1 \) and \( \nu \equiv 0 \) (mod. \( \omega \)) let the region of \( G_\nu \) \((\nu < x < \nu + \omega)\) containing the point \( (\omega_1, 0) \) be the set of all points following \((\nu, 0)\) and let the region containing the point \((\omega_1*, 0)\) be the set of all points preceding \((\nu*, 0)\). By Theorem 2, segment \([(\nu + 1)*, 0), (\nu + 1, 0)]\) is homeomorphic with the real number segment \((0, 1)\); let \( C_\nu \) denote a continuous 1-to-1 correspondence between the two segments. If \( P \) is a point of interval \([(\nu*, 0), (\nu, 0)]\), let a region of collection \( G_\nu \) \((x = \nu + n, n \text{ being finite})\) containing \( P \) be any sub-segment of segment \([(\nu + 1)*, 0), (\nu + 1, 0)]\) containing \( P \) and whose image under \( G_\nu \) is of length less than or equal to \( 1/n \). Let collection \( G_\nu \) be the same as collection \( G_{\nu+1} \). Let \( F \) denote the family of all collections \( G_\mu \) thus obtained \((0 < \mu < \omega_1)\). \( F \) satisfies the conditions of Axiom 1(\( \mathfrak{K} \)).

It follows that space \( U_{1,\omega_1} \) also satisfies Axiom 1(\( \mathfrak{K} \)).

Space \( U_{1,\omega_1} \) is a connected, locally connected, locally perfectly compact uniform space \( I(\mathfrak{K}_1) \). Space \( U_{1,\omega_1} \) has all these properties and in addition is perfectly compact. It will be noted that every point of \( U_{1,\omega_1} \) is a type \( \omega \) point.

Suppose that \( \mu \) and \( \xi \) are ordinal numbers. We shall call \( \bar{U}_{\mu,\xi} \) the space consisting of the set of all type \( \mu \) sequences of elements of \( K_\xi \) and \( U_{\mu,\xi} \) the space consisting of all type sequences of elements of \( K - [(\xi*, 0) + (\xi, 0)] \). If \( \mu \) is a natural number \( n \) and \( A(a_1, a_2, \ldots, a_n) \) and \( B (b_1, b_2, \ldots, b_n) \) are two points such that for each \( i \) \((0 < i < n)\) \( a_i < b_i \) we shall say that rectangle \( AB \) is the set of all points \( X(x_1, x_2, \ldots, x_n) \) satisfying the inequality \( a_i < x_i < b_i \). A point \( P \) in a space \( \bar{U}_{\mu,\xi} \) is said to be a limit point of a point set \( M \) if and only if every rectangle containing \( P \) contains a point of \( M \) distinct from \( P \).

We shall consider in particular spaces \( U_{2,\omega_1} \) and \( \bar{U}_{2,\omega_1} \). It follows without difficulty from what we have already proved that space \( U_{2,\omega_1} \) is a uniform space \( I(\mathfrak{K}_1) \). It is, furthermore, locally arc-wise connected, and locally perfectly compact. Space \( \bar{U}_{2,\omega_1} \) has all these properties except that it is not uniform. It fails to be uniform at the point \([(\omega_1, 0), (0, 0)\)]. Space \( \bar{U}_{2,\omega_1} \) is not a space \( A \) since, while \( P \) is a limit point of the set \( M \) of points \( P(x, y) \) such that \( x < (\omega_1, 0) \) and \( y > (0, 0) \) there exists no sequence of points of \( M \) converging to \( P \) and satisfying the axioms of spaces \( A \). This point-set \( M \) together with the point

\[^{15}\text{If } \rho, \mu, \text{ and } \xi \text{ are three ordinal numbers such that } 0 \leq \rho < \mu < \xi \text{ then we say that } \xi \equiv \rho \text{ (mod. } \mu) \text{ provided there exists an ordinal number } \chi \text{ such that } \xi = \mu \cdot \chi + \rho.\]
[(ω₁, 0), (0, 0)] constitutes (provided we take a region to be the com-
mon part of a rectangle of $U_{2\omega+1}$ and $M+[(ω₁, 0), (0, 0)]$ an example
of a connected, locally connected space $I(\aleph_0)$ satisfying Axioms 2
and 4, which, however, is not arc-wise connected. There does
not exist an arc from the point $(ω₁, 0), (0, 0)]$ to any other point
of the space. This space, however, does not satisfy Axiom 3 and
it fails to be locally compact at the point $(ω₁, 0), (0, 0)]$.

