The Extension of Algebraic Numbers,

by

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Introductory. An algebraic number has been defined to be that number which can be made the root of a rational integral algebraic equation with rational coefficients.

It is obvious from the definition that the roots of a rational algebraic equation (whether integral or not) with rational coefficients are algebraic numbers. And in particular if such an equation is reduced to the form \( f(x)/F(x) = 0 \) such that the coefficients of \( f(x) \) are rational, then, no matter what the coefficients of \( F(x) \) may be, the roots are algebraic numbers.

For the roots of \( f(x)/F(x) = 0 \) are such roots of \( f(x) = 0 \) as do not satisfy \( F(x) = 0 \); which establishes the proposition.

It has been proved in ‘The Elements of the Theory of Algebraic Numbers’ by Legh Wilber Reid, chap. 9, §2, Theorem 9, that the roots of a rational integral algebraic equation are algebraic integers or algebraic numbers, if the coefficients are themselves algebraic integers or algebraic numbers respectively, where an algebraic integer is defined to be the root of a rational integral algebraic equation whose coefficients are the integers of the rational realm (i.e. ordinary integers) and in which the coefficient of the highest degree term is unity.

Assuming the truth of the above theorem we shall prove that the roots of every algebraic equation (whether rational integral or not) whose coefficients are algebraic numbers are themselves algebraic numbers.

Irrational integral equations. The truth of the theorem is obvious in the case of non-integral rational algebraic equations. It is therefore sufficient to prove the theorem in the case of irrational algebraic equations.

To begin with we shall take up the discussion of irrational integral algebraic equations.

Case I. Let such an equation be of the type

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0, \tag{1} \]

where the indices are positive rational fractions (including integers)
and the coefficients are algebraic numbers. Also the terms are
arranged in descending order of the magnitude of the indices.

Let \( h \) be the highest common factor\(^1\) of \( v_0, v_1, v_2, v_3, \ldots, v_{n-1} \), so that

\[
\begin{align*}
v_0 &= h p_0, \\
v_1 &= h p_1, \\
v_2 &= h p_2, \\
v_3 &= h p_3, \\
&\vdots \\
v_{n-1} &= h p_{n-1},
\end{align*}
\]

where \( p_0, p_1, p_2, p_3, \ldots, p_{n-1} \) are positive integers.

Putting, therefore, \( y = x^h \) in (1), we have

\[
a_0 y^{p_0} + a_1 y^{p_1} + a_2 y^{p_2} + a_3 y^{p_3} + \ldots + a_{n-1} y^{p_{n-1}} + a_n = 0. \tag{2}
\]

Let the roots of (2) be \( y_1, y_2, y_3, \ldots, y_{p_0} \) which are obviously
algebraic numbers.

Now the roots of (1) are given by

\[
x^h = y_i, \text{ where } i = 1, 2, 3, \ldots, p_0.
\]

Also \( h \) being the H.C.F. of positive rational fractions is itself
a positive rational fraction.

Let \( h = \frac{p}{q} \) in its lowest terms, where \( p \) and \( q \) are positive integers.

Then we have

\[
x^p - y^{p_i} = 0, \text{ where } i = 1, 2, 3, \ldots, p_0,
\]

the coefficients of which are algebraic numbers, as \( y^{p_i} \) is an algebraic
number.

Thus the roots of \( x^p - y^{p_i} = 0 \) are algebraic numbers.

**Corollary.** If in (1) \( a_0 = 1 \), and \( a_1, a_2, a_3, \ldots, a_{n-1}, a_n \) are alge-
braic integers, then the roots of (1) are algebraic integers.

For we have the same values of \( a_0, a_1, \ldots, a_n \) in (2) also, there-
fore \( y_i \) and hence \( y^{p_i} \) is an algebraic integer. Hence the corollary.

