Semi-linear Equations,

by

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1. Introduction, Definition. An equation of the form

\[ u_0 + \sum_{i=1}^{n} m_i u_i = 0, \]

where the \( m \)'s are constants and where the \( u \)'s are (distinct) linear forms, we define as a semi-linear equation and say that it is of order \( n \) if each \( m_i \neq 0 \). We restrict ourselves to linear forms in two (or three homogeneous) real variables and hence to the case of two dimensions. Although it was while recently engaged in a study of certain non-linear integral equations(1) that we met, quite naturally, a system of equations of type (1.1), yet such an equation was considered by Söderblom(2) as early as 1899 and somewhat later by Riabounchinsky(3). Söderblom gives the equations of certain polygons but attempts no proofs. In the application of his "method of diagonals", which is essentially the same as that developed independently by the author, Söderblom fails to take proper account of the regions associated with the equations auxiliary to (1.1). As a result much of his fundamental work is incomplete and some of it incorrect—e.g. three of his four examples of non-convex polygons. Riabounchinsky was primarily interested in the application of the theory of absolute-values to certain problems in aerodynamics(4). The purpose of the present paper is to develop further the theory of semi-linear equations especially with regard to convex polygons.

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(3) D. P. Riabounchinsky, "Quelques considerations sur la fonction \(|x|\)" and "La fonction \(|x|\)", Bulletin de l'Institut Aérodynamique de Kouthino, Fascicule V, 1914, pp. 111-113 and pp. 114-142 respectively. Attention should also be called to a mimeographed lecture by Riabounchinsky given in 1919 at the University of Copenhagen in which he continues in the line of his previous articles.

(4) The author is aware of no other studies dealing directly with semi-linear equations.
2. Some preliminary considerations. Consider a semi-linear equation (1.1) of order $n$ and the $n$ lines $u_i = a_ix + b_iy + c_i = 0$. Now each of these lines $u_i = 0$ divides the plane into two regions, one where $u_i > 0$ and the other where $u_i < 0$; and hence the $n$ lines divide the plane up into a certain number of regions $(1)$ $R_i$ throughout everyone of which each function $u_i$ is of a definite fixed sign. For every region $R$ therefore we see, by removing the absolute-value signs in (1.1) and inserting $\pm$ before the $m_iu_i$ terms according as $u_i \geq 0$ in $R$, that this semi-linear equation reduces to a linear equation $U = Ax + By + C = 0$, which in general represents a straight line (which may or may not contain points of $R$) but which in particular may reduce to the identity $0 = 0$ in which case every point of $R$ satisfies (1.1).

Definitions. Such an equation $U = 0$ we call, with Söderblom, an auxiliary equation. If $U = 0$ contains points of $R$, we call the collection of those points a solution of the semi-linear equation in $R$. If $U = 0$ contains no point of $R$, we say that $R$ is a vacuous region and that $U = 0$ constitutes a shadow solution, or simply that there is no solution. The lines $u_i = 0$ we call the diagonals of the equation or of the graph, $u_n = 0$ the exterior diagonal.

For a given equation (1.1) it is a simple matter to draw the diagonals, to write down the auxiliary equations by means of the sign-sets associated with each region $R$, and hence to solve or to plot the graph of the equation. From the above considerations it is clear that this graph may be made up of isolated points, broken-line segments, area. Conversely the question arises, whether, given a particular configuration of points, lines, areas, there exists a semi-linear equation which would plot this configuration. We answer this question for certain configurations.

3. Convex polygons.

Theorem I. There exists a semi-linear equation of order $(n+1)$ whose solution is the perimeter of an arbitrary convex $(n+3)$-gon $(n \geq 1)$.

Proof. We assume that the origin is at one vertex and that the diagonal joining the two adjacent vertices is parallel to the y-axis (see Plate I, fig. 1). Let the coordinates of the vertices bo (in counter clockwise direction): $(0, 0), (a_i, \alpha_i), i = 0, 1, \ldots, n+1; \alpha_n = \alpha_{n+1}$.

(1) If the lines are as linearly independent as possible, i.e., no two parallel and no three concurrent, they divide the plane into $n(n+1)/2 + 1$ (a maximum number of) regions of which $(n-1)(n-2)/2$ are bounded $(n \geq 2)$. 