It will be noted that every rectangle of space $U_{2\omega+1}$ is topologically
equivalent to the number plane.

**Part VI. Spaces $V_{\mu, \xi}$.**

Suppose that $\xi$ is an ordinal number. Let $N_\xi$ denote the set of
all type $\xi$ sequences whose elements are the numbers 0 and 1 except
such sequences that have a residue of 1's immediately preceded by
a 0. The set $N_\xi$ is called a binary number system of type $\xi$, the
elements of $N_\xi$ are called numbers, and the elements of a number
of $N_\xi$ are called the digits of that number. If $\xi = \omega$, it is obvious
that $N_\xi$ is the ordinary binary system. It is clearly not necessary
that we restrict our attention to numbers of only two digits, though
we shall do so in order to simplify the treatment.

If $a(a = \sum_{s=1}^{\xi} A_s)$ and $b(b = \sum_{s=1}^{\xi} B_s)$ are two distinct numbers of system
$N_\xi$ and $\alpha$ is the smallest ordinal such that $A_\alpha \neq B_\alpha$ we shall say that
$\alpha$ is less than $b(a < b)$ provided $A_\alpha = 0$ and $B_\alpha = 1$. If $a < b$, we shall
say that $b$ is greater than $a(b < a)$. It can be shown without much
difficulty that these definitions satisfy the following relations:

1. If $a$ and $b$ are any two numbers of $N_\xi$ such that $a < b$, then
   it is false that $b < a$.

2. If $a$ and $b$ are any two numbers of $N_\xi$, they satisfy one
   and only one of the following relations: $a < b, a > b, a = b (a$ is identical
   with $b)$.

3. If $a, b,$ and $c$ are any three numbers of $N$ such that $a < b$
   and $b < c$, then $a < c$.

We shall now consider in particular number system $N_{\omega_1}$. The

---

(16) If $A$ and $B$ are two points, by an arc from $A$ to $B$ is meant a closed,
connected, perfectly compact point set $M$ containing $A$ and $B$ such that $M$ is dis-
connected by the omission of any point distinct from $A$ and $B$. A point set $K$ is
said to be arc-wise connected provided it is true that if $A$ and $B$ are any two dis-
tinct points of $M$, there exists an arc from $A$ to $B$ lying wholly in $K$. 
following definitions and theorems are understood to apply to this system.

The following theorem is immediately obvious:

**Theorem 1.** Number system $\mathbb{N}_\omega$ is of power $2^{\aleph_0}$.

**Definition.** A number of $\mathbb{N}_\omega$ is said to be rational provided it has a residue of 0's.

**Theorem 2.** The set $K$ of all rational numbers of $\mathbb{N}_\omega$ is of power $2^{\aleph_0}$.

It is obvious that $K$ is of power greater than or equal to $2^{\aleph_0}$. The theorem is evidently true provided the set $T$ of all type $\nu$ sequences of digits 0 and 1 is of power $2^{\aleph_0}$ for every $\nu$ belonging to the first or second number class.

We shall show first that if $\nu$ is any ordinal of the second number class then the set $T_\nu$ of all type $\nu$ sequences of digits 0 and 1 is of power $2^{\aleph_0}$. This is clearly true if $\nu=\omega$. If it is not true for all ordinals of $Z(\aleph_0)$ let $\beta$ denote the first one for which it is not true. Suppose that $\beta$ is of the first kind and let $\alpha$ denote the greatest ordinal less than $\beta$. Then the set of all type $\alpha$ sequences of digits $(0, 1)$ is of power $2^{\aleph_0}$. Hence the set of all type $\beta$ sequences of digits $(0, 1)$ is of power $2^{2^{\aleph_0}}$ which is $2^{\aleph_0}$.

