Repeated Sums of Certain Functions,

by

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Introduction. The first object of this paper is to generalize some of the results contained in a paper by R. D. Carmichael\(^1\), hereafter referred to as C.

We then apply these results to the equation

\[ \Delta_{\omega_1} \Delta_{\omega_2} F(x) = \frac{F(x + \omega_1 + \omega_2) - F(x + \omega_1) - F(x + \omega_2) + F(x)}{\omega_1 \omega_2} = \varphi(x). \]

We assume that \( \omega_1 \) and \( \omega_2 \) are complex numbers whose ratio is not real and we shall further suppose without loss of generality that the notation is so chosen that the imaginary part of \( \omega_2/\omega_1 \) is positive.

We obtain solutions of (1) in the case where \( \varphi(x) \) has a certain asymptotic form in suitably defined regions. We shall find that there are four solutions corresponding to the two given by Carmichael (l.c.). Having obtained these solutions we consider them in the case where \( \varphi(x) = 1/x \). We show that the results obtained may be combined so as to give rise to certain classic functions from which the whole theory of elliptic functions may be developed.

Other equations are treated briefly and connections with the Jacobian elliptic functions are indicated.

1. Solutions of the Equations \( \Delta_\omega F(x) = \varphi(x) \) and \( \nabla_\omega G(x) = \varphi(x) \).

Consider the equations

\[ \Delta_\omega F(x) = \frac{F(x + \omega) - F(x)}{\omega} = \varphi(x), \]

\( \nabla_\omega G(x) = \frac{G(x + \omega) + G(x)}{2} = \varphi(x). \)

We shall first obtain formal solutions in domains \( D_1(\omega) \). Let us suppose that \( \varphi(x) \) has the asymptotic form

\( ^1 \) R. D. Carmichael: Summation of Functions of a Complex Variable. Annals of Mathematics, vol. 34 (1933), pp. 349-378. The generalization relates to the results contained in § 1.2, § 1.3, and § 1.4, and is along the line indicated in the footnote in § 1.2.

\( ^2 \) Domains \( D_1(\omega) \) are defined in C, § 1.2.
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\[(4) \quad \varphi(x) \sim \sum_{s=0}^{\infty} (\log x)^s \left\{ p_s(x) + c_n + \frac{c_{1s}}{x} + \frac{c_{2s}}{x^2} + \ldots \right\},\]
in some domain \(D_t(\omega)\) where \(n\) is a positive integer and the \(p_s(x)\) are polynomials in \(x\) which without loss of generality we suppose to be such that \(p_n(0) = 0\). By the foregoing statement we mean that \(\varphi(x)\) is the sum of \((n+1)\) functions \(\varphi_k(x)\), \(k = 0, 1, \ldots, n\), such that

\[(5) \quad \varphi_k(x) \sim (\log x)^k \left\{ p_k(x) + c_{nk} + \frac{c_{1k}}{x} + \frac{c_{2k}}{x^2} + \ldots \right\},\]

all the asymptotic relations holding in \(D_t(\omega)\).

Now consider functions of the following form:

\[(6) \quad D_{\mu,v}^{(k)}(x) = \left\{ q_v^{(k)}(x) + a_v^{(k)} + \frac{a_v^{(k)}}{x} + \ldots + \frac{a_v^{(k)}}{x^{\mu-1}} \right\},\]

\(v = 0, 1, \ldots, k, \quad \mu = 1, 2, \ldots\)

Then there exist unique polynomials \(q_v^{(k)}(x)\) (with \(q_v^{(k)}(0) = 0\)), and a unique set of constants \(a_v^{(k)}, a_v^{(k)}, \ldots\) (with \(a_0^{(k)} = 0\)), such that the functions

\[(7) \quad Q_{\mu,v}^{(k)}(x) = \frac{c_{1k}}{k+1} (\log x)^{k+1} + \sum_{v=0}^{k} (\log x)^{v} D_{\mu,v}^{(k)}(x), \quad k = 0, 1, 2, \ldots, n,\]

have the property that

\[\Delta_v Q_{\mu,v}^{(k)}(x) = (\log x)^k \left\{ p_k(x) + c_{nk} + \frac{c_{1k}}{x} + \ldots + \frac{c_{nk}}{x^{\mu-1}} \right\} + E_{\mu,v}(x), \quad \mu = 1, 2, \ldots,\]

where \(E_{\mu,v}(x)\) is a generic notation indicating a function satisfying the inequality

\[|E_{\mu,v}(x)| < M_\mu \left| (\log x)^k \right|,\]

where \(M_\mu\) is a positive quantity dependent on \(\mu\) but independent of \(k\) and \(x\). Then the function \(Q_{\mu}(x)\),

\[(8) \quad Q_{\mu}(x) = \sum_{k=0}^{n} Q_{\mu,v}^{(k)}(x) = \frac{c_{1n}}{n+1} (\log x)^{n+1}
\quad + \sum_{v=0}^{n} (\log x)^v \left\{ \beta_v(x) + \gamma_v^{(k)} + \frac{\gamma_v^{(k)}}{x} + \ldots + \frac{\gamma_v^{(k)}}{x^{\mu-1}} \right\},\]
is uniquely determined (this may be seen upon consideration of (6) and (7)) and has the property that

\[(9) \quad \Delta_v Q_{\mu}(x) = \sum_{v=0}^{n} (\log x)^v \left\{ p_v(x) + c_{nv} + \frac{c_{1v}}{x} + \ldots + \frac{c_{nv}}{x^{\mu-1}} \right\} + E_{\mu,v}(x), \quad \mu = 1, 2, \ldots.\]
When $\mu$ becomes infinite in (8), the second member becomes the sum of a particular function and $(n+1)$ infinite series (each in general divergent); this sum we shall denote by $R_i(x)$. We shall say that $R_i(x)$ affords a formal solution of (2).