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It is important to point out that every side $s$ of the polygon must necessarily be isolated into a separate region $R$ by some system of diagonals. This follows from the considerations of paragraph 2. The question of what system of diagonals to choose is one which we discuss at some length later. We now take as diagonals the lines $x-a_i=0$, $y-\lambda_i x=0$, $i=1, 2, \ldots, n$, and wish to determine $a_0$, $b_0$, $c_0$, $m_0$, $m_1$, \ldots, $m_n$ so that

$$a_0x+b_0y+c_0+m_0|x-a_0|+\sum_{i=1}^n m_i|y-\lambda_i x|=0$$

shall be the equation of the polygon. Note that by the very nature of the problem we must have non-zero values for the $m$'s since otherwise some diagonal term would be missing. Making use of the sign-sets associated with the $(n+3)$ regions in which the sides of the polygon lie and substituting the coordinates of the vertices into the auxiliary equations, we obtain the following system of $(n+3)$ linear homogeneous equations in $(n+4)$ unknowns (the first sign goes with the first absolute-value term, etc.):

$$0\cdot m_1+\ldots+0\cdot m_n=0,$$

$$-\ldots-0\cdot a_0+0\cdot b_0+c_0+\alpha_0 m_0+$$

$$+\ldots+\alpha_0 a_0+\alpha_0 \lambda_0 b_0+c_0+0\cdot m_0-(\alpha_0 \lambda_0-\alpha_0 \lambda_1)m_1+\ldots-$$

$$(\alpha_0 \lambda_0-\alpha_0 \lambda_n)m_n=0,$$

$$+\ldots+\alpha_0 a_0+\alpha_0 \lambda_1 b_0+c_0+(\alpha_1-\alpha_0)m_0+$$

$$(\alpha_1 \lambda_1-\alpha_1 \lambda_n)m_n=0,$$

$$+\ldots+\alpha_0 a_0+\alpha_0 \lambda_n b_0+c_0+0\cdot m_0+(\alpha_n \lambda_{n+1}-\alpha_0 \lambda_1)m_1+$$

$$\ldots+0\cdot m_n=0,$$

$$-\ldots+\alpha_0 a_0+\alpha_0 \lambda_{n+1} b_0+c_0+0\cdot m_0+(\alpha_{n+1} \lambda_{n+1}-\alpha_0 \lambda_1)m_1+$$

$$\ldots+(\alpha_0 \lambda_{n+1}-\alpha_0 \lambda_n)m_n=0.$$

Let $c_0=1$; then from the first equation we get $m_0=-1/\alpha_0 \neq 0$. Omitting the first equation and transposing the columns involving $c_0$ and $m_0$ we have $(n+2)$ non-homogeneous equations in $(n+2)$ unknowns the determinant of whose coefficients is

$$\Delta=2^n \alpha_0 \alpha_1 \ldots \alpha_{n+1}(\lambda_1-\lambda_0)(\lambda_2-\lambda_1)\ldots(\lambda_{n+1}-\lambda_n) \neq 0.$$
where \( A(i-1, i, i+1) \) is the area of the triangle whose vertices are 
\((x_{i-1}, x_{i-1}, x_{i-1})\), \((x_i, x_i, x_i)\), \((x_{i+1}, x_{i+1}, x_{i+1})\);

\[
\begin{align*}
\alpha_0 &= \frac{\alpha_i\alpha_n(\lambda_i\lambda_n - \lambda_0\lambda_{n+1}) - \alpha_0\alpha_i\lambda_{n+1}(\lambda_i - \lambda_0) + \alpha_0\alpha_n\lambda_0(\lambda_{n+1} - \lambda_n)}{\alpha_i\alpha_n(\lambda_i - \lambda_0)(\lambda_{n+1} - \lambda_n)}, \\
b_0 &= \frac{\alpha_n(\alpha_i - \alpha_0)(\lambda_{n+1} - \lambda_n) - \alpha_i(\alpha_n - \alpha_0)(\lambda_i - \lambda_0)}{\alpha_i\alpha_n(\lambda_i - \lambda_0)(\lambda_{n+1} - \lambda_n)}.
\end{align*}
\]

Thus an equation (3.1) has been found whose solution in the \((n+3)\) regions used is the given polygon. It is clear from the figure that there can be no solution in the regions \( T'[-+-....-, -+ +-. ...
-.... -+ +....] \). But it is not so immediately obvious that the regions to the left of the \( y \)-axis are vacuous. To show that this is the case we first examine

\[
(3.3) \quad u_0 = a_0x + b_0y + c_0 = 0.
\]

Now \( a_0, b_0, c_0 \) involve only the coordinates \((0, 0)\), \((\alpha_0, \alpha_0\lambda_0)\), \((\alpha_1, \alpha_1\lambda_1)\), \((\alpha_n, \alpha_n\lambda_n)\) and \((\alpha_0, \alpha_0\lambda_{n+1})\) and further it turns out that \( u_0 = 0 \) is the join of the intersections of sides 1 and \( n+2 \), and 2 and \( n+3 \). In a region such as \( S \) equation (3.1) reduces to

\[
(3.4) \quad x(a_0 + P) + y(b_0 + Q) + 2 = 0,
\]

where \( P \) and \( Q \) are certain constants. In region \( S' \), since the sign sets are complementary to those of \( S \), the reduction leads to

\[
(3.5) \quad x(a_0 - P) + y(b_0 - Q) = 0.
\]

But the left hand members of (3.3), (3.4) and (3.5) are linearly dependent and the lines therefore are concurrent; thus (3.5) has just the one point \((0, 0)\) in common with \( S' \) and consequently region \( S' \) is essentially vacuous. This completes the proof. Only in case \( A(i-1, i, i+1) = 0 \) will \( m_i = 0 \). Thus if a diagonal is not needed it drops out automatically.