Suppose now that $\beta$ is of the second kind. Then there exists a sequence $\sum_{n=1}^\omega \beta_n$ of ordinals of $Z(\aleph_0)$ such that for each $n$, $\beta_n < \beta_{n+1}$ and such that $\lim_{n \to \omega} \beta_n = \beta$. For each $n$ call $T_{\beta_n}$ the set of all type $\beta_n$ sequences of digits 0,1 ; $T_{\beta_n}$ is of power $2^{\aleph_0}$. Let $H$ denote the set of all type $\omega$ sequences $\sum_{n=1}^\omega t_{\beta_n}$ such that for each $n$, $t_{\beta_n}$ is an element of $T_{\beta_n}$. Then $H$ is of power $(2^{\aleph_0})^{\aleph_0}$ which is $2^{\aleph_0}$. Let $T_{\beta}$ denote the set of all type $\beta$ sequences of digits 0,1. To each element $t$ of $T_{\beta}$ let correspond that element $h_t$ of $H$ having the property that every element of $h_t$ is an Abschnitt of $t$. Thus $T_{\beta}$ is in one-to-one correspondence with a subset of $H$, and thus the assumption that $T_{\beta}$ is of power greater than $2^{\aleph_0}$ has led to a contradiction.

The set of all sequences of digits 0,1 of ordinal number belonging to $Z(\aleph_0)$ is of power $\aleph_1(2^{\aleph_0})$ which is $2^{\aleph_0}$ since $\aleph_1 \leq 2^{\aleph_0}$. If to this set we add the set of all finite sequences of digits 0, 1, the set thus obtained is also of power $2^{\aleph_0}$. Hence $K$ is of power $2^{\aleph_0}$.

**Theorem 3.** If $a$ and $b$ are any two distinct numbers of $N$, $a < b$, then there is a rational number between them and an irrational number between them.
Let $\sum_{n=1}^{\alpha} A_n$ represent $a$ and $\sum_{n=1}^{\beta} B_n$ represent $b$. Let $\alpha$ denote the smallest ordinal such that $A_\alpha = B_\alpha$. Then $A_\alpha = 0$ and $B_\alpha = 1$. Let $\beta$ denote the smallest ordinal greater than $\alpha$ such that $A_\beta = 0$. $A_\beta$ exists, otherwise sequence $\sum_{n=1}^{\beta} A_n$ would have a residue of 1's immediately preceded by a 0 and would thus not be in system $N_{\omega_1}$. Let $c$ denote the sequence $\sum_{n=1}^{\beta} C_n$ such that $C_\beta = 1$ and such that if $\nu < \beta$, $c_\nu = A_\nu$ and if $\beta < \nu < \omega_1$, $C_\nu = 0$. Then $c$ is the required rational number. Let $d$ denote the sequence $\sum_{n=1}^{\beta} D_n$ such that if $\nu \leq \beta$, $D_\nu = C_\nu$; if $\beta < \nu < \omega_1$ and $\nu \equiv 0 \pmod{\omega}$, $D_\nu = 1$; and for all other values of $\nu < \omega_1$, $D_\nu = 0$. Then $d$ is the required irrational number.

