Two Elementary, Purely Projective and Synthetic Proofs for the Grace's Theorem concerning the Double Six and Two Consequences,

by

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The only purely projective and synthetic proof for the Grace's theorem\(^1\) concerning the double six, hitherto carried out within the scheme of points, lines and planes only, seems to be that due to Prof. T. Takasu\(^2\). But he uses double ratios more than positional relations. This paper will fulfil this lack giving two purely positional proofs to Lemma 4° of Prof. Takasu and two remarkable corollaries will be added.

Grace's Theorem. When six straight lines \(a_1, a_2, a_3, a_4, a_5\) and \(a_6\), no two of which meet, are met by another straight line \(g\), the common intersector \(b_{12} = b_{21}\) of \(a_3, a_4, a_5, a_6\) other than \(g\),

\[
\begin{align*}
  b_{13} &= b_{31} \text{ of } a_2, a_4, a_5, a_6, \\
  b_{14} &= b_{41} \text{ of } a_2, a_3, a_5, a_6, \\
  b_{15} &= b_{51} \text{ of } a_2, a_3, a_4, a_6, \\
  b_{16} &= b_{61} \text{ of } a_2, a_3, a_4, a_5,
\end{align*}
\]

are met by another straight line \(b_1\) by Lemma 3° for \(a_2, a_3, a_4, a_5\) and \(a_6\). Similarly, \(b_{21}, b_{23}, b_{24}, b_{25}\) and \(b_{26}\) are met by a straight line \(b_2\),

\[
\begin{align*}
  b_{31}, b_{32}, b_{34}, b_{35} \text{ and } b_{36} & \text{ are met by a straight line } b_3, \\
  b_{41}, b_{42}, b_{43}, b_{45} \text{ and } b_{46} & \text{ are met by a straight line } b_4, \\
  b_{51}, b_{52}, b_{53}, b_{54} \text{ and } b_{56} & \text{ are met by a straight line } b_5, \\
  b_{61}, b_{62}, b_{63}, b_{64} \text{ and } b_{65} & \text{ are met by a straight line } b_6,
\end{align*}
\]

and \(b_{ij} = b_{ji}\) being the common intersector other than \(g\) of four straight

\(^1\) J. H. Grace, Circles, Spheres, and Linear Complexes, Trans. Cambridge Phil. Soc., 16 (1897), p 153-190.

lines other than $a_1$, $a_4$ out of $a_1$, $a_2$, $a_3$, $a_4$, $a_5$ and $a_6$; and the six straight lines $b_1$, $b_2$, $b_3$, $b_4$, $b_5$ and $b_6$ are met by another straight line $h$, provided that, of the row of points $g(a_1, a_2, a_3, a_4, a_5, a_6)$ and the sheaf of planes $g(a_1, a_2, a_3, a_4, a_5, a_6)$, no corresponding quadruples of elements are mutually projective.

**Lemma 1°.** When four straight lines $a_1$, $a_2$, $a_3$ and $a_4$, no two of which meet, are met by a straight line $b_6$, there exists necessarily another straight line $b_5$ different from $b_6$, which meets all of $a_1$, $a_2$, $a_3$ and $a_4$, provided that the row of points $b_6(a_1, a_2, a_3, a_4)$ and the sheaf of planes $b_6(a_1, a_2, a_3, a_4)$ are not projective(1).

**Lemma 2°.** If for the six points $A$, $B$, $C$, $D$, $C'$ and $D'$ on a straight line the relation

$$ABCD \cong ABC'D'$$

holds, then

$$ABCC' \cong ABDD'.$$

**Proof.** We can find two perspectivities such that

$$ABCD \cong A_1BCD_1 \cong ABC'D'.$$

Hence

$$ABCC' \cong AA_1SS' \cong ABDD'.$$

**Lemma 3°.** (Double Six Theorem of Schlafli)(2).

**Lemma 4°.** The six straight lines $b_1$, $b_2$, $b_3$, $b_4$, $b_5$ and $b_6$, spoken of in the Grace's theorem, are met by another straight line $a_{56}$. If we take $a_i(i=1, 2, 3 \text{ or } 4)$ in place of $a_5$, there exists a straight line $a_{6j}$ which meets $a_i$ and $a_6$ as well as the four straight lines $b_j(j \neq i, 6)$.