Likewise consider functions of the form

$$P^{(k)}(x) = \left\{ r^{(k)}(x) + \alpha^{(k)}_0 \frac{c_{00}}{x} + \ldots + \frac{\alpha^{(k)}_{0\mu}}{x^\mu} \right\},$$

$$\nu = 0, 1, \ldots, k, \quad \mu = 0, 1, 2, \ldots$$

Then there exist unique polynomials $r^{(k)}(x)$ (with $r^{(k)}(0) = 0$), and a unique set of constants $\alpha^{(k)}_0, \alpha^{(k)}_1, \ldots$, such that the functions

$$P^{(k)}_\mu(x) = \sum_{\nu=0}^k (\log x)^\nu P^{(k)}_{\mu \nu}(x), \quad k = 0, 1, 2, \ldots n,$$

have the property that

$$\nabla_\omega P^{(k)}_\mu(x) = (\log x)^k \left\{ p_\mu(x) + c_{0\mu} + \frac{c_{1\mu}}{x} + \ldots + \frac{c_{\mu k}}{x^k} \right\} = \frac{E_{\mu, k}(x)}{x^{\mu+1}}, \quad \mu = 0, 1, 2, \ldots$$

Then the function $P_\mu(x)$,

$$P_\mu(x) = \sum_{k=0}^n P^{(k)}_\mu(x) = \sum_{k=0}^n (\log x)^k \left\{ p_k(x) + \delta_{0k} + \frac{\delta_{1k}}{x} + \ldots + \frac{\delta_{\mu k}}{x^k} \right\},$$

is uniquely determined and has the property that

$$\nabla_\omega P_\mu(x) = \sum_{k=0}^n (\log x)^k \left\{ p_k(x) + c_{0k} + \frac{c_{1k}}{x} + \ldots + \frac{c_{\mu k}}{x^k} \right\} + \frac{E_{\mu, n}(x)}{x^{\mu+1}},$$

$$\mu = 0, 1, 2, \ldots$$

When $\mu$ become infinite in (12), the second member becomes the sum of $(n+1)$ infinite series (each in general divergent) which we shall denote by $Y_i(x)$. We shall say that $Y_i(x)$ affords a formal solution of (3).

We shall now sketch the proof of the first of the following two theorems, omitting the similar proof of the second (1).

**Theorem I.** If $\varphi(x)$ has the asymptotic form (4) in some domain $D_1(\omega)$ and if for a given positive integer $\mu$ the function $Q_\mu(x)$ is defined as in (8), then equation (2) has the solution

$$F(x) = Q_\mu(x) - \omega \sum_{k=0}^\infty \{ \varphi(x + k\omega) - \Delta_\omega Q_\mu(x + k\omega) \}.$$
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This solution $F(x)$ is the only solution of (2) in $D_1(\omega)$ for which the difference $F(x) - Q_1(x)$ approaches zero as $x$ becomes infinite in any domain $D_1(\omega)$ contained in $D_1(\omega)$. It is therefore independent of $\mu$. If $R_1(x)$ is defined as above, then $F(x) \sim R_1(x)$ in $D_1(\omega)$.

**Theorem II.** If $\varphi(x)$ has the asymptotic form (4) in some domain $D_1(\omega)$ and if for a given non-negative integer $\mu$ the function $P_\mu(x)$ is defined as in (12), then equation (3) has the solution

$$G(x) = P_\mu(x) + 2 \sum_{k=0}^{\infty} (-1)^k \{ \varphi(x + k \omega) - \nabla \omega P_\mu(x + k \omega) \}.$$  

This solution $G(x)$ is the only solution of (3) in $D_1(\omega)$ for which the difference $G(x) - P_\mu(x)$ approaches zero as $x$ becomes infinite in any domain $D_1(\omega)$ contained in $D_1(\omega)$. It is therefore independent of $\mu$. If $Y_1(x)$ is defined as above, then $G(x) \sim Y_1(x)$ in $D_1(\omega)$.

The proof of all but the last statement of Theorem I involves only obvious changes from that of Theorem 1.4 of C and hence may be omitted. We do need to show however that $F(x) \sim R_1(x)$, that is, that $F(x)$ is the sum of $(n+1)$ functions having asymptotic forms of a type similar to (5)\(^{(1)}\). Consider equation (6). When $\mu$ becomes infinite, the second member becomes an infinite series (in general divergent) which we shall denote by $\bar{D}_\mu^{(k)}(x)$. The essence of the proof consists in showing that there exist functions asymptotic to $\bar{D}_\mu^{(k)}(x)$ and hence functions asymptotic to $(\log x)^k \bar{D}_\mu^{(k)}(x)$. From this it readily follows that there exist functions asymptotic to

$$\frac{c_{\nu}}{\nu+1} (\log x)^{\nu+1} + (\log x)^\nu \sum_{k=0}^{\nu} \bar{D}_\mu^{(k)}(x), \quad \nu = 0, 1, 2, \ldots, n,$$

whose sum is $F(x)$. For the sake of simplicity we give the proof in the case $n=1$. It will be seen that the method is general.