**Theorem 4.** If $M$ is a set of numbers of $N_{\omega_1}$, $M$ has a least upper bound and a greatest lower bound.

Suppose that $M$ is any set of numbers of $N_{\omega_1}$. Let $a$ denote the sequence $\sum_{n=1}^{\omega} A_n$ constructed as follows: $A_1$ is 1 provided 1 is the first digit of some number of $M$, otherwise $A_1$ is 0. $A_2$ is 1 provided some number of $M$ that has $A_1$ as its first digit has 1 as its second digit, otherwise $A_2$ is 0. In general, if $\sum_{n=1}^{\kappa} A_n[\mu < \omega_1]$ is an *Abschnitt* of the required sequence, the first element following all its elements is 1 provided there exists in $M$ a number of which an *Abschnitt* is $\sum_{n=1}^{\kappa} A_n$ otherwise it is 0. If $a$ belongs to $N_{\omega_1}$, then $a$ is evidently the least upper bound of $M$. But suppose that $a$ has a residue of 1's immediately preceded by a 0. Let $b$ denote the number obtained by replacing that 0 by 1 and each of the 1's of that residue by 0. Then $b$ is the least upper bound of $M$.

Let $H$ denote the set of all numbers $x$ such that $x$ precedes all the numbers of $M$ distinct from $x$. $H$ contains the number each of whose digits is 0. $H$ has a least upper bound $k$ which is the greatest lower bound of $M$.

**Theorem 5.** Number system $N_{\omega_1}$ satisfies the Dedekind postulate.

This follows immediately from the preceding theorem.

**Theorem 6.** If $\aleph_x$ denotes the power of the continuum, there does not exist a well-ordered sequence of numbers of $N_{\omega_1}$ of type greater than or equal to $\omega_{x+1}$ preserving the natural order, either ascending or descending.

Suppose there does exist such a sequence $A$. Let $\alpha$ denote the
ordinal number of that sequence. For each element \( a \) of \( A \) there exists a segment containing it whose end points are rational and which contains no element of \( A \) that follows \( a \). Thus there exists a one-to-one correspondence between the set of elements of \( A \) and a subset of the set of all segments with rational end-points which latter set is of power \( \aleph_\omega \). Hence it is false that \( A \) is of type \( \omega_{\omega^1} \).

**Theorem 7** There exists a set of numbers \( M \) of \( N_{\omega_1} \) of power \( 2^{\aleph_0} \) such that (1) there does not exist an increasing type \( \omega_1 \) sequence of elements of \( M \) and (2) there does not exist a decreasing type \( \omega_1 \) sequence of elements of \( M \).

Let \( M \) denote the set of all numbers \( \sum_{\nu=1}^{\omega_1} A_\nu \) such that for each \( \nu [\omega_1 \nu < \omega_1] A_\nu = 0 \). To each number \( m \) of \( M \) let correspond the real number represented in the ordinary binary notation by the type \( \omega \) Abschnitt of \( m \). Thus \( M \) is of power \( 2^{\aleph_0} \). The order of magnitude of the numbers of \( M \) is the same as the order of magnitude of the corresponding real numbers. Since conditions (1) and (2) of the theorem hold true for the set of real numbers, they evidently hold true of \( M \).

**Theorem 8.** If \( r \) is a rational number of \( N_{\omega_1} \), then there exists a monotonic descending type \( \omega_1 \) sequence \( \sum_{\nu=1}^{\omega_1} a_\nu \) of rational numbers greater than \( r \) such that \( \lim_{\nu \to \omega_1} a_\nu = r \).

Let \( r \) be represented by \( \sum_{\nu=1}^{\omega_1} R_\nu \). Let \( R_\alpha \) denote the first element of \( \sum_{\nu=1}^{\omega_1} R_\nu \) which is 0 and such that all the digits following it are 0. For each ordinal \( \nu < \omega_1 \) let \( a_\nu \) denote a number of \( N_{\omega_1} \) whose digit of subscript \( (\omega_1 + \nu) \) is 1 and all the remaining digits of which are the same as the corresponding digits of \( r \). Then the sequence \( \sum_{\nu=1}^{\omega_1} a_\nu \) is the required sequence.

**Theorem 9.** No segment of \( N_{\omega_1} \) is homeomorphic with the real number segment \((0, 1)\).

Since every segment contains a rational number it follows, with the help of the preceding theorem, that no segment is homeomorphic with the real number segment \((0, 1)\).