**Case II.** Let the equation be of the type

\[
A_0 \phi(x) \phi^{p_0} + A_1 \phi(x) \phi^{p_1} + A_2 \phi(x) \phi^{p_2} + A_3 \phi(x) \phi^{p_3} + \ldots \\
\ldots + A_{n-1} \phi(x) \phi^{p_{n-1}} + A_n = 0,
\tag{3}
\]

where \( A_0, A_1, A_2, A_3, \ldots, A_{n-1}, \) and \( A_n \) are algebraic numbers and the

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\(^1\) Any common factor will do, but H.C.F. is taken to simplify the investi-
gation.
indices are positive rational fractions including integers arranged in descending order and \( \phi(x) \) is of the form

\[
b_0x^r + b_1x^{r-1} + b_{r-1}x^2 + \ldots + b_{r-1}x + b_r,
\]

where \( b_i \) \((i = 0, 1, 2, \ldots, r-1, \text{and } r)\)

is a rational integral algebraic function of \( x \), whose coefficients are algebraic numbers, and the indices \( \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{r-1} \) are positive rational fractions including integers.

Let \( k \) be the greatest common measure of \( \mu_0, \mu_1, \mu_2, \mu_3, \ldots, \mu_{n-1} \), so that

\[
\mu_s = kq_s, \quad s = 0, 1, 2, \ldots, n-1,
\]

where \( q_0, q_1, q_2, q_3, \ldots, q_{n-1} \) are positive integers.

Put \([\phi(x)]^k = \eta\), then (3) becomes

\[
A_0\eta^0 + A_1\eta^1 + A_2\eta^2 + \ldots + A_{n-1}\eta^{n-1} + A_n = 0,
\]

the roots of which are \( \eta = q_i \), where \( i = 1, 2, 3, \ldots, q_0 \).

Thus \([\phi(x)]^k = \phi_i\).

Let now \( k = \pi/\kappa \), where \( \pi \) and \( \kappa \) are positive integers, then

\[
\phi(x) = (q_i)^{\pi/\kappa} = \xi_j \text{(say)},
\]

where \( j = 1, 2, 3, \ldots, \pi \) and \( i = 1, 2, 3, \ldots, q_0 \); so that \( \phi(x) \) has \( \pi q_0 \) values, and \( \xi_j \) is an algebraic number since \( q_i \) is an algebraic number.

But since \( \phi(x) \) is of the form (3a), whose coefficients are rational integral algebraic functions of \( x \), whose coefficients are algebraic numbers, therefore (3a) can always be reduced to the first number of (1) and therefore the equation \( \phi(x) = \xi_j \) can be reduced to the form (1).

Hence the roots of (3), whose solution depends on that of \( \phi(x) = \xi_j \) are algebraic numbers by Case I.

**Corollary.** If \( A_0 = 1 \), and \( A_1, A_2, A_3, \ldots, A_{n-1} \) and \( A_n \) are integers and if, when (3a) is reduced to (1), \( a_0 = 1 \), and \( a_1, a_2, a_3, \ldots, a_{n-1}, a_n \) are also integers, then the roots of (3) are algebraic integers.

**Case III.** Let some or all of the \( A \)'s be rational integral al-

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(1) Obviously any common measure will do if the simplicity be not under consideration.
geometric functions of $x$ and $\phi(x)$ whose coefficients are algebraic numbers, and let those $A$s which are constants be algebraic numbers.

Let the lefthand side of (3) be denoted by $\psi(x)$, where $\psi(x)$ may now without loss of generality be written as

$$A_0 + A_1[\phi(x)]^n + A_2[\phi(x)]^n + A_3[\phi(x)]^n + \ldots + A_{n-1}[\phi(x)]^{n-1}.$$  