One sees without difficulty that

$$\Delta \omega \left[ \frac{c_{\mu}}{2} (\log x)^2 + (\log x) D^{(1)}_\mu(x) \right]$$

$$= (\log x) \left\{ p_1(x) + \frac{c_{\mu}}{x} + \ldots + \frac{c_{\mu+1}}{x^{\mu+1}} \right\} + \frac{E_{\mu+1}(x)}{x^{\mu+1}}$$

\((1)\) By the methods of C one shows readily that

$$\lim_{x \to \infty} \frac{|x|^{\mu-1} |F(x) - Q_\mu(x)|}{\mu = 0, 1, 2, 3, \ldots}$$

It is evident that if we lessen the hypothesis on $\varphi(x)$ and interpret (4) to mean that

$$\lim_{x \to \infty} \frac{|x|^{\sigma} |\varphi(x) - \sum_{k=0}^{\sigma} (\log x)^k \left\{ p_\sigma(x) + \frac{c_{\sigma+1}}{x} + \ldots + \frac{c_{\sigma+1}}{x^{\sigma+1}} \right\} |}{\sigma = 0, 1, 2, \ldots,}$$

we can from (16) say that $F(x) \sim R_1(x)$ in $D_1(\omega)$ in this sense without further argument.
\[
\frac{\log (1 + \omega/x)}{\omega} D_{\mu_1}(x + \omega) + \frac{c_{11}}{2} [\log (1 + \omega/x)]^2.
\]

Hence
\[
\Delta_\omega [c_{11} \log x + D_{\mu_1}(x)] = \left\{ p_1(x) + c_{01} + \frac{c_{11}}{x} + \ldots + \frac{c_{\mu,1}}{x^{\mu}} \right\} + \frac{E_{\mu,0}(x)}{x^{\mu+1}}.
\]

Let
\[
K_1(x) = c_{11} \log x + D_{\mu_1}(x)
\]
\[
- \omega \sum_{k=0}^{\infty} \left\{ \varphi_{1}(x + k\omega) - \Delta_\omega [c_{11} \log (x + k\omega) + D_{\mu_1}(x + k\omega)] \right\}.
\]

Then by an argument analogous to that by which (16) is obtained it may be shown that
\[
H_1(x) = (K_1(x) - c_{11} \log x) \sim D_{\mu}(x).
\]

Now let
\[
f_1(x) = \varphi_1(x) + \frac{\log (1 + \omega/x)}{\omega} H_1(x + \omega) + \frac{c_{11}}{2} [\log (1 + \omega/x)]^2,
\]
and form
\[
A_1(x) = \frac{c_{11}}{2} (\log x)^2 + (\log x) D_{\mu_1}(x) - \omega \sum_{k=0}^{\infty} f_1(x + k\omega)
\]
\[
- \Delta_\omega \left[ \frac{c_{11}}{2} (\log (x + k\omega))^2 + \log (x + k\omega) D_{\mu_1}(x + k\omega) \right].
\]

Then
\[
A_1(x) \sim \frac{c_{11}}{2} (\log x)^2 + (\log x) D_{\mu}(x).
\]

Similarly, let
\[
f_0(x) = \varphi_0(x) - \frac{\log (1 + \omega/x)}{\omega} H_1(x + \omega) - \frac{c_{11}}{2} [\log (1 + \omega/x)]^2,
\]
and form
\[
A_0(x) = c_{10} \log x + D_{\mu_0}(x) + D_{\mu_0}(x)
\]
\[
- \omega \sum_{k=0}^{\infty} \left\{ f_0(x + k\omega) - \Delta_\omega [c_{10} \log (x + k\omega) + D_{\mu_0}(x + k\omega) + D_{\mu_0}(x + k\omega)] \right\}.
\]

Then
\[
A_0(x) \sim c_{10} \log x + D_{\mu_0}(x) + D_{\mu_0}(x).
\]

Then
\[
F(x) = A_0(x) + A_1(x) = Q_\mu(x) - \omega \sum_{k=0}^{\infty} \{ \varphi(x + k\omega) - \Delta_\omega Q_\mu(x + k\omega) \},
\]
and one has

$$F(x) \sim R_1(x).$$

Now suppose that $\varphi(x)$ has the asymptotic form (4) in some domain $D_1(-\omega)$. The formal solutions $R_1(x)$ and $Y_1(x)$ of (2) and (3) respectively remain unchanged; corresponding to Theorems I and II we have the following two which we state without proof (1):

**Theorem III.** If $\varphi(x)$ has the asymptotic form (4) in some domain $D_1(-\omega)$ and if for a given positive integer $\mu$ the function $Q_\mu(x)$ is defined as in (8), then equation (2) has the solution

$$F(x) = Q_\mu(x) + \omega \sum_{k=0}^{\infty} \left[ \varphi(x-k+1\omega) - \Delta_\omega Q_\mu(x-k+1\omega) \right].$$

This solution $F(x)$ is the only solution of (2) in $D_1(-\omega)$ for which the difference $F(x) - Q_1(x)$ approaches zero as $x$ becomes infinite in any domain $D_1(-\omega)$ contained in $D_1(-\omega)$. It is therefore independent of $\mu$. If $R_1(x)$ is defined as above, then $F(x) \sim R_1(x)$ in $D_1(-\omega)$.