**Theorem II.** There exists an equation of every order \( p \geq n+1 \) for an arbitrary convex \((n+3)\)-gon.

First take \( k \) constants \( a_j \) such that \( \lambda_{n+1} < a_1 < a_2 < \ldots < a_k \) but otherwise arbitrary, and find a set of \( k \) positive numbers \( \epsilon_j \) such that the larger of \( |\pm \epsilon_1 a_1 \pm \epsilon_2 a_2 \pm \ldots \pm \epsilon_k a_k| \) and \( |\pm \epsilon_1 \pm \epsilon_2 \pm \ldots \pm \epsilon_k| \) is small for every combination of signs. Then

\[
(3.6) \quad \left( a_0 + \sum_{j=1}^k \epsilon_j a_j \right)x + \left( b_0 - \sum_{j=1}^k \epsilon_j \right)y + c_0 + m_0|x - a_0| + \sum_{i=1}^n m_i|y - \lambda_i x| + \sum_{j=1}^k \epsilon_j |y - a_j x| = 0,
\]
a semi-linear equation of order \((n+1+k)\), will be the equation of the polygon since in the regions of the polygon it reduces to \((3.1)\) and in all other regions the solutions are shadow\(^1\).

**Corollary I.** The semi-linear equation of a convex-polygon is not unique.

**Corollary II.** There exists no semi-linear equation of maximum order for a convex-polygon.

There are, of course, system of diagonals other than that used in determining equation \((3.1)\) which would still isolate the sides into separate regions of the plane and some of these systems would contain fewer than \((n+1)\) diagonals. But, regardless of the diagonals used, the system of equations corresponding to \((3.2)\) would contain \((n+3)\) equations and thus, if there were fewer than \((n+4)\) unknowns (i.e., fewer than \((n+1)\) diagonals), there would be a solution only in special cases. Hence the

**Theorem III.** The order \((n+1)\) is the least order which can be used for every convex \((n+3)\)-gon.

4. The triangle. The triangle does not come under the above discussion since, strictly speaking, there is no set of diagonals for such a polygon.

**Lemma.** In regions \(R_1, R_2\) separated by but a single diagonal, the (line) solution \(U_1=0, U_2=0\) of a semi-linear equation have distinct slopes if the point of intersection of \(U_1=0, U_2=0\) on that diagonal is finite and unique.

For in passing from one region to the other only a single sign is changed; thus the auxiliary equations are of the form \(Ax+By+C±(Dx+Ey+F)=0\). And if the slope is to remain unaltered it follows that \(A:B=D:E\). But this would mean that in these regions the equation reduces to \(k_1u+k_2=0\) (where \(u=0\) is the equation of the separating diagonal) giving lines parallel to (or coincident with) \(u=0\). This is impossible by hypothesis.

However, it may well happen that the slope does not change when passing through the point of intersection of two or more diagonals; for here we pass between regions separated by at least two diagonals and hence there will be as many as two changes in the sign-sets.

\(^1\) This follows from the fact that in every region the solution is changed but little.
Let us examine this situation more generally. Consider the $2n$ regions formed by the $n$ concurrent lines (diagonals) $r_1x + s_1y = 0$. Slopes of half-lines issuing from the origin may be arbitrarily assigned in certain regions and the semi-linear equation of this configuration determined. Slopes in the remaining regions will then be fixed and may be computed in the following way. Let

\[ a_0x + b_0y + \sum_{i=1}^{n} m_i |r_i x + s_i y| = 0 \]

be the equation; and let $p_jx + q_jy = 0$, $j = 1, 2, \ldots, 2n$, be the lines of the configuration. Set $-r_i/s_i = \lambda_i$, $-p_i/q_i = \mu_i$. Then

\[ 2m_i = \frac{1}{r_i} (p_i - p_{i+1}) = \frac{1}{s_i} (q_i - q_{i+1}); \]

\[ q_{i+1} = q_i \frac{\lambda_i - \mu_i}{\lambda_i - \mu_{i+1}}; \]

\[ p_{i+1} = -\mu_{i+1} q_{i+1}; \quad i = 1, 2, \ldots, n. \]

From (4.2), (4.3) and (4.4) it is apparent that $p_i$, $q_i$ (hence $\mu_i$), and $\mu_{i+1}$ may be assigned arbitrarily ($\mu_i = \lambda_i$, $\mu_{i+1} = \lambda_i$, $\mu_i = \mu_{i+1}$) and that $q_{i+1}$ can then be uniquely determined. Thus we may assign slopes in $(n+1)$ angularly consecutive regions. If $\mu_i = \mu_{i+1} = \lambda_i$, equation (4.3) becomes indeterminate and is no longer used for computing $q_{i+1}$ which can then be chosen arbitrarily, and which, in turn, means that $\mu_{n+2}$ is arbitrary.

