An Extension of the Definitions of Perpendicular and Power,

by

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If the terms defined in the following lines are adopted it can be proved that a circle or circles in the theorem on circles may be replaced by a conic or homothetic conics.

**Definition.** The two straight lines parallel or coincident to a diameter and the conjugate chord of a conic respectively are said to be pseudo-perpendicular to each other, and one of them be the pseudo-perpendicular to the other. The point of intersection is said to be the foot of the pseudo-perpendicular.

If a segment of the straight line having the extremities at \( A \) and \( B \) is bisected by a straight line \( \chi \) pseudo-perpendicular to it, the points \( A \) and \( B \) are said to be pseudo-symmetrical with respect to the straight line \( \chi \), and the straight line \( \chi \) is called the axis of pseudo-symmetry. The definition of a pseudo-symmetrical figure may be inferred from that of an ordinary symmetrical figure.

Two straight lines are said to be pseudo-antiparallel with respect to a straight line parallel or coinciding to the axis of pseudo-symmetry. Especially if the two straight lines are parallel to each other the common pseudo-perpendicular may be considered to be parallel to the axis of pseudo-symmetry. A conic described by the extremity of a radius vector from its centre sweeping always in the same sense is said to be an oriented conic. The definition of the contact of two oriented conics or that of an oriented straight line with an oriented conic may be inferred from the definition of the contact of two oriented circles or that of an oriented straight line with an oriented circle.

The straight line which passes through the centre of an oriented conic homothetic to the oriented conic under consideration and touching the two oriented straight lines, and also passes through the point of intersection, is called the pseudo-bisector of the angle between the oriented straight lines.

Let \( l \) be a straight line making an angle \( \alpha \) with the directrix of a conic \( S \). The excess of the square of the distance from \( P \) to
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The focus of a point having an equal power in the direction of the same line with respect to two homothetic conics is called the radical axis(2) of the conics.

If $A, B$ are on a straight line passing through $P$, and $PA \cdot PB$ is equal to $\frac{\mu}{1 - e^2 \sin^2 \theta}$, where $\theta$ is the angle which the straight line makes with the directrix, then $A$ and $B$ are inverses(3) to each other, $P$ the centre and $\mu$ the constant of inversion.

The definition of the figures inverse to each other may be inferred from that of the ordinary inverse figures. As an example of the use of the terminology given above the proof of a relation which may be regarded as an extension of Steiner's theorem is given without making use of imaginary transformations, although it can be reached easier by the use of imaginaries. Before giving the proof of the theorem the definitions of a few terms and a lemma are added here.

Definition. The point of cointersection of the three pseudo-perpendiculars to the opposite sides from the vortices of a triangle is said to be the pseudo-orthocentre. Of the central conics touching the three sides of a triangle $ABC$, one whose centre is within the part common to any two angles, for example, $B$ and $C$, is called the inscribed conic. One whose centre is within the part common to the adjacent and supplement angles to $B$ and $C$ is called the escribed conic opposite to the angle $A$.

Lemma. If a quadrangle having four points for the vertices, the first one being on the side passing through one of two vertices

(1) If the straight line $l$ intersects the conic $S$ in two real points $A$ and $B$, the power of the point $P$ in the direction of the line $l$ is $PA \cdot PB$.

(2) The radical axis is a straight line.

(3) This is a modification of a definition given in my paper, Inversion with respect to a conic, Tôhoku Mathematical Journal, Vol. 6, p. 166. Take $P$ as the origin of rectangular coordinates whose $x$-axis is the straight line perpendicular to the directrix of the conic. Let the coordinates of two points inverse to each other be $x'y'$ and $x''y''$, then the relations between these coordinates are:

$$x'' = \frac{\mu x'}{(1 - e^2)x'^2 + y'^2}, \quad y'' = \frac{\mu y'}{(1 - e^2)x'^2 + y'^2}.$$
of the original quadrangle inscribed in a conic, the second being on
the other side passing through the other vertex, and the third and the
fourth being the two remaining vertices them-
selves, can be inscribed in a second conic which
is homothetic to the original one, the straight line
passing through a pair of the new points is parallel
to the side of the original quadrangle containing
the vertices first referred to.

In the case in which the original conic is a
hyperbola the second conic may be homothetic to
the conjugate hyperbola, or a set of two straight
lines parallel to the asymptotes of the original conic. In the case in
which the original conic is a parabola the second conic may be a set
of two straight lines parallel to the axis of the parabola.