**Theorem IV.** If $\varphi(x)$ has the asymptotic form (4) in some domain $D_1(-\omega)$ and if for a given non-negative integer $\mu$ the function $P_\mu(x)$ is defined as in (12), then equation (3) has the solution

$$G(x) = P_\mu(x) + 2 \sum_{k=0}^{\infty} (-1)^k \left[ \varphi(x-k+1\omega) - \Delta_\omega P_\mu(x-k+1\omega) \right].$$

The solution $G(x)$ is the only solution of (3) in $D_1(-\omega)$ for which the difference $G(x) - P_0(x)$ approaches zero as $x$ becomes infinite in any domain $D_1(-\omega)$ contained in $D_1(-\omega)$. It is therefore independent of $\mu$. If $Y_1(x)$ is defined as above, then $G(x) \sim Y_1(x)$ in $D_1(-\omega)$.

We now show that these solutions are the principal sums of $\varphi(x)$ in the sense of C, §1.1. From the results obtained in C, §1.4, it follows that we need only to prove that the principal sums of $\Delta_\omega x^n (\log x)^m [\Delta_\omega x^n (\log x)^m]$, where $n \geq 0$ and $m \geq 2$, are both equal to $x^n (\log x)^n + c x^n (\log x)^m$ where $c$ is a constant. We show this in the case of the first principal sum. The second principal sum and the first and second principal alternating sums may be similarly treated. For the sake of simplicity we give the proof for the principal case $m=2$. The method will be seen to be general.

Consider the following expression where $\eta$ is such that $\eta \omega > 0$:

$$-\omega \sum_{k=0}^{\infty} \Delta_\omega \{ (x+k\omega)^n (\log (x+k\omega))^m \} e^{-\eta(x+k\omega)}$$

(1) These theorems correspond to Theorems 1.6 and 1.7 respectively of C, §1.3.
\[ x^n (\log x)^2 e^{-\eta x} + \sum_{k=1}^{\infty} (x + k\omega)^n \left[ \log (x + k\omega)^2 \left\{ e^{-\eta (x + k\omega)} - e^{-\eta (x + k\omega - \omega)} \right\} \right] \]

\[ = x^n (\log x)^2 e^{-\eta x} + (1 - e^{\eta x}) \sum_{k=1}^{\infty} (x + k\omega)^n \left[ \log (x + k\omega) \right] e^{-\eta (x + k\omega)} \]

\[ = x^n (\log x)^2 e^{-\eta x} + (1 - e^{\eta x}) \ln_{n,1}(x, \eta), \quad \text{say.} \]

Now

\[ \ln_{n,2}(x, \eta) = \sum_{k=1}^{\infty} (x + k\omega)^n \left[ \log \omega + \log (k + x/\omega) \right] e^{-\eta (x + k\omega)} \]

\[ = (\log \omega)^2 \sum_{k=1}^{\infty} (x + k\omega)^n e^{-\eta (x + k\omega)} \]

\[ + 2\omega^n (\log \omega) \sum_{k=1}^{\infty} (k + x/\omega)^n \log (k + x/\omega) e^{-t(k + x/\omega)} \]

\[ + \omega^n l_{n,3}(x, s), \]

where

\[ l_{n,3}(x, s) = \sum_{k=1}^{\infty} (k + x/\omega)^n \left[ \log (k + x/\omega) \right]^2 e^{-s(k + x/\omega)}, \quad s = \eta \omega. \]

From the results of C (l.c.) it follows that it is sufficient to show that as \( s \) approaches zero thru positive values we have

\[ \lim_{s \to 0} (1 - e^{s}) \{ l_{n,3}(x, s) - l_{n,3}(0, s) \} = 0. \]

By the use of a standard formula for \( \log(k + x/\omega) \) one obtains

\[ [\log(k + x/\omega)]^2 = \int_0^\infty \left\{ e^{-(k+y)} - e^{-t} \right\} dy + e^{-(k+y)(k + x/\omega)} dt \cdot dy/ty, \quad R(k + x/\omega) > 0. \]

Now if \( D^n_s \) denotes the \( n \)-th derivative with respect to \( s \), we have since \( s > 0 \)

\[ l_{n,3}(x, s) = \sum_{k=1}^{\infty} (-1)^n D^n_s e^{-t(k + x/\omega)} [\log (k + x/\omega)]^2 \]

\[ = (-1)^n D^n_s \sum_{k=1}^{\infty} e^{-t(k + x/\omega)} [\log (k + x/\omega)]^2. \]

Using the foregoing expression for \( [\log(k + x/\omega)]^2 \) we have

\[ l_{n,3}(x, s) = (-1)^n D^n_s \sum_{k=1}^{\infty} e^{-t(k + x/\omega)} \int_0^\infty \left\{ e^{-(t+y)} - e^{-t} \right\} dy + e^{-(t+y)(k + x/\omega)} \]

\[ - e^{-t} \cdot e^{-t(k + x/\omega)} + e^{-(t+y)(k + x/\omega)} \] \[ dt \cdot dy/ty \]
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\[ =(−1)^n D_s^\infty \int_0^\infty \left\{ \frac{e^{−(t+\gamma)(1+s/\omega)} e^{−(t+\gamma)(1+1/\omega)}}{1−e^{−t}} − \frac{e^{−t} e^{−(t+\gamma)(1+s/\omega)}}{1−e^{−(t+\gamma)}} \right\} dt \cdot dy/ty. \]

Hence when \( s \) is positive, \( l_{n,2}(x, s) − l_{n,2}(0, s) \) is \( (−1)^n \) times the \( n \)-th derivative with respect to \( s \) of a function of \( s \) which is analytic at \( s = 0 \). Therefore \( l_{n,2}(x, s) \) has the property needed for the proof.