**Definition.** A side of a polygon may be considered as made up of $(m+1)$ (collinear) sides by the insertion of $M$ distinct points on that side. Such a point $M$ we shall call a sub-vertex. A sub-vertex will be called proper if there pass through it two diagonals neither of which coincides with that side containing the sub-vertex. Any finite system of lines which isolate into separate regions of the plane formed by those lines the sides of a polygon whose every subvertex is proper will be called a fundamental system of diagonals.

We are now in a position to prove

**Theorem IV.** There exists an equation (1.1) of order six for the triangle.

We employ homogeneous (areal) coordinates and take as triangle of reference the given triangle whose equation we seek. Let the co-ordinates of the vertices be $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ and let $X_1(0, \delta_i, 1-\delta_i)$, $X_2(1-\delta_i, 0, \delta_i)$, $X_3(\delta_i, 1-\delta_i, 0)$ be any three points on
(and interior to) BC, CA, AB respectively. We make use of the fundamental system of diagonals as shown in Plate I, fig. 2, and seek an equation of order six based on these diagonals. It can be checked that the equation turns out to be

\[(4.5) \quad \delta_0(1-\delta_1)x_1 + \delta_1(1-\delta_0)x_2 + \delta(1-\delta_2)x_3 = 0.
\]

The graph of this equation, for every choice of \(0 < \delta_1, \delta_2, \delta_3 < 1\), is the triangle \(ABC\) all regions other than those involving the sides of the triangle being vacuous. Note the cyclo-symmetry present in \((4.5)\). Also note from the figure that the sub-vertices \(x_1, x_2, x_3\) would still remain proper even though the diagonal \(x_1x_2\), say, were not used but that then no other diagonal could be dispensed with. However no semi-linear equation of order five exists for the triangle inasmuch as extraneous solutions creep in and can not be removed. If the diagonal \(x_1x_2\) were omitted there would be, in addition to the triangle, a halfline issuing from \(x_3\) and lying in either of the regions whose common boundary is the diagonal \(Cx_3\). Thus six is the least order for the triangle.

5. The quadrilateral. It is of interest to point out with Söderblom that in this case equation \((3.1)\), \((n=1)\), is of order two and \(u_0 = 0\) is the equation of the exterior diagonal.

6. The Hexagon. The hexagon is of special interest. Equation \((3.1)\) for \(n=3\) is of order four. It is with the possibility of determining an equation of order three for the hexagon that we are here concerned.

Consider six points in the plane so situated that the diagonals \((1,4), (3,6), (5,2)\) divide the plane up into six (or seven) regions and further such that, if no two of these diagonals are parallel, the point (or triangle) of intersection of these diagonals lies within the hexagon \((1, 2, 3, 4, 5, 6)\). If two are parallel, say \((3, 6)\) and \((5, 2)\), the hexagon to be considered is the hexagon \((1, 2, 3, 4, 5, 6)\). Now the equation is to be of the form \(u_0 + m_1u_1 + m_2u_2 + m_3u_3 = 0\) wherein there are six undeter-
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mined coefficients. When the coordinates of the six vertices are put into this equation we obtain six homogeneous equations in six unknowns, which system is in general inconsistent. For suppose that the points were so chosen that the determinant of the coefficients of this system were zero. Then, by holding five of the points fixed and varying the sixth point slightly (along the diagonal, of course), the determinant would not remain zero. Hence if the problem is possible at all it will be possible only for very specially located points.

If the diagonals are concurrent, it turns out that all centrally symmetric hexagons are capable of semi-linear representation of order three. Moreover the same is true for all hexagons which are symmetric with respect to an interior triangle—i.e., the sides of the hexagon are parallel in pairs to the diagonals which form a triangle interior to the hexagon. The centrally symmetric hexagons are a special case of the trianularly symmetric ones, and both groups belong to the group of Pascal Hexagons.

We shall now prove the following theorem concerning the location of the vertices of the hexagon:

**Theorem V.** Given six points, 1, 2, 3, 4, 5, 6 in the plane. If these points lie on a (non-degenerate) conic, three on either branch if the conic is a hyperbola, and are labeled in order around the ellipse or parabola, but around one branch of the hyperbola in one direction and around the other branch in the other direction, then and only then can the equation of the hexagon (1, 2, 3, 4, 5, 6) be written on the three diagonals (1, 4), (3, 6), (5, 2).