Then putting $y = \psi(x)$, and multiplying $y, y^2, \ldots, y^{n-2}, y^{n-1},$ and $y^n$ by $K_1, K_2, \ldots, K_{n-2}, K_{n-1},$ and 1 respectively, we have

$$K_1y = K_1A_0 + K_1A_1z^n + K_1A_2z^{2n} + K_1A_3z^n + \ldots + K_1A_{n-2}z^{n-2} + K_1A_{n-1}z^{n-1},$$

$$K_2y^2 = K_2B_0 + K_2B_1z^n + K_2B_2z^{2n} + K_2B_3z^n + \ldots + K_2B_{n-2}z^{n-2} + K_2B_{n-1}z^{n-1},$$

$$K_{n-2}y^{n-2} = K_{n-2}U_0 + K_{n-2}U_1z^n + K_{n-2}U_2z^{2n} + K_{n-2}U_3z^n + \ldots + K_{n-2}U_{n-2}z^{n-2} + K_{n-2}U_{n-1}z^{n-1},$$

$$K_{n-1}y^{n-1} = K_{n-1}V_0 + K_{n-1}V_1z^n + K_{n-1}V_2z^{2n} + K_{n-1}V_3z^n + \ldots + K_{n-1}V_{n-2}z^{n-2} + K_{n-1}V_{n-1}z^{n-1},$$

$$y^n = W_0 + W_1z^n + W_2z^{2n} + W_3z^n + \ldots + W_{n-2}z^{n-2} + W_{n-1}z^{n-1};$$

when $z$ denotes $\phi(x)$, and $B_1, \ldots, U_i, V_i,$ and $W_i, i$ being $0, 1, 2, \ldots, n-1$, are rational integral algebraic functions in $x$ and $\phi(x)$ arising obviously in the successive involutions of $y$.

The coefficients of these functions are obviously algebraic numbers.

Adding equations (4), we have

$$y^n + K_{n-1}y^{n-1} + K_{n-2}y^{n-2} + \ldots + K_2y^2 + K_1y = W_0 + K_{n-1}V_0 + K_{n-2}U_0 + \ldots + K_2B_0 + K_1A_0,$$

where $K$'s are so chosen that the coefficients of $z^n, z^{2n}, \ldots, z^{n-2}, z^{n-1}$ in the sum identically vanish, so that we have
Now equations (6) are \(n-1\) in number which, therefore, determine the \('n-1'\) \(K's\) uniquely, since they are linear, except in the critical case the discussion of which will be taken up later on, viz. the case when equations (6) become inconsistent.

Thus, except in the critical case, the \(K's\) will always be rational functions of \(x\) and \(\phi(x)\) with coefficients which are algebraic numbers.

This is obvious because the solution of the equations (6) involves only four fundamental rational algebraic operations.

The zeros of (3b), therefore, are the roots of \(y=0\) solved simultaneously with (5). Hence, the zeros of (3b) are the roots of

\[
W_0 + K_{n-1}V_0 + K_{n-2}U_0 + \ldots + K_2B_0 + K_1A_0 = 0,
\]

where the first member of (7) is rational in \(x\) and \(\phi(x)\) with coefficients, which are algebraic numbers, i.e., (7) is of the form

\[
\frac{G[x, \phi(x)]}{H[x, \phi(x)]} = 0,
\]

i.e. of the form

\[
\frac{D_0[\phi(x)]^n + D_1[\phi(x)]^{n-1} + \ldots + D_{n-1}[\phi(x)] + D_n}{C_0[\phi(x)]^n + C_1[\phi(x)]^{n-1} + \ldots + C_{n-1}\phi(x) + C_n} = 0,
\]

where \(D's\) and \(C's\) are rational integral functions of \(x\) with coefficients, which are algebraic numbers.

After the involutions of \(\phi(x)\) are evaluated, (9) reduces to

\[
\frac{E_0\omega^0 + E_1\omega^1 + E_2\omega^2 + \ldots + E_{n-1}\omega^{n-1} + E_n}{G_0\omega^0 + G_1\omega^1 + G_2\omega^2 + \ldots + G_{m-1}\omega^{m-1} + G_m} = 0,
\]

where \(E's\) and \(G's\) are algebraic numbers, and \(\rho's\) and \(\mu's\) positive rational fractions.

The zeros of (3b) are, therefore, the roots of (10) although the converse is not true, except those zeros of (3b), which make the denominator of the first member of (10) vanish.