**First Proof.** First of all let us state a pair of lemmas, which

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(1) For an elementary, purely projective and synthetic proof, which holds good even within the scheme of real elements, see C. Yamasita, An Elementary, Purely Projective and Synthetic Proof and Three Others for the Double Six Theorem of Schlafli, this journal, this volume, p. 382, Lemma.

(2) For a purely projective and synthetic proof carried out within the scheme of points, lines and planes only, see my paper last cited.

I have already proved (1).

(i) Every straight line, which meets three straight lines out of four, all meeting three other straight lines in skew positions, always meets the remaining fourth.

(ii) Let three straight lines \(a, c\) and \(d\) be three common intersecors of three straight lines \(b, e\) and \(f\) in skew positions. Then for the sheaves of planes the following relation holds:

\[
a(CDEF) \equiv b(FEDC),
\]

where \(C, D, E\) and \(F\) are the points of intersection of \(\pi\) with \(c, d, e\) and \(f\) respectively.

Now let us come to the first proof of Lemma 4°. The four straight lines \(b_1, b_2, a_5\) and \(a_6\) meets \(b_{12}\); let the other common intersector of \(b_1, b_2, a_5\) and \(a_6\) be \(a_{56}\). Let the points, in which the six straight lines \(b_{12}, b_{13}, b_{23}, g, a_1\) and \(a_6\) are cut by a plane passing through \(a_{56}\) be \(B_{12}, B_{13}, B_{23}, G, A_5\) and \(A_6\) respectively. The straight line \(a_{56}\) passes through \(A_5\) and \(A_6\).

Applying Lemma (ii) to the six straight lines \(b_{14}, b_{12}, g; a_3, a_5, a_6\), we see that for the sheaves of planes the relation

\[
b_{14}(B_{12}GA_5A_6) \equiv a_3(A_6A_5GB_{12})
\]

holds.

Taking \(b_{24}\) instead of \(b_{14}\), we have for the sheaves of planes:

\[
b_{24}(B_{12}GA_5A_6) \equiv a_1(A_6A_5GB_{12}).
\]

Hence for the sheaves of planes:

(1) \[
b_{14}(B_{12}GA_5A_6) \equiv b_{24}(B_{12}GA_5A_6).
\]

Similarly, taking instead of \(a_1\) the straight lines \(a_2\) and \(a_1\) in succession, we obtain for the sheaves of planes:

(2) \[
b_{14}(B_{13}GA_5A_6) \equiv b_{24}(B_{13}GA_5A_6),
\]

(3) \[
b_{34}(B_{23}GA_5A_6) \equiv b_{24}(B_{23}GA_5A_6).
\]

Applying (i) to the three straight lines \(b_{13}, b_{13}\) and \(a_{56} = A_4A_6\) out of \(b_{13}, b_{13}, b_{14}\) and \(a_{56}\), all meeting the three straight lines \(b_1, a_3\) and \(a_6\), we see that the straight line \(B_{12}B_{13}\), which meets \(b_{13}, b_{13}\) and \(a_{56}\), meets another straight line \(b_{14}\). Thus the sheaves of planes \(b_{14}(B_{12}GA_5A_6)\) and \(b_{14}(B_{13}GA_5A_6)\) are one and the same.

Similarly, taking \(b_2\) in place of \(b_1\), we see that the straight

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(1) C. Yamashita, An Elementary, Purely Projective and Synthetic Proof and Three Others for the Double Six Theorem of Schläfli, loc. cit., p. 392, Lemma 1 and 2
line $B_{12}B_{23}$ meets $b_{21}$, so that the sheaves of planes $b_{24}(B_{12}GA_5A_6)$ and $b_{24}(B_{23}GA_5A_6)$ are one and the same. Hence from (1), (2) and (3), it follows that

$$b_{24}(B_{12}GA_5A_6) \cong b_{24}(B_{23}GA_5A_6),$$

so that the planes $b_{24}A_{13}$ and $b_{34}A_{13}$ coincide and the straight line $B_{13}B_{23}$ meets $b_{24}$.