The converse of the lemma is also true.

Dem. Take the points E and F each on the pair of the sides
BC and AD of the quadrangle ABCD inscribed in a conic. Let the
quadrangle ABEF be inscribed in a conic homothetic to the conic
ABCD, then the lines EF and CD are parallel to each other.

Firstly, if a pair of the opposite sides, BC and AD, are not
parallel to one another and K is the point of intersection of the
sides, then by a property of homothetic conics

\[
\frac{KA \cdot KD}{KB \cdot KC} = \frac{KA \cdot KE}{KB \cdot KE'}, \quad \frac{KD}{KC} = \frac{KF}{KE'}
\]

Hence EF is parallel to CD.

Secondly, if BC and AD are parallel to one another, then by a
property of conic, AB and CD are pseudo-antiparallel with respect
to a diameter of a conic ABCD having BC and AD as conjugate
chords.

AB and EF are pseudo-antiparallel with respect to a diameter
of the conic ABEF having BE and AF as conjugate chords. More-
over, by a property of homothetic conics these two diameters are
parallel to one another. Therefore each of CD and EF is pseudo-
antiparallel to the same line AB with respect to a straight line, and
hence they are parallel to one another.

In the case in which the conic ABCD is a hyperbola, ABEF
may be homothetic to the conjugate hyperbola of ABCD, or a set of
two straight lines AE and BF, each parallel to the asymptotes of
the hyperbola ABCD.
In the case in which the conic $ABCD$ is a parabola, $ABEF$ may be a set of two straight lines $AE$ and $BF$, each parallel or coinciding to the axis of the parabola. Conversely, if $EF$ and $CD$ are parallel to one another the quadrangle $ABEF$ can be inscribed in a conic homothetic to the conic $ABCD$.

In the case in which the conic $ABCD$ is a hyperbola, if only one or all of the sides of the triangle $ABE$ intersect different branches of the former hyperbola $ABCD$, the latter hyperbola $ABEF$ will be homothetic to the hyperbola conjugate to the former hyperbola. There the case may occur in which $ABEF$ is a set of two straight lines parallel to the asymptotes of the former hyperbola. If $ABCD$ is a parabola, also the case may occur, in which $ABEF$ is a set of two straight lines $AE$ and $BF$ parallel or coinciding to the axis of the parabola.

Dem. The demonstration of the converse of this lemma is not given here.

Corollary. Let $ABCD$ be a quadrangle inscribed in a conic, and $KAB$ be a triangle having $BC$ and $AD$ as sides, and $AB$ as the base. The tangent at $K$ to a conic circumscribed to the triangle $KAB$, and homothetic to the original conic, is parallel to the remaining side $CD$ of the quadrangle. If $ABCD$ is a hyperbola $C$ and $D$ must lie on the same branch of it.

Steiner's theorems(1).

Theorem 1. The conics circumscribed to each of the four triangles formed by the four straight lines taken three by three, whereof neither two are parallel nor any three concurrent, pass the same point $P$.

Dem. Let $AB$, $BC$, $CD$ and $DA$ be the four lines; $E$ be the point of intersection of the first and the third line, and $F$ be that of the second and the fourth line. Call the triangles, $DFC$, $AED$, ....the first, second, ....triangle. Also call the conics circumscribed to these triangles the first, second,....conics. Let $D$ and $P$ be the points of intersections of the first and second conics and draw four lines $PA$, $PB$, $PD$ and $PF$.

(1) The conics stated in Steiner's theorems other than the first of them are all central conics. Two or more conics stated there are considered to be homothetic. When the conic is a hyperbola, the hyperbola homothetic to conjugate hyperbola or a set of two straight lines parallel or coincident to the asymptotes is sometimes to be used instead of the homothetic conic. When the conic is a parabola a set of two straight lines parallel or coincident to the axis of it is sometimes to be used instead of the homothetic conic.
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Draw a straight line passing through $E$ and parallel to $BC$, and let it intersect $PF$ at $G$. Then, since $EG$ is parallel to $CF$, one of the sides of a quadrangle $CFPD$ inscribed in the first conic, it is known by the converse of the lemma that the quadrangle $EGPD$ can be inscribed in a conic, or in other words, the second conic $AED$, that is, $PED$ passes $G$. Next, since $BF$ is parallel to $EG$, one of the sides of a quadrangle $Egap$ inscribed in the second conic, the quadrangle $BFAP$ can be inscribed in a conic, or in other words, the third conic $BFA$ also passes through $P$.