Proof. We again employ homogeneous (areal) coordinates and choose as triangle of reference the triangle (1, 3, 5) (Plate I, fig. 3). If the diagonals (1, 4), (3, 6), (5, 2) form seven regions we take + + + as the sign-set associated with the triangular region. The coordinates of the vertices of the hexagon are taken (in opposite pairs) as 1(1, 0, 0), 4(a1, a2, a3); 3(0, 1, 0), 6(b1, b2, b3); 5(0, 0, 1), 2(c1, c2, c3). The equations of the diagonals are:

\[(1,4) \quad a_2x_3-a_3x_2=0,\]
\[(3,6) \quad b_3x_1-b_1x_3=0,\]
\[(2,5) \quad c_1x_3-c_3x_1=0.\]

The determinant \(\Delta\) of the coefficients of the reduced system is
the vanishing of which is a necessary and sufficient condition that the six points 1, 2, 3, 4, 5, 6 lie on a conic.

The calculations lead to the equation

\[
\begin{vmatrix}
1 & 0 & 0 & 0 & b_3 & c_2 \\
c_1 & c_2 & c_3 & a_3c_2 - a_2c_3 & b_2c_1 - b_1c_3 & 0 \\
0 & 1 & 0 & a_3 & 0 & c_1 \\
a_1 & a_2 & a_3 & 0 & b_1a_3 - b_2a_1 & a_2c_1 - a_1c_2 \\
0 & 0 & 1 & a_3 & b_1 & 0 \\
b_1 & b_2 & b_3 & a_2b_3 - a_3b_2 & 0 & b_1c_2 - b_2c_1
\end{vmatrix}
\]

\[= 8 \begin{vmatrix}
a_2c_3 & b_1c_3 & c_1c_2 \\
a_2a_3 & a_1b_3 & a_2c_3 \\
a_3b_2 & b_1b_3 & b_2c_1
\end{vmatrix},
\]

and the hexagon which it represents is a Pascal Hexagon.

Now \( u_0 = 0 \), it is readily determined, is the equation of Pascal’s Line, which, in a sense, is the exterior diagonal of the hexagon. Hence the

**Corollary.** In the equation (6.1) of a Pascal Hexagon \( u_0 = 0 \) is the equation of Pascal’s Line.

It should be kept in mind that the above discussion applies to the case of the diagonals (1,4), (2,5), and (3,6). It so happens that representations on other diagonals may be possible when representations on these particular diagonals are not possible. For example, a necessary and sufficient condition that there should exist an equation

\[
(6.2) \quad a_0x + b_0y + c + m_1y + m_2x + m_3|x - \lambda| = 0,
\]

which shall plot the hexagon in Plate I, fig. 4 is that \((c_1d_2 - c_2d_1) = 0\). Here even convexity is not an essential condition.

For the general hexagon the condition which must be satisfied in order that there should exist an equation of order three based on the diagonals (1, 4), (3, 5), (2, 6) appears to be of little interest.

**7. Regular polygons.** The equation (based on the diametral-set of diagonals) of the regular polygon of 2n sides with center at \((0, 0)\)
and one vertex on the $x$-axis at $(\rho, 0)$ is $^{(1)}$:

$$ (7.1) \quad \sum_{i=1}^{n} y \cos(i-1) \frac{\pi}{n} - x \sin(i-1) \frac{\pi}{n} = \rho \cot \frac{\pi}{2n}. $$

The equations of the diagonals are in normal form. The fact that from this equation it follows that the sum of the (plus) distances from a point on a regular $2n$-gon to the diametral diagonals is a constant was known to Söderblom and is readily proved from simple geometric considerations$^{(2)}$.

**Definition.** By the $j$-th set of diagonals of a regular $n$-gon we shall mean the set of every diagonal joining two vertices which are separated by $j$ other vertices. If $n=2m$, then the $(m-1)$-st set will be the diametral-set; if $n=2m+1$, there will be no diametral-set. By the 0-th set we shall understand the set coinciding with the sides of the polygon themselves.

**Theorem VI.** For every point $P$ on the perimeter of regular $n$-gon the sum of the perpendiculars from $P$ to the members of the $j$-th set of diagonals is a constant $K$.

This constant $K$ depends, of course, upon $\rho$, $j$, and $n$. For the case $j=0$ it is clear that the theorem is true not only for every point on the perimeter of, but also for every point interior to, the polygon. In general if $n$ is odd there is an even number of diagonals in the $j$-th set; if $n$ is even there is an odd number of diagonals in the $j$-th set. With reference to (the points of) a side $AB$, we point out that the diagonals may be so paired that members of any pair form an isosceles triangle with $AB$ (produced). We include here the case where a pair of diagonals may be perpendicular to $AB$. There will then remain at most two diagonals but these will be parallel to $AB$ and therefore will present no difficulty. If $n$ is odd there is always one diagonal parallel to $AB$; if $n$ is even there will be two ($j$, even), or no ($j$, odd) diagonals parallel to $AB$. By making use of these facts, the theorem is readily deduced. It is also true that for the particular $K$ associated with a given $j$-th set of diagonals ($j\neq0$), the sum of the perpendiculars upon these diagonals equals $K$ for no point other than points on the perimeter of the polygon. Hence

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$^{(1)}$ The left-hand member of this equation is due to Söderblom.