Applying the Lemma (i) to the three straight lines $a_5$, $a_6$ and $B_{13}B_{13}$ out of $b_3$, $a_5$, $a_6$ and $B_{13}B_{23}$, all meeting the three straight lines $b_{13}$, $b_{23}$ and $b_{34}$, we see that the straight line $A_5A_6\equiv\alpha_{56}$, which meets $a_5$, $a_6$ and $B_{13}B_{23}$, meets also $b_3$.

Similarly, it may be proved that $a_{56}$ meets $b_4$.

Second Proof. Let the seven straight lines $g$, $b_{23}$, $b_{12}$, $b_{11}$, $b_{21}$ and $b_{34}$ be met by $a_5$ in the points $G$, $G_1$, $G_2$, $G_3$, $H_1$, $H_2$ and $H_3$ respectively and by $a_6$ in $G'$, $G'_1$, $G'_2$, $G'_3$, $H'_1$, $H'_2$ and $H'_3$ respectively.

Let the points, in which the three planes $a_4H'_1$, $a_4H'_2$ and $a_4H'_3$ are cut by $a_6$, be $H''_1$, $H''_2$ and $H''_3$ respectively. Then for the rows of points the following relation holds:

$$a_5(GG_1G_2G_3H''_1H''_2H''_3) \cong a_6(G'_1G'_2G'_3H'_1H'_2H'_3).$$

If $ijk$ be a permutation of 1, 2, 3, for the rows of points there holds the perspectivity

$$a_5(GG_1H_1H_2) \cong a(G'_1H'_1H'_2)$$

by means of the sheaf of planes $a_5(gb_1b_2b_3)$.

From (4) and (5), we conclude that

$$a_5(GG_1H_1H_2) \cong a_5(GG_1H''_1H''_2),$$

whence by Lemma 2°, it follows that

$$a_5(GG_1H_1H_2) \cong a_5(GG_1H_1H_2).$$

Let the points $G_1^*$ and $G_2^*$ be such that by the perspectivity

$$a_5(GG_1^*G_2^*G_3H_1H_2),$$

the correspondence

$$a_5(GG_1^*G_2^*G_3H_1H_2) \cong a_5(GG_1^*G_2^*G_3H_1H_2)$$

holds.

Let the points $G_2^*$ and $G_3^*$ be such that by the projectivity

$$a_5(GG_1^*H_2H_2) \cong a_5(GG_1^*H_2H_2),$$

the correspondence
holds.

Hence

\[ a_5(GG_2H_1H_1'') \cap a_5(G\overline{G_2}H_2H_2'') \]

Now

\[ a_5(GG_2H_1H_1') \cap a_5(G\overline{G_2}H_3H_3'') \]

Hence we see that \( G_2 = \overline{G_2} \), so that

\[ a_5(G_1G_1G_2G_3G_4H_1H_1'') \cap a_5(GG_1G_2G_3G_4H_2H_2'') \cap a_5(GG_1G_2G_3G_4H_3H_3'') \]

Let \( G^*, H \) and \( H'' \) be such that by the projectivity

\[ a_5(G_1G_2G_3) \cap a_5(G_1G_1G_2) \]

the correspondence

\[ a_5(G_1G_2G_3H_1H_1'') \cap a_5(G_1G_2G_3HH'') \]

holds. Thus if \( ijk \) be a permutation of 1, 2 and 3, we have

\[ a_5(GG_kG_iH_kH_k'') \cap a_5(GG_kG_iHH''), \]

i.e.

\[ a_5(G_iG_jH_kH_k'') \cap a_5(G_iG_jHH'') \]

Hence by Lemma 2°, we have

\[ a_5(G_iG_jH_kH) \cap a_5(G_iG_jH_k''H'') \]