2. The Equation \( \Delta_{\omega_1} \Delta_{\omega_2} F(x) = \varphi(x) \). We shall first obtain formal solutions of (1). By a domain \( D_1(\pm \omega_1, \pm \omega_2) \) we shall mean a region of the complex \( x \)-plane which is a domain \( D_1(\pm \omega_1) \) and at the same time a domain \( D_1(\pm \omega_2) \). For convenience in notation in what follows we shall denote a domain \( D_1(\omega_1, \omega_2) \) by \( D_{11}(\omega_1, \omega_2) \); \( D_1(\omega_1, −\omega_2) \) by \( D_{12}(\omega_1, \omega_2) \); \( D_1(−\omega_1, \omega_2) \) by \( D_{13}(\omega_1, \omega_2) \); \( D_1(−\omega_1, −\omega_2) \) by \( D_{14}(\omega_1, \omega_2) \).

Let us now suppose that \( \varphi(x) \) has the asymptotic form (4) in some domain \( D_{11}(\omega_1, \omega_2) \). Then from (1), there exist uniquely determined functions \( Q_{\mu}(x) \) of the form (8) such that \( \Delta_{\omega_1} Q_{\mu}(x) \) is equal to the second member of (9). Further there exist uniquely determined functions

\[ Q_{\mu}(x) = \frac{c_{1n}}{n+1} (\log x)^{n+1} \]

\[ + \sum_{\ell = 0}^n (\log x)^\ell \left\{ h_{\ell}(x) + d_{\ell x} + \frac{d_{\ell x}}{x} + \ldots + \frac{d_{\mu-\ell x}}{x^{\mu-\ell}} \right\}, \quad \mu = 2, 3, \ldots, \]

such that

\[ \Delta_{\omega_2} Q_{\mu}(x) = \frac{c_{1n}}{n+1} (\log x)^{n+1} \]

\[ + \sum_{\ell = 0}^n (\log x)^\ell \left\{ h_{\ell}(x) + \frac{d_{\ell x}}{x} + \gamma_{\mu} + \ldots + \frac{\gamma_{\mu-\ell x}}{x^{\mu-\ell}} \right\} + \frac{E_{\mu,n}(x)}{x^\mu}, \]

and hence

\[ \Delta_{\omega_1} \Delta_{\omega_2} Q_{\mu}(x) \]

\[ = \sum_{\ell = 0}^n (\log x)^n \left\{ p_{\ell}(x) + c_{1n} + \frac{c_{1n}}{x} + \ldots + \frac{c_{1n}}{x^{\mu}} \right\} + \frac{E_{\mu,n}(x)}{x^{\mu+1}}. \]

When \( \mu \) becomes infinite in (20), the second member becomes the sum of a particular function and \( (n + 1) \) infinite series (each in general divergent); which sum we shall denote by \( R_2(x) \). We shall say that \( R_2(x) \) affords a formal solution of (1).
Applying Theorem I twice we have the solution
\[ F_1(x) = Q_{\mu_2}(x) - \omega_2 \sum_{n=0}^{\infty} \left\{ Q_{\mu_1}(x + n \omega) - \omega_1 \sum_{m=0}^{\infty} \left[ \varphi(x + m \omega_1 + n \omega_2) \right] \right\} - \Delta_{\omega_1} Q_{\mu_1}(x + n \omega_2) \]
\[ = Q_{\mu_2}(x) - \omega_2 \sum_{n=0}^{\infty} \left\{ \Delta_{\omega_1 \omega_2} Q_{\mu_2}(x + m \omega_1 + n \omega_2) - \omega_1 \sum_{m=0}^{\infty} \left[ \varphi(x + m \omega_1 + n \omega_2) \right] \right\} - \Delta_{\omega_1} Q_{\mu_2}(x + n \omega_2) \]
\[ = Q_{\mu_2}(x) + \omega_1 \omega_2 \sum_{m, n=0}^{\infty} \left\{ \varphi(x + m \omega_1 + n \omega_2) \right\} (1), \text{ since } \mu \geq 2. \]

Hence applying (1) Theorem I twice (for \( F_1(x) \)), (2) Theorem I and then Theorem III (for \( F_2(x) \)), (3) Theorem III twice (for \( F_3(x) \)), (4) Theorem III and then Theorem I (for \( F_4(x) \)), we obtain the following theorem:

**Theorem V.** Let

(22) \[ F_1(x) = Q_{\mu_2}(x) + \omega_1 \omega_2 \sum_{m, n=0}^{\infty} \left\{ \varphi(x + m \omega_1 + n \omega_2) \right\} - \Delta_{\omega_1} Q_{\mu_2}(x + m \omega_1 + n \omega_2), \]

(23) \[ F_2(x) = Q_{\mu_2}(x) - \omega_1 \omega_2 \sum_{m, n=0}^{\infty} \left\{ \varphi(x + m \omega_1 - n + 1 \omega_2) \right\} - \Delta_{\omega_1 \omega_2} Q_{\mu_2}(x + m \omega_1 - n + 1 \omega_2), \]