But the roots of (10) are obviously such zeros of the numerator
of the first member as do not make the denominator vanish; and
such roots are obviously algebraic members by Case I.

Thus the zeros of (3b) are algebraic numbers in this case too,
except such zeros of (3b) that make the denominator of the first
member of (10) vanish. And these excepted zeros will be taken up
under the critical case.

Case IV. Let the equation be of the type

\[ f \left( \frac{1}{\phi_1(x)^{n_1}}, \frac{1}{\phi_2(x)^{n_2}}, \frac{1}{\phi_3(x)^{n_3}}, \ldots, \frac{1}{\phi_t(x)^{n_t}} \right) = 0, \quad (11) \]

where the first member of (11) is a rational integral algebraic func-
tion of \( \frac{1}{\phi_i(x)^{n_i}}, \ i = 1, 2, 3, \ldots, t, \) whose coefficients are
algebraic numbers and

\[ \phi_i(x) = b_{i_1}x^{i_1} + b_{i_2}x^{i_2} + \ldots + b_{i_t}x^{i_t}, \]

\( i = 1, 2, 3, \ldots, t, \) where \( b_{ij} \) is a rational integral algebraic
function of \( x, \) whose coefficients are algebraic numbers.

The form of \( \phi_i(x) \) being settled, (11) can be written as

\[ A_1 + A_{n_1} \frac{1}{\phi_1(x)^{n_1}} + A_{n_2} \frac{1}{\phi_2(x)^{n_2}} + \ldots + A_{n_t} \frac{1}{\phi_t(x)^{n_t}} \]

\[ + B_{n_1} \frac{1}{\phi_1(x)^{n_1}} + B_{n_2} \frac{1}{\phi_2(x)^{n_2}} + \ldots + B_{n_t} \frac{1}{\phi_t(x)^{n_t}} \]

\[ + B_{n_2} \frac{1}{\phi_1(x)^{n_1}} \phi_2(x)^{n_2} + \ldots + \ldots \ldots \]

\[ + \ldots \ldots \ldots \ldots \ldots \ldots = 0, \quad (12) \]

where the coefficients denoted by capital letters are rational integral
functions of \( x \) and \( \phi(x) \) whose coefficients are algebraic numbers.

In Case III we were able to rationalise the equation \( \psi(x) = 0 \)
by considering \( \psi(x) \) as linear in \( [\phi(x)]^{1/n}, \) where \( i = 1, 2, 3, \ldots, (n - 1). \)

It is therefore obvious that we can utilize the same method of
rationalisation by considering the first member of (12) as linear in

\[ [\phi_1(x)]^{n_1}, [\phi_2(x)]^{n_2}, \ldots, [\phi_t(x)]^{n_t}, \]

\[ [\phi_1(x)]^{2/n_1}, [\phi_2(x)]^{2/n_2}, \ldots, [\phi_t(x)]^{2/n_t}, \]
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\[ [\phi_1(x)]^{\nu_1}, [\phi_2(x)]^{\nu_2}, \ldots \text{ etc.} \]

i.e. we may first rationalise it with respect to \( \phi(x) \), then with respect to \( \phi_2(x) \) etc. This will introduce \( n_1 + n_2 + n_3 + \ldots + n_t - t \) undetermined multipliers.

Or we may rationalise simultaneously with respect to \( \phi_1(x), \phi_2(x), \ldots, \phi_t(x) \). This will introduce \( L - 1 \) multipliers, \( L \) being the lowest common multiple of \( n_1, n_2, \ldots, n_t \), which are positive integers.

Of the two methods that one is preferable which leads to the rationalised equation of the lower degree and involves the smaller number of undetermined multipliers.

In general,

\[ L > n_1 + n_2 + n_3 + \ldots + n_t, \]
\[ L - 1 > n_1 + n_2 + n_3 + \ldots + n_t - t. \]

Hence exactly by the same reasoning as in case III, the roots of (12) are algebraic numbers except in the critical case, in which the simultaneous linear equations analogous to those of the system (6) become inconsistent and except such roots of (12) that are analogous to those of case III, which make the denominator of the first member of (10) vanish.