If we take a point \( H' \) on \( a_6 \) such that the plane \( a_iH' \) is cut by \( a_6 \) in \( H'' \), then

\[ a_5(G_iG_jH_k''H'') \cap a_6(G_iG_jH_iH') \]

Hence, for the rows of points, we have

\[ a_6(G_iG_jH_kH) \cap a_6(G_iG_jH_k'H') \]

Now, since \( b_k \) meets the three straight lines \( G_iG_jH_kH, G_jG_iH_k'H \) and \( II_kH_k' = b_k' \),

\[ a_5(G_iG_jH_kH) \cap b_k(b_k b_k b_k H), \]

\[ a_6(G_iG_jH_k'H') \cap b_k(b_k b_k b_k H'), \]

so that for the sheaves of planes the following relation holds:

\[ b_k(b_k b_k b_k H) \cap b_k(b_k b_k b_k H'). \]

Hence the planes \( b_kH \) and \( b_kH' \) must coincide, so that the straight line \( HH' \) meets \( b_k \) (\( k=1, 2, 3 \)).

Thus we have obtained the common intersector \( HH' \) of the five straight lines \( a_6, a_6, b_1, b_2 \) and \( b_3 \).
Similarly, the common intersector of the five straight lines \(a_5, a_6, b_1, b_2\) and \(b_4\) may be obtained.

Now besides \(b_{12}\), there is only one common intersector of \(a_5, a_6, b_1\) and \(b_2\) and \(b_{12}\) does not meet \(b_3\) and \(b_4\).

Therefore the two common inter sectors just obtained must coincide and is nothing but the common intersector \(a_{56}\) of \(b_1, b_2, b_3, b_4, a_5\) and \(a_6\).

Proof for the Grace's Theorem. By Lemma 1°, the four straight lines \(b_1, b_2, b_3\) and \(b_4\) have, besides \(a_{56}\), which was obtained by the proof of Lemma 4°, another common intersector. Let it be \(h\).

If we take \(a_i(i=1, 2, 3, 4\text{ or }6)\) in place of \(a_5\) in Lemma 4°, there exists a straight line, which meets \(a_i\) and \(a_5\) as well as the four straight lines \(b_j\) \((j \neq i, 6)\). Let it be \(a_{16}\).

Applying Lemma 3° to the five straight lines \(a_{16}, a_{26}, a_{36}, a_{46}\) and \(a_{56}\) with the common intersector \(a_6\), since the common intersector other than \(a_6\) of the four straight lines \(a_{j6}(j \neq k; k=1, 2, 3, 4\text{ or }5)\) is \(b_k(k=1, 2, 3, 4\text{ or }5)\), we see that the five straight lines \(b_1, b_2, b_3, b_4\) and \(b_5\) have a common intersector. Since, this common intersector is not \(a_{56}\), it is \(h\) itself. Hence \(h\) meets \(b_5\).

Similarly, it may be proved that \(h\) meets \(b_6\).

Cor. 1°. In the configuration of the Grace's theorem, the following relations hold:

\[\text{row of points} \quad g(a_1a_2a_3a_4a_6) \cong \text{sheaf of planes} \quad h(b_1b_2b_3b_4b_6)\]
\[\cong \text{sheaf of points} \quad a_{16}(a_{14}a_{13}a_{12}a_{16}a_{16}) \cong \text{sheaf of planes} \quad b_1(b_1b_2b_3b_4b_5b_6),\]

\(a_{16}\) and \(b_{16}\) not arising.

Proof. We refer to the facts arising in the second proof of Lemma 4°. Let three points \(E_1, E_2\) and \(E_3\) be so taken on a plane passing through \(a_5\), which contains \(G, G_1, G_2, G_3, H_1, H_2\) and \(H_3\), that the three sides \(E_1E_3, E_3E_1\) and \(E_1E_2\) pass through \(G_1, G_2\) and \(G_3\) respectively. Let the point of intersection of \(E_1H_1\) and \(E_2H_2\) be \(E\) and let the points, in which the three straight lines \(G_1E_2E_3, G_2E_3E_1\) and \(G_3E_1E_2\) meet the straight line \(GE\) be \(F_1, F_2\) and \(F_3\) respectively.