By the same reason the fourth conic $CEB$ also passes through $P$.

Remarks on the special cases in the above demonstration are given below:—In the case in which the first conic is a hyperbola, if the first straight line $AB$ which is not one of the sides of the first triangle is parallel or coincides to one of the asymptotes of the hyperbola, the set of this side of the second triangle and the straight line passing through the vertex $D$ opposite to the side and parallel or coinciding to the other asymptote may be taken instead of the second conic. In the case in which the first conic is a parabola, if the first straight line is parallel or coincides to the axis of it, the set of this straight line and the line passing through $D$ parallel or coinciding to the axis of the parabola may be taken instead of the second conic. The same is true for the third and fourth conics. In these special cases, since the first line is in the part common to the second, third and fourth conics the point of intersection of this line and the first conic is the point common to the four conics.

**Theorem 2.** The centres of these four conics and $P(1)$ are on a same conic.

**Dem.** Let the centres of the first, second, third and fourth conics be $O_1, O_2, O_3$ and $O_4$ and let $P_1, P_2, P_3$ and $P_4$ be the diametrically opposite points in these conics to $P$. Join $A$ to $P_1, P_2, P_3$ and

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(1) The alphabets used in the present paper have the same significations, except in the part from the beginning to Theorem 1, excluding the theorem itself.
P. Then, since two lines $P_2A$ and $P_3A$ are pseudo-perpendicular to the common chord $PA$ of the second and third conics, they are in the same straight line. Similarly, the straight line $P_3P_1$ passes through $F$ and the line $P_1P_2$ through $D$. By applying Theorem 1 to the four lines, that is, the three sides of the triangle $P_1P_2P_3$ and the straight line $DF$, it is seen that a conic circumscribed to the triangle $P_1P_2P_3$, the first conic $DFP_1$ and the second conic $ADP_2$ intersect at the same point $P$. Since $P_1, P_2, P_3$ and $P$ are all on the same conic, $P$ and the middle points $O_1, O_2$ and $O_3$ of the segments $PP_1, PP_2$ and $PP_3$ are also on the same conic.

In the other words $O_1$ is on the conic $PO_2O_3$. Similarly $O_4$ is also on the conic $PO_2O_3$.

Alternative demonstration. Since Theorem 3 can be demonstrated independently of Theorem 2, the demonstration of the second theorem may be given basing on the third theorem. Inverse the whole figure with respect to $P$. Then the four original straight lines become four conics passing through $P$, and each of the four original conics becomes a straight line. Any three of the original conics, for example, the first, second and third conics pass the three points two by two, at which the fourth line intersects the other three lines. The conic circumscribed to a triangle formed by the inverse figures of the three conics is the inverse of the fourth line, therefore passes $P$.

The inverses of the points diametrically opposite to $P$ in these conics are the feet of the pseudo-perpendiculars from $P$ to the straight lines, the inverse figures of the conics. Since, the feet of these pseudo-perpendiculars lie, by Theorem 3, on a straight line, the original points lie on a same conic.

**Theorem 3.** The feet of the pseudo-perpendiculars on these four straight lines from $P$ are on the same line $l$.

**Dem.** Let $L_1, L_2, L_3$ and $L_4$ be the feet of the pseudo-perpendiculars from $P$ on the straight lines $AB, BC$ or $FC, CD, DA$ or $DF$. 
Then since $PL_2$ and $PL_3$ are pseudo-perpendicular to $FC$ and $CD$ respectively, $L_2$ and $L_3$ are on the conic on $PC$ as diameter. Similarly $L_2$ and $L_4$ are on the conic on $PF$ as diameter. Now let $L_4'$ be the point of intersection of $L_2L_3$ and $DF$, then the three conics $DEC$, $L_2L_3C$ and $L_2L_4'F$ meet at $P$ by Theorem 1. Since the last conic $L_2L_4'F$ and the conic on $PF$ as diameter have three points $L_2$, $F$ and $P$ in common, they have the same figure, therefore $L_4$ and $L_4'$ coincide. In other words $L_4$ is on the line $L_2L_3$. Similarly $L_1$ is also on the line $L_2L_3$.

**Theorem 4.** The pseudo-orthocentres of these triangles lie on the straight line $l'$.

*Dem.* The demonstration of this theorem is given in that of Theorem 5.

**Theorem 5.** The straight lines $l$ and $l'$ are parallel and $l$ passes through the middle point of the pseudo-perpendicular on $l'$ from $P$.