**Definition.** Suppose that \( r \) is a rational number. Let \( \alpha \) denote the smallest ordinal such that digit \( R_\alpha \) of \( r \) is 0 and such that all the digits of \( r \) of greater subscript are also 0. We shall say that \( r \) is a rational number of the first kind provided \( \alpha \) is an ordinal
24 C.W. VICKERY:

number of the first kind; otherwise, \( r \) is said to be a rational number of the second kind.

**Theorem 10.** If \( a \) and \( b \) are two distinct numbers, then there exists a rational number \( r_1 \) of the first kind and a rational number \( r_2 \) of the second kind between \( a \) and \( b \).

This theorem can be proved by a simple modification of the argument used in proving Theorem 3.

**Theorem 11.** If \( r_1 \) is a rational number of the first kind, there exists a monotonic increasing type \( \omega_1 \) sequence \( \sum_{n=1}^{\omega_1} A_n \) of rational numbers less than \( r_1 \) such that \( \lim_{r \to r_1} a_r = r_1 \).

Let \( \alpha \) denote the smallest ordinal number such that every digit of \( r_1 \) of subscript greater than or equal to \( \alpha \) is 0 and let \( \beta \) denote the greatest ordinal less than \( \alpha \). For each ordinal number \( \nu < \omega_1 \) let \( a_r \) denote a number of \( \mathbb{N}_{\omega_1} \) such that each of its digits of subscript less than \( \beta \) is the same as the corresponding digit of \( r_1 \), its digit of subscript \( \beta \) is 0, each of its digits of subscript greater than \( \beta \) and less than or equal to \( \beta + \nu \) is 1, and all the rest of its digits are 0. Sequence \( \sum_{n=1}^{\omega_1} a_n \) is the required sequence.

**Theorem 12.** If \( r_2 \) is a rational number of the second kind, then there exists a monotonic increasing type \( \omega \) sequence \( \sum_{n=1}^{\omega} a_n \) of rational numbers less than \( r_2 \) such that \( \lim_{n \to r_2} a_n = r_2 \).

Let \( r_2 \) be represented by \( \sum_{n=1}^{\omega_1} R_n \). Let \( \alpha \) denote the smallest ordinal such that every digit of \( r_2 \) of subscript greater than or equal to \( \alpha \) is 0. Since \( \alpha \) is of the second kind and belongs to \( \mathbb{Z}(\mathbb{N}_\omega) \) there exists a type \( \omega \) monotonic increasing sequence of ordinals \( \sum_{n=1}^{\omega_1} \alpha_n \) such that \( \lim_{n \to \omega_1} \alpha_n = \alpha \). For each natural number \( n \), let \( h_n \) denote a number of \( \mathbb{N}_{\omega_1} \) every digit of which of subscript less than or equal to \( \alpha_n \) is the same as the corresponding digit of \( r_2 \) and every other digit of which is 0. There exists an order-preserving subsequence \( \sum_{n=1}^{\omega_1} h_n' \) of distinct elements of \( \sum_{n=1}^{\omega_1} h_n \). \( \sum_{n=1}^{\omega_1} h_n' \) is the sequence required.

Suppose that \( \mu \) and \( \xi \) are two ordinal numbers. We shall call space \( V_{\mu, \xi} \) the set of all type \( \mu \) sequence of elements of \( \mathbb{N}_\xi \).

We shall now give particular consideration to space \( V_{1, \omega_1} \), whose elements are the same as the elements of \( \mathbb{N}_{\omega_1} \). We shall call every segment of \( V_{1, \omega_1} \) a region. A point \( P \) is said to be a limit point of
Spaces with uncount. converg. sequences of points.