In what follows we shall discuss the critical cases of III and IV and it will be seen that the method covers all the excepted roots under III and IV. Thus the method applied to the critical case is quite general.

The critical case V. Under case III we have remarked that those zeros of (3b), which make the denominator of the first member of (10) vanish have not been proved to be algebraic by what is said there. Since the coefficients involved in the determination of the undetermined coefficients \( K \)'s from the system of equations (6) in case III and from an analogous system of equations in case IV are rational integral algebraic functions of a certain function of \( x \), it is obvious that the exceptional portion of case III and the analogous one of case IV are respectively included in the two critical cases, in which the two systems of linear equations for determining the undetermined multipliers become inconsistent.

To state the condition of inconsistency it will be sufficient to concentrate our attention to case III, in which the \((n - 1)\) linear equations (6) become inconsistent if, and only if, the matrix of the system
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\[
\begin{array}{cccc}
A_1 & B_1 & \ldots & U_1 & V_1 \\
A_2 & B_2 & \ldots & U_2 & V_2 \\
A_3 & B_3 & \ldots & U_3 & V_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{n-2} & B_{n-2} & \ldots & U_{n-2} & V_{n-2} \\
A_{n-1} & B_{n-1} & \ldots & U_{n-1} & V_{n-1} \\
\end{array}
\]

has a lower rank than the augmented matrix

\[
\begin{array}{cccc}
A_1 & B_1 & \ldots & U_1 & V_1 & W_1 \\
A_2 & B_2 & \ldots & U_2 & V_2 & W_2 \\
A_3 & B_3 & \ldots & U_3 & V_3 & W_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
A_{n-2} & B_{n-2} & \ldots & U_{n-2} & V_{n-2} & W_{n-2} \\
A_{n-1} & B_{n-1} & \ldots & U_{n-1} & V_{n-1} & W_{n-1} \\
\end{array}
\]

We say 'if and only if' because the condition is both necessary and sufficient the proof of which can be found in any standard treatise on Modern Higher Algebra.

As a necessary consequence of this necessary and sufficient condition we have the determinant

\[
\begin{array}{cccc}
A_1 & B_1 & \ldots & U_1 & V_1 \\
A_2 & B_2 & \ldots & U_2 & V_2 \\
A_3 & B_3 & \ldots & U_3 & V_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{n-2} & B_{n-2} & \ldots & U_{n-2} & V_{n-2} \\
A_{n-1} & B_{n-1} & \ldots & U_{n-1} & V_{n-1} \\
\end{array}
= 0. \quad (13)
\]

In this case, therefore, it is obvious that at least some of the zeros of (3b) have not been proved to be algebraic numbers.

In this critical case we shall, therefore, proceed as follows: —

Let the irrational integral algebraic equation be of the type

\[A_0 + A_1[\phi(x)]^1 + A_2[\phi(x)]^2 + A_3[\phi(x)]^3 + \ldots + A_{n-1}[\phi(x)]^{n-1} = 0, \quad (3c)\]

where the first member is the same as (3b).

Let us form the equations
(y - A_0)^n - A^n_0[\phi(x)] = 0,
\frac{y^n}{A_0^n}[\phi(x)]^2 = 0,
\frac{y^n}{A_1^n}[\phi(x)]^3 = 0,
\frac{y^n}{A_2^n}[\phi(x)]^4 = 0,
\ldots
\frac{y^n}{A_{n-1}^n}[\phi(x)]^n-1 = 0, \quad (14)

one of whose roots is respectively, when solved for \( y \), \( A_0 + A_1[\phi(x)]^{\frac{1}{n}} \), \( A_2[\phi(x)]^{\frac{2}{n}} \), \( A_3[(x)]^{\frac{3}{n}} \), \ldots, \( A_{n-1}[\phi(x)]^{\frac{n-1}{n}} \).

