The Calculation of Centroids,

by

E. C. Stopher, Iowa, Iowa, U.S.A.

1. Introduction. The object of this paper is to develop formulae for finding the centroids of volumes and of the lateral surfaces of homogeneous solids which are associated with a given curve, $s$, in such a manner that a plane through any point, $p$, of $s$ and perpendicular to $s$ will intersect the solid in a circle with $p$ at the center. The discussion will be limited to solids for which $s$ is a continuous curve with a continuous derivative. Furthermore, the evolute of $s$ must lie outside the solid.

Examples of such solids include cylinders, cones, the torus, cornucopiae, and many others. General formulae will be presented and they will then be applied to some of these examples.

2. Volumes. Consider the volume (fig. 1) associated with the curve $s$ and divided into $n$ parts by planes perpendicular to $s$. Let $X_I$ represent any one of the triple, $(X_1^I, X_2^I, X_3^I)$, of coordinates of the centroid of the $i^{th}$ section and $A(s_i)$ some intermediate cross sectional area of that part. Then the coordinates, $X$, of the centroid of $V$ are given by (1)

$$X = \frac{\sum_{i=1}^{n} X_i A(s_i) \Delta s_i}{\sum_{i=1}^{n} A(s_i) \Delta s_i}.$$

The number $n$ of dividing planes may be increased without limit. In so doing, by Duhamel's theorem or, preferably, by E. V. Huntington's substitute theorem (2), the summation process may


THE CALCULATION OF CENTROIDS.

replaced by an integration process

\[ \bar{X} = \frac{\int_{s_1}^{s_2} X(s)A(s) \, ds}{\int_{s_1}^{s_2} A(s) \, ds}. \]  

(1)

Since this formula was obtained by letting \( \Delta s \) approach zero, then for any value \( s \) of the curve, \( A(s) \) is the cross sectional area at that point and \( X(s) \) is any one of the triple of coordinates giving the limiting position of the centroid of that wedge containing the point \( s \) as the two bounding planes approach coincidence at \( s \). The next problem is to find this position given by \( X(s) \).

This can readily be accomplished by relying on a proposition by Harry Hart(1), “If a wedge be cut out of a homogeneous solid by two planes intersecting in \( YY' \) and inclined at an infinitely small angle \( \theta \), then the distance from \( YY' \) of the center of gravity of the wedge, multiplied by its volume, is equal to the moment of inertia about \( YY' \) of the corresponding portion of the area, multiplied by \( \theta \).” This can be rigorously established by a limiting process based on a triple integration.

In the case at hand \( YY' \) passes through the center of curvature of the curve \( s \), “the corresponding portion of the area” is a circular cross section, and its moment of inertia about \( YY' \) is equal to its moment of inertia about an axis through its center plus the product of its area and the square of the distance from \( YY' \) to its center(2). If the radius of the circular cross section be denoted by \( r \) and the radius of curvature denoted by \( R \), then the moment of inertia of the circle with respect to \( YY' \) would be given by

\[ I = \pi r^2 \left( R^2 + \frac{1}{4} r^2 \right). \]

The infinitesimal volume \( dv \) of such a wedge-shaped slice is, by Pappus' theorem(3), equal to the cross sectional area times the length of the path of the centroid of the area:

\[ dv = \pi r^2 R \, d\theta, \]

(2) Smith and Longley, Theoretical Mechanics, section 11-13.
(3) Ibid, section 10.
where $d\theta$ represents the infinitesimal angle of the wedge.

Therefore, if $h$ represent the distance from $YY'$ to the centroid of the infinitesimal wedge-shaped slice,

$$h = \frac{1}{\pi r^2 R} \left( R^2 + \frac{1}{4} r^2 \right) R + \frac{r^2}{4R}.$$

The distance to the centroid from the curve $s$ outward along the principal normal is equal to $r^2/4R$. The coordinates of the centroid of the infinitesimal slice are, therefore, equal to the corresponding coordinates of the curve $s$ plus $r^2/4R$ times the proper direction cosine. The direction cosines of the principal normal directed outward are equal to $-Rx''$.$(1)$. The coordinates of the centroid of the infinitesimal slice are given by

$$X = x(s) - \frac{1}{4} r^2 x'' \quad \text{or} \quad x(s) - \frac{Ax''}{4\pi}.$$