(24) \[ F_3(x) = Q_{\mu_2}(x) + \omega_1 \omega_2 \sum_{m, n=0}^{\infty} \left\{ \varphi(x - m + 1 \omega_1 - n + 1 \omega_2) \right\} - \Delta_{\omega_1 \omega_2} Q_{\mu_2}(x - m + 1 \omega_1 - n + 1 \omega_2), \]

(25) \[ F_4(x) = Q_{\mu_2}(x) - \omega_1 \omega_2 \sum_{m, n=0}^{\infty} \left\{ \varphi(x - m + 1 \omega_1 + n \omega_2) \right\} - \Delta_{\omega_1 \omega_2} Q_{\mu_2}(x - m + 1 \omega_1 + n \omega_2). \]

Then if for a particular value \( i \) of the set \( i = 1, 2, 3, 4 \), \( \varphi(x) \) has the

asymptotic form (4) in some domain $D_{41}(\omega_1, \omega_2)$ and if for a given positive integer $\mu \geq 2$ the function $Q_{2\mu}(x)$ is defined as in (20), equation (1) has the solution $F_t(x)$. The solution $F_t(x)$ is the only solution of (1) in $D_{41}(\omega_1, \omega_2)$ for which the difference $F_t(x) - Q_{2\mu}(x)$ approaches zero as $x$ becomes infinite in $D_{41}(\omega_1, \omega_2)$. It is therefore independent of $\mu$. If $R_t(x)$ is defined as above, then $F_t(x) \sim R_t(x)$ in $D_{41}(\omega_1, \omega_2)$.

These solutions are principal repeated sums of $\varphi(x)$ in that e.g., $F_t(x)$ is a principal sum of a principal sum. It follows that the principal repeated sums are unique except for an additive arbitrary linear function of $x$ which comes in since at each step an arbitrary constant may be added to the principal sum$^{(1)}$.

### 3. The Equation $\Delta_{\omega_1}\Delta_{\omega_2}F(x) = 1/x$.

When $\varphi(x) = 1/x$ one finds that

$$R_1(x) = \log x - \sum_{\mu=1}^{\infty} (-1)^\mu \frac{B_\mu \omega_1^\mu}{\mu x^\mu},$$

where the $B$'s are the Bernoulli numbers$^{(2)}$, and that

$$R_2(x) = -x + \left(x - \frac{\omega_1 + \omega_2}{2}\right) \log x + \sum_{\mu=1}^{\infty} b_\mu \omega_\mu x^\mu,$$

where the $b$'s are given by the recursion formula

$$\frac{\omega_1^k(1-k)}{2k(k+1)} - \frac{\omega_1 \omega_2^{k-1}}{2k} + \sum_{r=1}^{k-1} (-1)^{r-1} \left(\frac{b_1}{r-1}\right) b_r \omega_2^{k-r-1} = B_k \omega_1^k / k,$$

$k = 2, 3, \ldots$

From (26) we have

$$Q_{2\mu}(x) = -x + \lfloor x - (\omega_1 + \omega_2)/2 \rfloor \log x.$$

Using Theorem V we have the four solutions $F_t(x)$, $i = 1, 2, 3, 4$, where

$$F_t(x) = Q_{2\mu}(x) + \omega_1 \omega_2 \sum_{m,n=0}^{\infty} \left\{ \frac{1}{x + m\omega_1 + n\omega_2} \right\}$$

$$- \Delta_{\omega_1} \Delta_{\omega_2} Q_{2\mu}(x + m\omega_1 + n\omega_2),$$

and $F_t(x)$, $i = 2, 3, 4$, may be likewise written out in detail from Theorem V.

The logarithm is made single-valued by a cut along the line $0, -\omega_2, \infty$ for $F_t(x)$ and $F_4(x)$ and along the line $0, \omega_2, \infty$ for $F_2(x)$

$^{(1)}$ If we modify $Q_{2\mu}(x)$ as given by (20) so that $h_\epsilon'(0) = 0$, then $F_t(x)$ is independent of the order of summation.

$^{(2)}$ The notation is that of Nörlund, i.e., p. 18.
and $F_3(x)$. The principal determination of the logarithm in all cases is taken on the line bisecting the angle $\omega_10\omega_2$.

Let $S_1$ be any sector drawn from the origin and containing in its interior the rays $(0, -\omega_1, \infty), (0, -\omega_2, \infty)$, (the point 0 excepted). Let $S_2$ be any sector similarly drawn containing in its interior the rays $(0, -\omega_1, \infty), (0, \omega_1, \infty)$, (the point 0 excepted); $S_3$, $(0, \omega_2, \infty), (0, \omega_1, \infty); S_4$, $(0, \omega_1, \infty), (0, -\omega_2, \infty)$. To fix the ideas we shall take $S_i$ such that its sides make a small positive angle $\theta$ with the rays in question. Then $F_i(x) \sim R_i(x)$ outside $S_i$.