Also we can easily see that the coefficients of the terms in \( y \), or independent of \( y \), of (14) are rational integral algebraic functions of \( x \) and \( \phi(x) \), whose coefficients are algebraic numbers.

Multiplying the equations (14), we have,

\( y^{n(n-1)} + S_1 y^{n(n-1) - 1} + S_2 y^{n(n-1) - 2} + \ldots + S_{n(n-1)} = 0, \quad (15) \)

where \( S_1, S_2, \ldots, S_{n(n-1)} \) are obviously the rational integral algebraic functions of \( x \) and \( \phi(x) \).

Now each of the roots of (14) is also the root of (15), so that \( A_0 + A_1[\phi(x)]^{\frac{1}{n}}, A_2[\phi(x)]^{\frac{2}{n}}, A_3[(x)]^{\frac{3}{n}}, \ldots, A_{n-1}[\phi(x)]^{\frac{n-1}{n}} \) are among the roots of (15).

Let us now form an equation, whose roots are the sums of the roots of (15) taken \( n - 1 \) at a time.

Thus we have

\( y^p + R_1 y^{p-1} + R_2 y^{p-2} + \ldots + R_{p-1} y + R_p = 0 \) \quad (16)
as the required equation, where

\[ P = \binom{n(n-1)}{n-1}, \text{ i.e. } = \frac{\{n(n-1)\}!}{(n-1)!\,(n-1)^{n-1}}. \]

and \( R \)'s being rational integral symmetrical algebraic functions of the roots of (15) are by Newton's theorem rational integral algebraic functions of \( S \)'s, also the coefficients in (15), and since these \( S \)'s themselves in their turn are the rational integral algebraic functions of \( x \) and \( \phi(x) \), our \( R \)'s in (16) are also rational integral algebraic functions of \( x \) and \( \phi(x) \).

Also it is obvious that

\( y = A_0 + A_1[\phi(x)]^{\frac{1}{n}} + A_2[\phi(x)]^{\frac{2}{n}} + A_3[\phi(x)]^{\frac{3}{n}} + \ldots + A_{n-1}[\phi(x)]^{\frac{n-1}{n}} \)
is a root of (16). Therefore the roots of (3c) are given by \( y = 0 \) simultaneously with (16).

Therefore all the roots of (3c) are given by
although the converse is not true.

But $R_p$ is a rational integral algebraic function of $x$ and $\phi(x)$, whose coefficients are algebraic numbers.

Therefore the equation $R_p=0$ is easily reducible to the form (1) under case I, whence all the roots of $R_p=0$ are algebraic numbers and therefore those of (3c) are also so.

A similar demonstration exactly on the same lines can be given for case IV.

It is no doubt true that the demonstration given in the present section is applicable even to the non-critical cases and thus the demonstrations given in cases III and IV may be regarded as alternative proofs under non-critical conditions.

But at any rate the methods outlined in cases III and IV lead to a rationalised equation of the lowest degree and therefore are more suitable to practical needs unless we have the critical case.

**Irrational integral equations concluded.** Thus we have arrived at the conclusion that the roots of an irrational integral algebraic equation, whose coefficients are algebraic numbers are also algebraic numbers. And, the various cases taken, I think, are exhaustive.

**Irrational non-integral equations.** Every such equation is reducible to the form $\phi(x)/\psi(x)=0$, where each of $\phi(x)$ and $\psi(x)$ is an integral irrational algebraic function of $x$.

Now the roots of $\phi(x)/\psi(x)=0$ are obviously those roots of $\phi(x)=0$, which do not occur in $\psi(x)=0$. But by foregoing discussion all the roots of $\phi(x)=0$ are algebraic numbers if coefficients occuring in it are algebraic numbers.

Therefore the roots of $\phi(x)/\psi(x)=0$ are algebraic numbers even if the coefficients of $\psi(x)$ are non-algebraic numbers provided only that the coefficients of $\phi(x)$ are algebraic numbers.

19. 5. 1933.