*Dem.* Let $H_1$ and $H_2$, etc. be the pseudo-orthocentres of the triangles $DFC$ and $AED$, etc. respectively. Let $D$ and $K$ be the points of intersection of the pseudo-perpendicular from the vertex $D$ of the triangle $DFC$ on the opposite side $FC$ with the circumscribed conic of the triangle. First will be given a proof that $H_1$ and $K$ is pseudo-symmetrical with respect to $FC$.  

Let $D'$ be the diametrically opposite point of $D$ in the conic $DFC$. Join $D'$ to $H_1$, $F$, $C$ and $K$. Then $FH_1$ and $D'C$ are pseudo-perpendicular to $CD$, and therefore are parallel to one another. $CH_1$ and $D'F$ are also parallel to each other. Hence $H_1FD'C$ is a (1) In the special case in which $DK$ passes through one of the extremities of the line segment $FC$ or the middle point of it some of procedures in the demonstration become unnecessary.
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parallelogram, and the diagonals $H_1D'$ and $FC$ bisect each other. Also $FC$ and $D'K$ are pseudo-perpendicular to $DK$, and therefore they are parallel to one another. Since $FC$ is parallel to the base $D'K$ passing through the middle point of the side $H_1D'$ of the triangle $H_1D'K$ it passes through the middle point of the remaining side $H_1K$. Hence $H_1$ and $K$ are pseudo-symmetrical with respect to $FC$.

Now let $Q$ be the point of pseudo-symmetry of $P$ with respect to $FC$. As it is just proved, $K$ and $H_1$ are pseudo-symmetrical with respect to $FC$. Hence $PK$ and $QH_1$ are pseudo-antiparallel with respect to $FC$.

Let $P$ and $P'$ be the intersecting points of $PQ$ with the conic $DFC$. The diameter having the parallel chords $PP'$ and $KD$ as conjugate chords is the axis of pseudo-symmetry of $PK$ and $P'D$. Since this diameter is parallel or coincides to the line $FC$, the two lines $PK$ and $P'D$ are pseudo-antiparallel with respect to $FC$. Thus two lines $QH_1$ and $P'D$ are pseudo-antiparallel to $PK$ with respect to the same line $FC$, hence they are parallel to each other. As shown in the demonstration of Theorem 3 the quadrangle $CPL_2L_3$ is inscribed in a conic. This quadrangle and the quadrangle $CPP'D$ inscribed in the conic $PCD$ have two vertices $C$ and $P$ in common, and the sides $CL_3$ and $PL_2$ of the former quadrangle, which pass through the vertices also pass through the remaining vertices $D$ and $P'$ of the latter quadrangle. Hence, by the lemma, the line $L_2L_3$, that is, $l$ is parallel or coincides to $P'D$; therefore $l$ is parallel to $QH_1$ and since $L_2$ is the middle point of the segment $PQ$, $l$ passes through the middle point of the segment $PH_1$. Similarly for $H_2$, $H_3$ and $H_4$. Therefore $H_1$, $H_2$, etc. lie all on the line $l'$ parallel to $l$, and $l$ passing through the middle point of the pseudo-perpendicular on $l'$ from $P$.

**Theorem 6.** The middle points of the three diagonals of a complete quadrilateral formed by the four lines lie on the same line $l''$.

**Dem.** The proof of this theorem is given in that of Theorem 7.

**Theorem 7.** The line $l''$ is the pseudo-perpendicular common to both $l$ and $l'$.

**Dem.** Let $X$, $Y$ and $Z$ be the middle points of the diagonals $AC$, $BD$ and $EF$ of a complete quadrilateral $ABCDEF$. Again let $D'$, $F'$ and $C'$ be the feet of the
pseudo-perpendicular from the vertices $D$, $F$ and $C$ on the opposite sides of a triangle $DFC$. Then, since the conic on $BD$ as diameter passes through $D'$ the power of $H_1$ in the direction of $DD'$ with respect to the conic is $H_1D\cdot H_1D'$. Similarly the power of $H_1$ in the direction of $FF'$ with respect to the conic on $EF$ as diameter is $H_1F\cdot H_1F'$. On the other hand the conic on $DF$ as diameter passes through the four points $D, D', F$ and $F'$.

