A Characteristic Property of the Normal Curvature,

by

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Let \( \xi(u^1, u^2) \), \( (\xi_1 \xi_2 \xi_3 = 1) \) be the direction-cosines of the surface normal of a surface \( x(u^1, u^2) \) and put as usual:

\[
D_{hk} = -(x_h \xi_k)_3 = -(x_h \xi_k)_3, \quad G_{hk} = (x_h x_k)_3, \quad \mathcal{G}_{hk} = (\xi_h \xi_k)_3.
\]

Then the quantity

\[
(1) \quad \frac{1}{R} = \frac{D_{hk} du^h du^k}{G_{hk} du^h du^k}
\]

is called the normal curvature corresponding to the line-element \( du^1 : du^2 \). When \( du^1 : du^2 \) corresponds in particular to one of the principal directions, the corresponding value

\[
R_i^{-1} \text{ of } R^{-1},
\]

\((i=1, 2)\), is called the principal curvature.

It is known that the relations

\[
(2) \quad R_i^{-1} = P_i
\]

hold.

The tangential sphere with the radius

\[
R
\]

of the surface is called the semi-osculating sphere corresponding to the line-element \( du^1 : du^2 \).

Now let us prove the following

**Theorem.** Among the \( \infty^1 \) tangential spheres of a surface \( x(u^1, u^2) \) at the surface point \( x(u^1, u^2) \), the semi-osculating sphere is characterized by the property that the common tangential segment between the consecutive tangential spheres at \( x(u^1, u^2) \) and \( x(u^1 + du^1, u^2) \),
\( u^2 + du^2 \) becomes minimum.

**Proof.** Let the radius of the tangential sphere at \( x(u^1, u^2) \) be \( r \) and that of the tangential sphere at \( x(u^1 + du^1, u^1 + du^2) \) \( r + dr \). Then their centers are \( (x + r\xi) \) and \( (x + dx + (r + dr)(\xi + d\xi)) \) respectively and the square of the distance between them is given by

\[
\sum \{x + dx + (r + dr)(\xi + d\xi) - (x + r\xi)\}^2 = \sum (dx + rd\xi + \xi dr + drd\xi)^2.
\]

Hence from

\[
\sum (dx + rd\xi + \xi dr + drd\xi)^2 = r^2 + (r + dr)^2 - 2r(r + dr) \cos \theta, \quad \Rightarrow \quad \{r + dr - r\}^2 = F,
\]

where

\( \theta \) is the angle

\[
F = \text{the square of the segment in question, we have}
\]

\[
f = 4\sin^2 \theta \quad \Rightarrow \quad F = (dx dx)_3 + 2(r + dr)(dx d\xi)_3 + (r + dr)^2 (d\xi d\xi)_3.
\]

Hence from

\[
0 = \frac{1}{2} \frac{\partial f}{\partial p} = p(dx dx)_3 + (dx d\xi)_3,
\]

we have

\[
p = \frac{1}{r} \frac{(dx d\xi)_3}{(dx dx)_3},
\]

which becomes

\[
\frac{1}{R} = \frac{D_{nk} dx^i dx^k}{G_{nk} dx^i dx^k}.
\]

Since

\[
\frac{1}{2} \frac{\partial^2 f}{\partial p^2} = (dx dx)_3 > 0,
\]

the extreme value obtained above is the minimum.

Substituting the special value of

\[
p \quad \Rightarrow \quad r
\]

in the expression for

\[
f, \quad \Rightarrow \quad F,
\]

we obtain
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Taking \( r' \) in place of \( r + dr \) and considering \( r \) and \( r' \) as independent variables, we reach the same result.

\[
0 \leq \min f = \mathcal{H}_{ik} \, du^i \, du^k \quad \frac{(D_{nk} \, du^k \, du^k)^2}{G_{nk} \, du^i \, du^k} \quad \frac{(D_{nk} \, du^i \, du^k)^2}{\mathcal{H}_{nk} \, du^i \, du^k} \leq 0 \leq \min F = G_{nk} \, du^i \, du^k
\]

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