By (6) and from the figure we see that for rows of points the following relations hold:

\[GF_2F_3E \cong (GG_2G_3H_1) \cap (G^*G_2G_3H),\]
\[GF_1F_3E \cong (GG_1G_3H_2) \cap (G^*G_1G_3H).\]
Hence

(7) \((GF_1F_2F_3E) \cong (G^*G_1G_2G_3H)\).

Now by (6),

\((GG_1G_2H) \cong (G^*G_1G_2H)\).

Hence

\((GF_1F_2E) \cong (GG_1G_2H)\).

From the figure, we have

\((GF_1F_2) \cong (GG_1G_2)\).

Thus the straight line \(EH_3\) passes through \(E_3\).

Let the point of intersection of \(E_1E_3\) and \(E_2E\) be \(D\). Then from the figure we see that for the rows of points the relations

\((H_1H_2H_3G) \cong (E_1D) \cong (E_2) \cong (F_3EF_1F_2)\)

hold. Hence

\((II_1H_2H_3G) \cong (F_3EF_1F_2) \cong (F_1F_2F_3E)\).

By (7) and (8), it follows for the rows of points on \(a_5\), that

\((H_1H_2II_5G) \cong (G_1G_2G_3H)\),

row of points \(a_5(a_{14}a_{23}b_{34}g) \cong \) row of points \(a_5(b_{23}b_{31}b_{12}a_{56})\).

Projecting \(a_5(b_{23}b_{31}b_{12}a_{56})\) from \(a_6\),

row of points \(a_6(b_{14}b_{23}b_{34}g) \cong \) sheaf of planes \(a_6(b_{23}b_{31}b_{12}a_{56})\).

If we take \(a_6\) in place of \(a_5\),

row of points \(a_6(b_{14}b_{23}b_{34}g) \cong \) row of points \(a_6(b_{23}b_{12}a_{56})\).

If we project the row of points on the lefthand side from \(a_6\),

sheaf of planes \(a_6(b_{14}b_{23}b_{34}g) \cong \) row of points \(a_6(b_{23}b_{12}a_{56})\).

Hence

row of points \(a_5(b_{14}b_{23}b_{34}g) \cong \) sheaf of planes \(a_6(b_{23}b_{12}a_{56})\).

Applying the Dixson's theorem(1) to the double six

\[ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \]
\[ b_{14} \ b_{24} \ b_{34} \ g \ b_{46} \ b_{64} \]

we obtain

Similarly from the double six

\[ b_1 \ b_2 \ b_3 \ a_1 \ a_2 \ a_3, \]
\[ b_2 \ b_3 \ b_4 \ a_4 \ a_5 \ a_6, \]

we obtain

\[ \text{row of points } a_6(b_1b_2b_3a_4), \]
\[ \text{sheaf of planes } a_6(b_2b_3b_4a_5). \]

Hence

\[ \text{row of points } a_4(b_1b_2b_3a_4), \]
\[ \text{sheaf of planes } b_4(a_1a_2a_3b_4). \]

Taking 6 in place of 3, we obtain

\[ \text{row of points } a_6(b_1b_2b_3a_4), \]
\[ \text{sheaf of planes } a_6(b_2b_3b_4a_5). \]

Hence

\[ \text{row of points } a_4(b_1b_2b_3a_4), \]
\[ \text{sheaf of planes } a_4(b_1b_2b_3a_5). \]

Now

\[ \text{row of points } a_6(b_1b_2b_3b_4), \]
\[ \text{sheaf of planes } b_4(a_1a_2a_3a_4). \]

Hence

\[ \text{row of points } g(a_1a_2a_3a_4), \]
\[ \text{sheaf of planes } h(b_1b_2b_3b_4). \]

Similarly

\[ \text{row of points } g(a_1a_2a_3a_4), \]
\[ \text{sheaf of planes } h(b_1b_2b_3b_4). \]

From the last relations, we obtain

\[ \text{row of points } g(a_1a_2a_3a_4a_5a_6), \]
\[ \text{sheaf of planes } h(b_1b_2b_3b_4b_5b_6). \]

Thus the first part of the proposition is proved.