Now let $\theta$ be the angle which the straight line $DD'$ makes with the directrix of the conic $BDD'$. Let $\sigma_1$ be the power corresponding to an angle $\alpha$ of the point $H_1$ with respect to the conic, $\sigma$ that with respect to a new conic $DFD'$, then each of $\frac{\sigma_1}{H_1D\cdot H_1D'}$ and $\frac{\sigma}{H_1D\cdot H_1D'}$ equals $\frac{1-e^2\sin^2\theta}{1-e^2\sin^2\alpha}$. Hence $\sigma_1$ is equal to $\sigma$. Similarly the power corresponding to $\alpha$ of $H_1$ with respect to the conic $EFF'$ is also equal to $\sigma$. Therefore, the powers of $H_1$ in the direction of the same lines with respect to the conics $BDD'$ and $EFF'$ are equal. Similarly for the conics $EFF'$ and $ACC'$. Hence $H_1$ is a point common to any two of the radical axes of the three conics. The points $H_2$, $H_3$ and $H_4$ have the same properties. Thus the radical axes of any two of the three conics have two or more points in common. Hence they are a coincident line. Therefore the three centres $X$, $Y$ and $Z$ lie on a straight line, and this line is pseudo-perpendicular to the line $l'$ on which the points $H_1$ and $H_2$, etc. lie.

**Theorem 8.** The sixteen centres of the conics inscribed and escribed to the four triangles lie four by four on a conic, giving rise to eight new conics. In the case in which the conics are hyperbolas the original four lines do not intersect different branches of the conic under consideration.
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Dem. Giving the sense to each of the four lines a complete quadrilateral is obtained. Call this the basal complete quadrilateral. Let $I_1$, $I_2$, $I_3$, and $I_4$ be the oriented conics touching the three sides of the first, second, third and fourth triangles of the complete quadrilateral, and denote their centres by the same notations. Again let $Y$ and $Z$ be the points of contact of the oriented conic $I_1$ touching the third and fourth lines, $CD$ and $DF$, and let $Y'$ and $Z'$ be those of the oriented conic $I_2$ touching the two lines mentioned above. Denote the point at which the first conic $I_1$ touches the second line $FC$ by $X$, and that at which the second conic $I_2$ touches the first line $AE$ by $X'$. Then these two conics are homothetic, the point $D$ of intersection of the third and fourth lines being the centre, and $Y$ being the corresponding point to $Y'$ and $Z$ to $Z'$. Let $X_1$ be the corresponding point of $X'$, and $U$ and $U_1$ be the diametrically opposite points of $X$ and $X_1$ in the conic $XYZ$. Join $U$ and $U_1$ to $Y$ and $Z$, and draw a straight line passing through $U$, parallel or coinciding to $I_1I_2$. Denote the point of intersection of this straight line and $ZU_1$ by $V$. Draw a straight line passing through $V$, parallel to $U_1Y$, intersecting with $UY$ at $W$. Then $UZ$ and $I_1I_2$ are each a pseudo-perpendicular to $XZ$, hence are parallel to each other. Similarly $YU$ and $I_3I_4$ are parallel to each other.

$ZU_1$ and $I_3I_4$ which are pseudo-perpendicular to the parallel lines $X_1Z$ and $X'Z'$ are parallel to each other. Similarly $U_1Y$ and $I_2I_4$ are parallel to each other, therefore $WV$ is parallel to $I_1I_2$. Hence the two quadrangles $I_1U_1VW$ and $UZVW$ are similar to each other.

Since the side $VW$ of the quadrangle $UZVW$ is parallel to the side $U_1Y$ of the quadrangle $UZU_1Y$ inscribed in the conic $XYZ$, the quadrangle $UZVW$ can by the converse of the lemma be inscribed in a conic. So also the quadrangle $I_1I_2I_3I_4$ can be inscribed in a conic.

Now by changing the sense of the sides (*) of the basal complete quadrilateral one by one are got four complete quadrilaterals, from each of which are obtained the conics as $I_1I_2I_3I_4$. Call these the conics of the second group. Next, by changing one by one the sense of the side of one of the four complete quadrilaterals are obtained four new complete quadrilaterals, one of which being the basal complete quadrilateral. From these complete quadrilaterals are got the conics as $I_1I_2I_3I_4$, which are called the conics of the first group. Thus there are eight conics, both groups taken together.