Now it is easily shown that $F_i(x)$ may be expressed as follows:

$$F_i(x) = \lim_{k \to \infty} \left[ Q_{2i}(x + k\omega_1) - Q_{2i}(x + k\omega_1 + l\omega_3) \right] + \omega_1\omega_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{x + m\omega_1 + n\omega_2}.$$}

Using this result one shows without difficulty that

$$F_1(x) = F_2'(\omega_2) + (x - \omega_2)(\log \omega_1 - \gamma) + \omega_2 \sum_{n=0}^{\infty} \left\{ \frac{\omega_1}{x + m\omega_1 + n\omega_2} - \frac{\omega_1}{m\omega_1 + (n+1)\omega_2} \right\},$$

where $\gamma$ is Euler's constant. If one omits the term $m=n=0$, it is easy to see that except near the poles of $F_i(x)$ the general term of the series is dominated by $M|m\omega_1 + n\omega_2|^{-3}$ where $M$ is a positive constant independent of $x$ and hence that the series converges absolutely and uniformly throughout any finite region of the $x$-plane from which the poles of $F_i(x)$, which are simple poles of residue $\omega_1\omega_2$ at the points $x = -m\omega_1 - n\omega_2$, $m, n = 0, 1, 2, \ldots$, are excluded by small circles of radius $\delta$ about these points as centers.

Similarly one obtains analogous expressions for $F_i(x)$, $i=2, 3, 4$.

One finds that $\Delta_{\omega_2} F_1(x) = \Delta_{\omega_2} F_2(x)$ and hence that $F_1(x) - F_2(x)$ is a periodic function of period $\omega_2$.

Now consider $F_1(x) - F_4(x)$ where $F_1(x)$ is given by (28) and $F_4(x)$ by the similar formula obtained by application of Theorem V. By a method analogous to that by which (29) was arrived at one finds that

$$F_1(x) - F_4(x) = \omega_2 \sum_{n=0}^{\infty} \left\{ \pi i + \pi \cot \frac{\pi}{\omega_1} (x + n\omega_2) \right\},$$
and one sees that $F_1(x) - F_4(x)$ is a periodic function of period $\omega_1$.
Similarly one finds that

$$F_3(x) - F_2(x) = \omega_2 \sum_{n=0}^{\infty} \left\{ -\pi i + \pi \cot \frac{\pi}{\omega_1} (x - n + 1 \omega_2) \right\}.$$ 

One sees that $F_3(x) - F_2(x)$ is periodic of period $\omega_1$.

Let the cuts for $F_i(x)$ and the principal determination of the logarithm be as stated in the foregoing. Then the four solutions $F_i(x)$ ($i=1, 2, 3, 4$) have the following properties:

1) $F_i(x)$ ($i=1, 2, 3, 4$) is single-valued and analytic throughout the finite plane except for simple poles with residue $(-1)^{i+1} \omega_1 \omega_2$ at the points. (1) $x = -m \omega_1 - n \omega_2$ for $F_1(x)$. (2) $x = -m \omega_1 + (n+1) \omega_2$ for $F_2(x)$. (3) $x = (m+1) \omega_1 + (n+1) \omega_2$ for $F_3(x)$. (4) $x = (m+1) \omega_1 - n \omega_2$ for $F_4(x)$, $m, n = 0, 1, 2, \ldots$

2) $\lim_{x \to \infty, x \to -\infty} \frac{F_i(x) - [(x - \omega_1 + \omega_2/2) \log x - x]}{\omega_1 \omega_2} = 0$, $i = 1, 2, 3, 4$.

3) $F_1(x)$ is periodic of period $\omega_2$.
4) $F_3(x)$ is periodic of period $\omega_1$.
5) $F_2(x)$ is periodic of period $\omega_1$.

The functions $F_i(x)$ are completely characterized by these properties in the sense that if $g_i(x)$ are functions having the properties of $F_i(x)$ respectively, then $g_i(x) \equiv F_i(x)$.

Now consider $F_1(x)$ as given by (29). Since the series converges uniformly, it may be differentiated term-by-term. Let $F_1(x) - \omega_1 \omega_2 A(x) \equiv A(x)$. Then $A(x)$ is analytic at $x = 0$. One readily verifies that

$$|F_1(x) - x A'(0) - A(0)| / \omega_1 \omega_2 \equiv \xi_1(x)$$

$$= \frac{1}{x} + \sum_{m,n=0}^{\infty} \left\{ \frac{1}{x + m \omega_1 + n \omega_2} - \frac{1}{m \omega_1 + n \omega_2 + [m \omega_1 + n \omega_2]^2} \right\};$$

$$\xi_1(x) \sim \{F_1(x) - x A'(0) - A(0)\} / \omega_1 \omega_2 \text{ outside } S_1.$$

Similarly with $F_i(x)$, $i = 2, 3, 4$, one obtains the following relations:

$$|F_2(x) - x F_2'(0) - F_2(0)| / \omega_1 \omega_2 \equiv -\xi_2(x)$$

$$= - \sum_{m,n=0}^{\infty} \left\{ \frac{1}{x + m \omega_1 - (n+1) \omega_2} - \frac{1}{m \omega_1 - (n+1) \omega_2 + [m \omega_1 - (n+1) \omega_2]^2} \right\};$$

$$|F_3(x) - x F_3'(0) - F_3(0)| / \omega_1 \omega_2 \equiv \xi_3(x)$$

$$= \sum_{m,n=0}^{\infty} \left\{ \frac{1}{x - (m+1) \omega_1 - (n+1) \omega_2} - \frac{1}{(m+1) \omega_1 - (n+1) \omega_2} \right\};$$
\[
\begin{align*}
\{ F_n(x) - xF'_n(0) - F_n(0) \} / \omega_1 \omega_2 & \equiv - \zeta_n(x) \\
= & - \sum_{m,n=0}^{\infty} \left\{ \frac{1}{x - (m+1)\omega_1 + n\omega_2} - \frac{1}{-(m+1)\omega_1 + n\omega_2} \right. \\
& \left. + \frac{x}{[-(m+1)\omega_1 + n\omega_2]^3} \right\}; \\
\zeta_1(x) & \sim (-1)^{i+1} \left\{ R_2(x) - xF'_1(0) - F_1(0) \right\} / \omega_1 \omega_2 \text{ outside } S_i, \quad i = 2, 3, 4.
\end{align*}
\]