From the double six

\[ b_1 \ a_1 \ a_2 \ \ldots, \ (a_j \ \text{not arising}), \]
\[ g \ b_1 \ b_2 \ \ldots, \ (b_j \ \text{not arising}), \]

we obtain

\[ \text{row of points } g(a_1a_2a_3a_4a_5a_6), \]
\[ \text{sheaf of planes } b_1(b_1b_2b_3 \ldots). \]

and from the double six

\[ a_1 \ b_1 \ b_2 \ \ldots, \ (b_1 \ \text{not arising}), \]
\[ h \ a_1 \ a_2 \ \ldots, \ (a_1 \ \text{not arising}), \]
Thus the second part of the proposition is proved.

**Cor. 2°.** In the configuration of the Grace's theorem, the following relation holds:

row of points \( a_i(a_1a_2b_1b_2) \) \( \succ \) sheaf of planes \( h(b_1b_2) \).

**Proof.** The formula (8) is

\[
\text{row of points} \quad a_i(b_1b_2b_4b_6a_4) \quad \succ \quad \text{sheaf of planes} \quad b_i(a_1a_2a_3a_5b_5).
\]

Interchanging 4 and 5,

\[
\text{row of points} \quad a_i(b_1b_2b_3b_6a_5) \quad \succ \quad \text{sheaf of planes} \quad b_i(a_1a_2a_3a_6b_6).
\]

From (9) and (10) we obtain

\[
\text{row of points} \quad a_i(b_1b_2b_3b_4a_4) \quad \succ \quad \text{sheaf of planes} \quad b_i(a_1a_2a_5a_6b_6).
\]

Similarly,

\[
\text{row of points} \quad a_i(a_1a_2b_1b_2) \quad \succ \quad \text{sheaf of planes} \quad b_i(b_1b_2a_1a_2).
\]

**N.B.** In similar manners, we may deduce several projectivities concerning five straight lines meeting \( a_i \) and five straight lines meeting \( b_i \).

**Schematic Expression of the Straight Lines arising in the Grace’s Theorem.** The straight lines arising in the Grace’s theorem may be schematically expressed by the following figures.

![Schemata for (i)](image-url)
The totality of the straight lines arranged on the curve in the schemata starting from

\[ g \text{ and } h \quad | \quad h \text{ and } g \quad | \quad a_y \text{ and } b_y \quad | \quad b_y \text{ and } a_y \]

is the totality of the straight lines meeting

\[ g \text{ and } h. \quad | \quad h \text{ and } g. \quad | \quad a_y \text{ and } b_y. \quad | \quad b_y \text{ and } a_y. \]

The totality of the straight lines arranged on the straight lines (in the schemata) passing through

\[ b_i \quad | \quad a_i \quad | \quad b_f \quad | \quad a_f \]

is the totality of the straight lines meeting

\[ b_i \quad | \quad a_i \quad | \quad b_f \quad | \quad a_f \]

and not meeting

\[ a_i. \quad | \quad b_i. \quad | \quad a_f. \quad | \quad b_f. \]

The straight lines meeting

\[ b_y \quad | \quad a_y \quad | \quad a_{yk}(\neq a_y) \quad | \quad b_{yk}(\neq b_y) \quad | \quad a_{yk}(\neq a_y) \quad \text{or} \quad b_{yk}(\neq b_y) \quad \text{or} \quad a_{yk}(\neq a_y) \quad \text{or} \quad b_{yk}(\neq b_y) \]

are

\[ b_i \text{ and } b_j \quad | \quad a_i \text{ and } a_j \]

on the two straight line passing through

\[ b_{ij} \quad | \quad a_{ij} \]

to be read out from the schemata, being guided by

(i). \quad (ii).
(in the schemata) and
\[ a_k \quad \mid \quad b_k \]
with other index \( k \).

**Literatures on the Grace's Theorem.**


11°. This paper.