$^{(2)}$ This theorem is undoubtedly to be found elsewhere in the literature. The proof is based on the fact that the sum of the distances from every point on the base of an isosceles triangle to the equal sides is equal to one and the same constant.
Theorem VI is true only for points on the perimeter.

With these things in mind we turn to the derivation of the semi-linear equation of a regular n-gon referred to the j-th set of diagonals. We choose \( n \) points \((\alpha_i, \beta_i)\) where \( \alpha_i = \rho \cos \left(\frac{(i-1)2\pi}{n}\right), \beta_i = \rho \sin \left(\frac{(i-1)2\pi}{n}\right) \).

The normal form of the equation of the member of the j-th set of diagonals joining \((\alpha_i, \beta_i)\) and \((\alpha_{i+j+1}, \beta_{i+j+1})\) is:

\[
(7.2) \quad x \cos \left(\frac{2i+j-1}{n}\right) + y \sin \left(\frac{2i+j-1}{n}\right) - \rho \cos \left(\frac{j+1}{n}\right) = 0.
\]

The equation of the polygon is, therefore (Plate I, fig. 5):

\[
(7.3) \quad \sum_{i=1}^{n} \left| x \cos \left(\frac{2i+j-1}{n}\right) + y \sin \left(\frac{2i+j-1}{n}\right) - \rho \cos \left(\frac{j+1}{n}\right) \right| = K(\rho; j, n),
\]

where \( n \) is either of the form

\( n=2m, \)

or

\( n=2m+1, \quad (m \geq 2); \)

and where

\[ K = 2\rho \sum_{i=1}^{n} \left| \sin \left(\frac{i+j}{n}\right) \frac{\pi}{n} \sin \left(\frac{i-1}{n}\right) \frac{\pi}{n} \right|. \]

If \( j=0 \), (7.3) is the equation of all points interior to and on the perimeter of the polygon, or simply the equation of the area of the polygon. Since \( m \geq 2 \), equation (7.3) does not contain the equation of the equilateral triangle.

If \( n=2m \), then the members of the sum in (7.3) are equal in pairs for the \( j=(m-1) \)-st set of (diametral) diagonals and hence the number of absolute-value terms, ordinarily \( n \) in number, is reduced to \( n/2 \). That \( n/2 \) is the least order of a semi-linear equation representing a regular polygon of an even number \( n \) of sides is seen at once; for, since each side must be enclosed in a separate region, it follows that there must be at least one diagonal-line (absolute-value term) through each vertex and no fewer than \( n/2 \) lines can in this way bring about this separation.

If \( n=2m+1 \), there is no diametral-set and consequently no such reduction in the order of the equation. The least order of an equation of a regular polygon of an odd number of sides, \( n=2m+1 \) is \( (n+1)/2 \).

In passing we point out that (7.3) will yield many interesting trigonometric identities.
It is possible to write down an equation (1.1) of a regular polygon based upon many other sets of diagonals. For example we may use certain subsets of diagonals or again may add extra diagonals to what is already a fundamental system.

The equation

\[(7.4) \quad |x + (1 + \sqrt{2})y - \rho + (1 + \sqrt{2})x - y - \rho| + |(1 + \sqrt{2})x + y - \rho| + |x + (1 + \sqrt{2})y + \rho| = 4 + 2\sqrt{2},\]

is the equation of the octagon with one vertex at \((\rho, 0)\) (Plate I, fig. 6), and is based on a subset of diagonals. The equation

\[(7.5) \quad 2|y| + 2 \left| \frac{1}{2} y - \frac{\sqrt{3}}{2} x \right| + 2 \left| \frac{1}{2} y + \frac{\sqrt{3}}{2} x \right| - \frac{2}{\sqrt{3}} x - \frac{1}{2} = 0,\]

is the equation of a hexagon; the first three terms are the diametral diagonals, while the last three terms are three lines as indicated in Plate I, fig. 7.

The equation

\[(7.6) \quad 2|y| + 2 \left| \frac{1}{2} y - \frac{\sqrt{3}}{2} x \right| + 2 \left| \frac{1}{2} y + \frac{\sqrt{3}}{2} x \right| - \frac{2}{\sqrt{3}} x - \frac{1}{4} \rho = \frac{\sqrt{3}}{2} \rho,\]

is that of the equilateral triangle (Plate I, fig. 8) with vertices at \((\rho, 0), (-\rho/2, 3\rho/2), (-\rho/2, -3\rho/2).\) We note that the left hand members of (7.5) and (7.6) are identical for \(\rho = 2.\)

8. Non-convex polygons. No semi-linear equation of order two exists for a non-convex quadrilateral. However, by making use of sub-vertices as in the case of the triangle, it can be shown that there exists an equation of order six for every simple non-convex quadrilateral.

