Some Arithmetical Functions,

by

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1. We define the functions $\delta(k)$, $\epsilon(k)$ and $\theta(k)$ as follows:

1) Let us write

\[(1) \quad (m)^k = (n)^k,\]

when there exist positive integers $x_t(s \leq m)$, $y_t(t \leq n)$ such that

\[(2) \quad \sum_{s \leq m} x_t^k = \sum_{t \leq n} y_t^k,\]

where

\[(3) \quad (x_1, \ldots, x_m, y_1, \ldots, y_n) = 1,
(4) \quad x_t(s \leq m) = y_t(t \leq n).\]

If (1) is true infinitely often we write

\[(5) \quad (m)^k = (n)^k \text{ i. o.}\]

Then $\epsilon(k)$ denotes the least value of $m + n$ such that (5) is true. Since, trivially,

\[(6) \quad (k + 1)^k = (k + 1)^k \text{ i. o.,}\]

we have

\[(7) \quad \epsilon(k) \leq 2k + 2.\]

2) $\delta(k)$ denotes the least value of $(m + n)$ such that the equation

\[(8) \quad c = \sum_{s \leq m} x_t^k - \sum_{t \leq n} y_t^k\]

has infinitely many solutions satisfying (4), where $c$ is a constant (different from zero) depending only on $k$.

3) $\theta(k)$ is defined as the least value of $s$ such that there exists a constant $c$ depending only on $k$ with the property that the equation

\[c = \sum_{m=1}^s e_m u_m^k \quad (each \ e_m = +1 \ or \ -1)\]

has infinitely many solutions in rational $u_m(m \leq s)$.

It is immediately obvious that

\[(9) \quad \theta(k) \leq \text{Min} [\delta(k), \epsilon(k)].\]

Since\(^{(1)}\)

\(^{(1)}\) A proof of this result is given in my paper "Some problems of Waring's type" to appear in Proc. Indian Acad. Sc.
we have

\[ \theta(k) \leq 2k. \]

Concerning \( \delta(k) \) it is not even known whether (for example)

\[ \delta(k) \leq k \log \log k \]
is true for all large \( k \).

We observe that if the probable result

\[ (k)^2 = (k)^k \text{ i. o.} \]
is true, then

\[ \epsilon(k) \leq 2k. \]

On the other hand if (13) is false, then Hypothesis \( K \) of Hardy
and Littlewood\(^1\) is true. It is known that\(^2\)

\[ \epsilon(5) \leq 6, \epsilon(8) \leq 16, \delta(8) \leq 16, \epsilon(9) \leq 17, \text{ etc.} \]

To these I add

\[ \delta(5) \leq 5, \]
and

\[ \theta(8) \leq 8. \]

2. Proof (B). We have

\[ \begin{aligned}
& [[20(2^{12} - 2^3)^5 z^5 + 16]^5 \\
& - [[20(2^{12} - 2^3)^5 z^5 - 16]^5 \\
& - 2[20(2^{12} - 2^3)^5 z^5 + 1]^5 \\
& + [2[20(2^{12} - 2^3)^5 z^5 - 1]^5 \\
& - [20(2^{12} - 2^3)^5 z^5] = 2(2^{39} - 2),
\end{aligned} \]

so that \( \delta(5) \leq 5. \)

3. Proof of (C). We have

\[ \begin{aligned}
& (x^5 + \frac{2^{19}}{x^3})^8 - (x^5 - \frac{2^{19}}{x^3})^8 - (2^6 x^5 + \frac{4}{x^3})^8 \\
& + \left(2^6 x^5 - \frac{4}{x^3}\right)^8 - (2^6 x^5)^8 + (2^9 x^4)^8 \\
& \quad - \left(2^{11} \frac{x^5}{x^2}\right)^8 + \left(\frac{2^3}{x^3}\right)^8 = 7(2^{32} - 2^n),
\end{aligned} \]

whence \( \theta(8) \leq 8. \)

\(^1\) Math. Zeitschr. 23 (1925), 1-37 (4).

\(^2\) For references see my paper already cited. See also foot-note in p. 127.
4. We observe that from (C) it follows immediately that (1)

\[(8)^* = (8)^* \text{ i. o.,} \]

and thus that

\[\epsilon(8) \equiv 16.\]

But from (17) or (18) we cannot deduce \(\theta(8) \equiv 8.\)

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