(* Of course, without changing the positions of the sides.)
Some remarks are given below:

Remark 1. The conics of the first group are got from the complete quadrilaterals got by changing one by one the sense of the sides of any one of the four complete quadrilaterals which are got by changing one by one the sense of the sides of the basal complete quadrilateral. For instance consider the two complete quadrilaterals got by changing the sense of the first and second sides of the basal complete quadrilateral, and form eight complete quadrilaterals by changing the sense of sides. Denote the first, second, .... sides of the basal complete quadrilateral by $l_1, l_2, ....$ and the sides whose sense is changed by $l_1', l_2', ....$. From the first of the two complete quadrilaterals $l_1l_2l_3l_4; l_1'l_2'l_3'l_4; l_1l_2l_3l_4'; l_1'l_2l_3l_4'$ are obtained. From the second are got $l_1'l_2'l_3l_4; l_1l_2l_3l_4; l_1l_2'l_3'l_4; l_1l_2'l_3l_4'$. The two complete quadrilaterals $l_1l_2l_3l_4$ and $l_1'l_2'l_3l_4$ are contained in both groups. Moreover, the remaining complete quadrilaterals $l_1'l_2l_3l_4$ and $l_1l_2'l_3l_4'$ of one group are got by changing the sense of all the sides of the remaining complete quadrilaterals, $l_1l_2'l_3l_4$ and $l_1'l_2l_3l_4'$ of the other group, and vice versa. The angles between two oriented lines and the angles which these lines with sense reversed make with each other have a common pseudo-bisector.

Remark 2. Just as the conics of the first group are obtained from those of the second group, the conics of the second group are got from those of the first group. Because changing the sense of all the sides of complete quadrilaterals got from changing the sense of the three sides of the basal complete quadrilateral is the same as changing the sense of one side of the latter complete quadrilateral.

**Theorem 9.** The eight new conics are grouped into two groups, the four conics of one of the groups cut those of the other group pseudo-perpendicularly. The centres of the conics of these groups lie on the two lines pseudo-perpendicular to each other.

*Dem.* Consider one of the conics of the first group and any one of the conics of the second group. For instance, let the former one be the conic $I_1I_2I_3I_4$ got from the basal complete quadrilateral, and let the latter be a conic as the conic $I_1'I_2'I_3'I_4$, formed by changing the sense of the first side of the basal complete quadrilateral, then $I_1$ is the first conic common to both the new and basal complete quadrilaterals. Let $I'_1$ and $I'_2$ be the centres of the second and third conics of the new complete quadrilateral.

Again let $O_1$, $O_2$ and $O_3$ be the centres of the conics circumscribing
the three triangles $I_1I_2I_3$, $I_1'I_2'I_3$ and $AI_2I_2'$ of the four triangles formed by three by three of the four lines, that is, the line $I_3I_3'$ and sides of the triangle $I_1I_2I_3$. Denote the common point of these conics by $p$, and the diametrically opposite points in the three cones to $p$ by $p_1$, $p_2$ and $p_3$. Draw the tangents $pt$ and $pt'$ at $p$ to the conics $I_1I_2I_3$ and $I_1'I_2'I_3'$ respectively. Then as shown in the demonstration of Theorem 2 each of the sets $p_2, p_3, I_2'$; $p_3, p_1, I_2$; $p_1, p_3, I_1$ lie on a straight line. Hence the four points $p_1, p_2, p_3$ and $p$ lie on a same conic.

Since the two lines $AI_2$ and $AI_2'$ are pseudo-perpendicular to each other, $I_2I_2'$ is a diameter of the conic $I_2Ap_3I_3'$. Hence the supplementary chords $p_3I_2$ and $p_3I_2'$ of this conic are pseudo-perpendicular. Since the two lines $p_3p_1$ (or $p_3I_3$) and $p_3p_2$ (or $p_3I_2'$) are a pair of the supplementary chords of the conic $pp_3p_3p$, the line $p_3p_2$ is a diameter of the conic, a pair of the supplementary chords $pp$ and $pp_2$ are pseudo-perpendicular. These two chords are diameters of the conics $O_1$ and $O_2$; therefore the tangents $pt$ and $pt'$ pseudo-perpendicular to the chords are also pseudo-perpendicular to each other.
Similarly each of the other three conics of the first group intersects each of the conics of the second group pseudo-perpendicularly. Thus the conics of one group cut those of the other group pseudo-perpendicularly, and therefore the conics of each group are coaxal. The centres of the conics of one group lie on the radical axis of those of the other group.

**Theorem 10.** The two lines referred to in Theorem 9 intersects at the point $P$ referred to in Theorem 1.

As the demonstration of this theorem is very tedious is not given here.