Substituting this value in (1), the formula for the centroid of the volume $V$ becomes

$$\bar{X} = \frac{\int_{s_1}^{s_2} A(s) x(s) \, ds - \int_{s_1}^{s_2} x'' A \, ds}{\int_{s_1}^{s_2} A(s) \, ds}.$$

3. It is interesting to note the simplification occurring when the cross sectional area is constant. In that case (2) becomes

$$\bar{X} = \frac{\int_{s_1}^{s_2} x(s) \, ds - \int_{s_1}^{s_2} A \, ds}{\int_{s_1}^{s_2} dx''} \left|_{s_1}^{s_2} \right.$$

$$= \bar{x} - \frac{A}{4\pi L} \left. x'' \right|_{s_1}^{s_2},$$

where $\bar{x}$ is any one of the coordinates giving the centroid of the curve $s$, and $L$ is the length of $s$.

Examination of this formula reveals the interesting fact that if the two ends of the solid are parallel but oppositely directed, the centroid of the solid coincides with the centroid of the curve through the centers of cross section.

4. If the cross sections of a solid are constant circular rings

$(1)$ Snyder and Sisam, Analytic Geometry of Space, p. 247.
instead of constant circular areas, it can be considered as the difference of two solids of the type already discussed. If $A_1$ and $A_2$ denote the areas of the inner and the outer circles respectively, the principle of addition of moments, together with formula (3), gives

$$\overline{x} = \overline{r} - \left(\frac{A_2 + A_1}{4\pi L}\right) x'_s \bigg|^{s_2}_{s_1}.$$  

As the thickness of this shell approaches zero, its centroid approaches that of its lateral surface. Letting $A_1$ equal $A_2$ in the above expression and dropping the subscripts, we have for the centroid of the lateral surface

$$\overline{x} = \overline{r} - \frac{A}{2\pi L} x'_s \bigg|^{s_2}_{s_1}.$$  \hspace{1cm} (3')

5. Lateral Surface. A formula for the centroid of the lateral surface of a solid with a varying circular cross section will be developed by methods similar to those used for the volume.

Again consider the figure as divided into $n$ parts by planes perpendicular to the curve $s$. Then let $\Delta s_i$ represent the length of the $i^{th}$ interval of $s$; $r(s_i)$ some intermediate value of the radius in that interval; and $X(s_i)$ any one of the triple of coordinates of the lateral surface belonging to that interval

$$\overline{x} = \frac{\sum_{i=1}^{n} X(s_i) 2\pi r(s_i) \Delta s_i}{\sum_{i=1}^{n} 2\pi r(s_i) \Delta s_i}.$$  

As before, letting $n$ increase without limit and replacing the summation process by an integration process, an equation corresponding to (1) results

$$\overline{x} = \frac{\int_{s_1}^{s_2} X(s) r(s) \, ds}{\int_{s_1}^{s_2} r(s) \, ds}.$$  \hspace{1cm} (1')

Hart gave a proposition about areas corresponding to the one about volumes by which the $X(s)$ of the preceding equation can be found. $X(s)$ are the coordinates of the limiting position of the centroid of the lateral surface of a wedge as the planes perpendicular to the curve $s$ and bounding the wedge approach coincidence. If $2d\theta$ is the angle between the planes bounding the wedge, then the
distance $h$ from the center of curvature of the curve at the point under consideration to the centroid of the infinitesimal band is given by

$$h = 2I \sin d\theta |dA|.$$

Letting $R$ represent the radius of curvature at the point, the moment of inertia $I$ of the circle about an axis at a distance $R$ from the center of the circle is equal to $2\pi r \left( R^2 + \frac{1}{2} r^2 \right)$. The infinitesimal area $dA$ of the surface generated between the two bounding planes is equal to the perimeter of the circle times $ds$, or times $2Rd\theta$, the distance traveled by the centroid of the circle.