Then
\[
\zeta_1(x) + \zeta_2(x) + \zeta_3(x) + \zeta_4(x) \equiv \zeta(x)
= \frac{1}{x} + \sum_{m,n=-\infty}^{\infty} \left\{ \frac{1}{x + m\omega_1 + n\omega_2} - \frac{1}{m\omega_1 + n\omega_2} + \frac{x}{[m\omega_1 + n\omega_2]^3} \right\}.
\]

Now let
\[
\psi_i(x) \equiv - \varphi_i(x), \quad i = 1, 2, 3, 4.
\]

Then
\[
\psi_1(x) + \psi_2(x) + \psi_3(x) + \psi_4(x) \equiv \psi(x),
\]

where \( \psi(x) \) is the Weierstrass \( \psi \)-function;
\[
\psi_1(x) \sim - \left\{ R_2(x) - A'(0) \right\} / \omega_1 \omega_2 \text{ outside } S_i;
\]
\[
\psi_i(x) \sim (-1)^i \left\{ R_2(x) - F'_i(0) \right\} / \omega_1 \omega_2 \text{ outside } S_i, \quad i = 2, 3, 4.
\]

We thus see that the solutions \( F_i(x) \) of the equation \( \Delta_{\omega_1} \Delta_{\omega_2} F(x) = 1/x \) may be combined so as to give certain classic functions from which the whole theory of elliptic functions may be developed.

4. Other Equations. It is evident that by modifications of the methods used in obtaining Theorem V analogous theorems may be obtained for the equations
\[
\nabla_{\omega_1} \Delta_{\omega_2} F(x) \equiv \frac{F(x + \omega_1 + \omega_2) - F(x + \omega_1) + F(x + \omega_2) - F(x)}{2\omega_1} = \varphi(x),
\]
\[
\Delta_{\omega_1} \nabla_{\omega_2} F(x) \equiv \frac{F(x + \omega_1 + \omega_2) + F(x + \omega_1) - F(x + \omega_2) - F(x)}{2\omega_1} = \varphi(x),
\]
\[
\nabla_{\omega_1} \nabla_{\omega_2} F(x) \equiv \frac{F(x + \omega_1 + \omega_2) + F(x + \omega_1) + F(x + \omega_2) + F(x)}{4} = \varphi(x).
\]

As an example we shall treat briefly the equation
\[
\nabla_{\omega_1} \nabla_{\omega_2} G(x) = 1/x.
\]
Let \( Y_2(x) \) correspond to \( Y_1(x) \) (as defined in § 1) as \( R_2(x) \) corresponds to \( R_1(x) \). One finds

\[
Y_2(x) = \sum_{\mu=1}^{\infty} \frac{d_{\mu}}{x^{\mu}},
\]

where the \( d' \)s are given by the recurrence relation

\[
\sum_{s=0}^{r} (-1)^{s} \binom{r}{s} d_{s+1} \omega_2^{r-s} + d_{r+1} = (-1)^r 2E_r(0)\omega_1, \quad r=0, 1, 2, \ldots,
\]

where \( E_r(x) \) is the \( r \)-th Euler polynomial \(^1\).

One finds four solutions

\[
G_1(x) = 4 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{x + m\omega_1 + n\omega_2};
\]

\[
G_2(x) = 4 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{x + n\omega_1 + (n+1)\omega_2};
\]

\[
G_3(x) = 4 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{x - (m+1)\omega_1 - (n+1)\omega_2};
\]

\[
G_4(x) = 4 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{x - (m+1)\omega_1 + n\omega_2}.
\]

Moreover one has \( Gi(x) \sim Y_2(x) \) outside \( S_i, \ i=1, 2, 3, 4 \).

Form

\[
T(x) = \frac{1}{4} \sum_{i=1}^{4} (-1)^{i+1} G_i(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{x + m\omega_1 + n\omega_2}.
\]

The function \( T(x) \) is an elliptic function with two simple poles in a cell with residues 1 and -1. It satisfies the equations

\[
T(x + \omega_1 + \omega_2) = T(x + 2\omega_1) = T(x); \quad T(x + \omega_1) = T(x + \omega_2) = -T(x).
\]

Now let \( \omega_1 = 2K, \ \omega_2 = 2iK' \), and form \(-ik^{-1}T(u - iK')\). Upon consideration of the zeros and poles of this function one sees that

\[
-ik^{-1}T(u - iK') = cn(u)
\]

and that

\[
T(u) = ik \cdot cn(u + iK') = ds(u) = dn(u)/sn(u),
\]

where \( sn(u), \ cn(u), \) and \( dn(u) \) are the Jacobian elliptic functions \(^2\).

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\(^1\) The notation is that of Nörlund, I.c. p. 24.

\(^2\) See, for instance, Whittaker and Watson: Modern Analysis, Chap. XXII.