A point-set $M$ if and only if every region containing $P$ contains a point of $M$ distinct from $P$. A region is said to be rational provided its end-points are rational. The set of rational regions is thus of power $2^{\aleph_0}$. Let $H$ denote the collection of all triplets $(R, A, B)$, where $R$ is a rational region and $A$ and $B$ are rational points of $R$. $H$ is of power $2^{\aleph_0}$. Let $F, D, C, E, G$ denote rational points of $R$ such that $F < D < A < C < B < E < G$. Let $C_{(R, A, B)}$ denote the collection consisting of segments $DC, CE, AB, FA, BG$ together with all segments that contain no point of interval $DE$. Let $F$ denote the family of all such collections $C_{(R, A, B)}$. Then $F$ is of power $2^{\aleph_0}$. It can be seen without difficulty that $F$ satisfies the conditions of Axiom 1$(2^{\aleph_0})$ of Part IV. It can be shown, by an argument similar to that used in showing that space $U_{\omega_1}$ does not satisfy Axiom 1$(\aleph_1)$, that space $V_{\omega_1}$ does not satisfy Axiom 1$(\aleph_1)$. If the hypothesis of the continuum is true, space $V_{\omega_1}$ is thus a space $\text{I}(\aleph_1)$, in which the set of points in the space is of power $(2^{\aleph_1})$ greater than the subscript of $\text{I}(\aleph_1)$.

**Theorem 13.** Space $V_{\omega_1}$ is connected.

This is easily proved with the help of Theorem 5.

**Theorem 14.** Every segment of space $V_{\omega_1}$ is connected.

By an argument similar to one by means of which the preceding theorem can be established it can be proved that every interval is connected. Since a segment is the sum of a set of intervals all containing a given point of that segment, it follows that every segment is connected.

**Theorem 15.** Space $V_{\omega_1}$ is compact in the sense of Part IV.

This follows by making use of Theorem 5 and a generalization of the method used by Weierstrass to prove the compactness, in the ordinary sense, of the real number interval $(0, 1)$.

It follows from Theorems 8, 10, and 12 that every segment of space $V_{\omega_1}$ contains a point at which the space fails to be uniform. Thus no connected subset of $V_{\omega_1}$ is metric either in the sense of Fréchet or in the extended sense of Part III.

If we say that a sequence $B$, of type $\omega$ or $\omega_1$, of points converges to a point $P$ if and only if every segment containing $P$ contains a residue of $B$, then space $V_{\omega_1}$ is a space $\Sigma$.

Spaces $V_{\omega_1}$ and $V_{\omega_1}$ are connected, locally connected, spaces satisfying Axioms 0, 1$(2^{\aleph_0})$, 2, and 4, provided we define region as we did for space $U_{\omega_1}$. If the hypothesis of the continuum be true,
then they are spaces \( I(\aleph) \). These spaces, however, are not uniform and \( V_{\omega_1} \) is not a space \( A \), as can be shown by an argument similar to the one used in showing that a space \( U_{\omega_1} \) is not a space \( A \).

Since every interval of \( V_{\omega_1} \) is an arc, it is evident that Axioms 0, 1(2\( \omega \)), 2, and 4 do not necessitate that every arc be homeomorphic with the real number interval (0, 1) nor that every arc should contain a subset homeomorphic with real number interval (0, 1), nor do they necessitate that every two arcs be homeomorphic.

It seems desirable to have an example of a space \( I(\aleph_0) \) such that \( \aleph_0 < \aleph \) and such that the power of the space is greater than \( \aleph_0 \). If the hypothesis of the continuum be false, then spaces \( U_{\omega_1} \) and \( U_{\omega_1} \) constitute such examples, since the power of each of these spaces is \( 2^{\aleph_0} \) and since each of them is a space \( I(\aleph_1) \). If the hypothesis of the continuum be true, then spaces \( V_{\omega_1} \) and \( V_{\omega_1} \) constitute such examples. For if the hypothesis of the continuum be true, then spaces \( V_{\omega_1} \) and \( V_{\omega_1} \) are spaces \( I(\aleph_1) \) each of which is of power \( 2^{\aleph_1} \).