Substituting,

$$h = \frac{2 \sin d\theta 2\pi r \left( R^2 + \frac{1}{2} r^2 \right)}{2d\theta 2\pi r R} = R + \frac{r^2}{2R}.$$

The coordinates of the centroid of this infinitesimal band are equal to the corresponding coordinates of the curve $s$ plus the product of the distance $h$ and the proper direction cosine of the principal normal directed outward:

$$X = x(s) - \frac{1}{2} r^2 x''.$$

Substituting in (1'),

$$\overline{X} = \frac{\int_{s_1}^{s_2} x(s) r(s) ds - \frac{1}{2} \int_{s_1}^{s_2} r^2(s) x'' ds}{\int_{s_1}^{s_2} r(s) ds}. \quad (2')$$

The formula (3') already obtained is a special case of (2'), as is readily seen if the $r$ of (2') is considered constant.

$$\overline{X} = \frac{\int_{s_1}^{s_2} x(s) ds - \frac{1}{2} r^2 \int_{s_1}^{s_2} x'' ds}{\int_{s_1}^{s_2} ds} = \overline{x} - \frac{r^2}{2L} x'' \bigg|_{s_1}^{s_2}. \quad (3')$$

6. Examples. The centroids both of the volumes and of the
lateral surfaces of several solids will be found by use of these formulae.

Consider the half torus, for example. By symmetry, \( X_1 \) and \( X_2 \) will be zero both for the volume and for the lateral surface. \( X_3 \) in the case of the volume is given by formula (3).

\[
X_3 = \bar{x}_3 - \frac{s^2}{4L} x'_3 \bigg|_{s_1} = \frac{2R}{\pi} + \frac{r^2}{2\pi R}.
\]

For the centroid of the lateral surface,

\[
X_3 = \bar{x}_3 - \frac{r^2}{2L} x'_3 \bigg|_{s_1} = \frac{2R}{\pi} + \frac{r^2}{\pi R}.
\]

If the solid in question is associated with a semi-ellipse but has constant circular cross sections, \( X_1 \) and \( X_2 \) are again zero by symmetry. For the volume,

\[
X_3 = \frac{2eb^2 + 2ab \sin^{-1} e + er^2}{4ae E\left(\frac{1}{2}, \pi, e\right)}
\]

where \( a \) and \( b \) are the semi-major and the semi-minor axes respectively and \( e \) is the eccentricity of the ellipse.

For the lateral surface,

\[
X_3 = \frac{eb^2 + ab \sin^{-1} e + er^2}{2ae E\left(\frac{1}{2}, \pi, e\right)}
\]

If the curve \( s \) is an arch of a cycloid and the circular cross sections are constant, \( X_1 \) and \( X_2 \) would again be zero. For the volume,

\[
X_3 = 4a/3 + r^2/16a,
\]

where \( a \) is the constant associated with the equations of the cycloid curve.

For the lateral surface,

\[
X_3 = 4a/3 + r^2/8a.
\]

The more general formulae (2) and (2') were used to find the centroid of the volume and of the lateral surface of the cornucopia.
mentioned by J. B. Reynolds in a recent article (1). This cornu-
copia is associated with a spiral curve, the coordinates of which can
be expressed parametrically in terms of the arc s as follows:

\[ x = 4a \cos s/5a, \ 4a \sin s/5a, \ 3s/5. \]

The variable radius of cross section is equal to \( s/8 \). In considering
the figure associated with one quarter of a revolution of the spiral,
the limits in the integration are 0 and \( 5\pi a/2 \). In the case of the
volume, formula (2) gives

\[
\bar{X}_1 = \frac{3\pi a}{128} + \frac{183a}{8\pi} - \frac{183a}{\pi^3},
\]
\[
\bar{X}_2 = \frac{3a}{16} + \frac{183a}{2\pi} - \frac{183a}{\pi^3},
\]
\[
\bar{X}_3 = \frac{9\pi a}{8}.
\]

For the lateral surface,

\[
\bar{X}_1 = \frac{\pi a}{32} + \frac{61a}{4\pi} - \frac{61a}{2\pi^3},
\]
\[
\bar{X}_2 = \frac{3a}{16} + \frac{61a}{2\pi^2},
\]
\[
\bar{X}_3 = \pi a.
\]

(1) J. B. Reynolds, A New Formula for Volume, American Mathematical