The methods used in paragraph 7 and as there applied to the case of the regular even polygons will permit us to consider more general polygons of an even number of sides. Little in the analysis will be changed if we consider centrally symmetric polygons, or stars. The point of symmetry is the point of intersection of the diagonals. We have the

Theorem VII. There exists an equation (1.1) of the \(n\)-th order for every centrally symmetric polygon of \(2n\) sides. Moreover in such
an equation \( u_0 \) reduces to a constant.

The condition of convexity in paragraph 3 can be given up provided the resulting non-convex polygon is of such character that (3.3), (3.4) and (3.5) do not intersect in \( S \). This will be the case when and only when (3.3) and (3.1) have no point in common; i.e., when the exterior diagonal does not intersect the polygon.

Equations (1.1) of many varieties of polygons may be obtained from existing equations by inserting coefficients into these equations and then varying these as parameters. A general property of centrally symmetric polygons and their equations is expressed in the following

**Theorem VIII.** If, in the equation (1.1) of the \( n \)-th order of a centrally symmetric polygon of \( 2n \) sides, parameters \( M_i \) are introduced as multipliers of \( u_i \), then the figure remains symmetric no matter how these parameters are made to vary\(^{(1)}\).

The essence of the proof lies in the fact that regions of opposite sign-sets (and all the sign-sets occur in such pairs) play like roles. Or again directly from the equation itself we see that the graph will be symmetric with respect to the origin.

The problem of changing just one parameter in the equation of a regular polygon is of interest.

Consider, in equation (7.3), \( K \) as a parameter instead of a fixed constant. For fixed values of \( \rho, j \) and \( n \), there are \( n \) absolute-value terms and hence \( n \) diagonals. Since the (plus) sums are being taken of the absolute-values of these terms, it is at once obvious that \( K \) must be equal to or greater than zero in order that there exist a geometric representation. As a matter of fact there is a least permissible value of \( K (>0) \) except when \( j=(m-1), (n=2m) \), which value of \( j \) gives the diametral-set of diagonals. In this latter case the polygon is an even polygon and it is readily seen that as \( K \) increases (from zero) the size of the polygon increases (from a point).

Having dispensed with this case we turn next to that of \( j=0 \). Consider a regular polygon with center at \((0, 0)\) and one vertex at \((\rho, 0)\). With respect to this polygon, equation (7.3) now represents all interior and boundary points provided that \( K=2A/L \) where \( A \) is the area of the polygon and \( L \)=length of a side of the polygon. This is the least permissible value of \( K \).

\(^{(1)}\) Of course configurations other than polygons may be obtained by this procedure.
Now let $K$ increase continuously. First of all we note that no point interior to or on the perimeter of the original polygon now satisfies the equation. Next consider the closed region $S$ formed by a diagonal (side) and the two adjacent diagonals (sides) extended. These boundary lines of this region form an isosceles triangle with one side of the polygon as base. Now in such a region it is clear that the coefficient of $x$ and the coefficient of $y$ remain unchanged as $K$ increases from $K=2A/L$; and since the constant term increases, it follows that the line which constitutes the solution in this region is moved away from the origin and parallel to itself. This line is a side of the polygon itself. (There is, to be sure, a jump from an areal solution to a linear solution). Since the increase in $K$ affects in a like manner every such region $S$, it follows that the sides of the $n$-gon decrease in length (they are squeezed down between the diagonals) and move parallel to themselves away from the center of the polygon and that, because of continuity properties, $n$ other segments must connect these sides through the intervening regions. Thus it happens that the (regular) $n$-gon swells into an irregular $2n$-gon, into a regular $2n$-gon, and on into a regular $n$-gon again with vertex corresponding to $(\rho, 0)$ changed (by translation and rotation) into vertex at $(\rho_i, y_i)$ where $y_i/\rho_i=\tan \pi/n$. This is to be expected because of the fact that the sides of a regular $n$-gon extended intersect (in pairs) in points which are vertices of another regular $n$-gon. Indeed, for an $n$-gon there exactly $m-2$ (where $n$ is either $2m$ or $2m-1$) such $n$-gons and for increasing $K$ all of these are realized. Note that the $j=0$-th set of diagonals of the first $n$-gon become the $j=1$-st set of diagonals for the second $n$-gon, etc. (See Plate II, fig. 9).

9. Other configurations. Semi-linear equations exist for many combinations of points, lines and areas. We shall indicate only a few of the possible configurations. We have seen above that, for $j=0$, equation (7.3) represents an area. This generalizes immediately to the

Theorem IX. For every convex $n$-polygonal area there exists a semi-linear equation(1) of order $n$.

The proof is immediate. Let $L_i=$ lengths of the sides of the polygon, $u_i=0$ the equations of the sides (diagonals) in normal form, $A=$ area. Then it is obvious that from every interior point we have

(1) Statement due to Söderblom.
It can be shown that all other regions are essentially vacuous. Thus (9.1) is the equation of the area of the polygon.

If $L_i$ are the sides of a non-convex polygon, equation (9.1) may represent a convex area, line segment, one point, or no point (Plate II, figs. 10, 11, 12). This follows from the fact that dissection into triangles may be possible for only a limited (line-segment, etc.) set of points.

An equation of every order $n \geq 2$ exists which represents a single point. For, given a point $P$, if we select $n$ lines $u_i=0$ each passing through $P$, then $\sum |u_i|=0$ will be such an equation. By making use of convex polygons as above, we can determine an equation of every order $n \geq 4$ of the form (9.1) which will represent a single point.

The equation (1) $y=\sum |x-\alpha_i|$ plots a broken-line graph. Equations of the form

$y=\sum_{i=1}^{n} m_i |x-\alpha_i|$, 

may be used for approximations to the graphs of certain single-valued functions (Plate II, fig. 13).

The equation of $n$ unit-segments horizontally placed one above the other is (Plate II, fig. 14):

$y=-n+|x|+|x-1|+|y|-|y-1|+|y-2|+\ldots+|y-2(1-n)|=0$. 

The equation $-1+|x|+|y|=0$ represents a unit square. The equation of $n$ squares placed as in Plate II, fig. 15 is:

$y=-n+|y|+|x|-|x-1|+|x-2|+\ldots+|x-2(n-1)|=0$. 

Each of the equations

$-1+|x|+|x-1|+ky=0$, $k>0$, 

$-3+|x|+|x-1|+|y|+|y+1|+|y|+|y-1|=0$, 

$y+1-|x|-3|y|-|x+y-1|=0$, 

represents the unit-segment $(0,1)$. The equation of order $3n$ of $n$ line-segments as indicated in Plate II, fig. 16 is:

$-\lambda_n+|x|+|x-\lambda_i|-2|x-\frac{\lambda_1+\lambda_2}{2}|+|x-\lambda_2|+|x-\lambda_3|$ 

(1) Due to Riabounchinsky.
SEMI-LINEAR EQUATIONS.

\[-2\left|x - \frac{\lambda_3 + \lambda_4}{2}\right| + |x - \lambda_4| + \ldots + |x - \lambda_{n-1}| + |x - \lambda_n| + |y| = 0.\]

Note the presence of the absolute-value terms \(|x - (\lambda_1 + \lambda_2)/2|\), etc.; these are necessary in order to separate the two end points \((0, \lambda_1), (0, \lambda_2)\) since otherwise they would be in the same region. If the sum \(|x - \lambda_1| - 2|x - (\lambda_1 + \lambda_2)/2| + |x - \lambda_2| \to 0\) and hence if \(\lambda_1 \to \lambda_2\), the segment \([(0, \lambda_1)]\) connects with segment \([(\lambda_2, \lambda_3)]\) thereby forming a longer segment \([(0, \lambda_3)]\). If \(\lambda_2 \to \lambda_3\), the segment \([(\lambda_2, \lambda_3)]\) \to \(0\) and, if \(\lambda_2 = \lambda_3\), reduces to a point. Hence equation (9.8) is the equation of any combination of \(n\) line-segments and points along the positive \(x\)-axis.

Equation (9.8) might be called the equation of the Morse Code.

Both end segments in the representation of the equation

\[(9.9) \quad |x| - 2\left|x - \frac{\lambda_1}{2}\right| + |x - \lambda_1| + |x - \lambda_2| - 2\left|x - \frac{\lambda_3 + \lambda_4}{2}\right|

+ \ldots + |x - \lambda_n| + |y| = 0,\]

are half lines while only the left end segment in the graph for

\[(9.10) \quad x - \lambda_n + |x| - 2\left|x - \frac{\lambda_1}{2}\right| + |x - \lambda_2| - 2\left|x - \frac{\lambda_3 + \lambda_4}{2}\right|

+ \ldots + |x - \lambda_n| + |y| = 0,\]

is a half line the right end segment being of length \((\lambda_n - \lambda_{n-1}).\)

**Theorem XI.** A semi-linear equation (1.1) exists for every arrangement of points and line-segments lying in one and the same straight line. A semi-linear equation, a part of whose solution is a given configuration \(H\) made up of points, lines, and area, exists for every \(H\) for which there is a fundamental system of diagonals. For, since an arbitrary number of additional diagonals may be used, it follows that a sufficient number of unknowns \((m_i's)\) may in this way be introduced as to result in a consistent system of auxiliary equations and hence in the desired semi-linear equation.