On the Operational Solutions of Differential and Difference Equations(1),

by

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1. Introduction.

The notation \( p \) in this paper represents the Heaviside's operator with the operand zero(2), and the notation \( P \) represents the Milne-Thomson's operator with the operand zero(3).

If the relation

\[ u(p) \cdot 0 = y(x) \] (1)

exists and if \( y(x) \) is the solution of differential equation, we call \( u(p) \) the operational solutions of this differential equation.

If the relation

\[ v(p) \cdot 0 = z(x) \] (2)

holds when \( z(x) \) is the solution of difference equation, we call \( v(p) \) the operational solution of this difference equation.

Since we can easily prove

\[ x^\mu \{ p^m \cdot 0 \} = \frac{\Pi(\mu - m - 1)}{\Pi(-m - 1)} p^{m-\mu} \cdot 0, \] (3)(4)

this result is translated into

\[ x^\mu p^m = \frac{\Pi(\mu - m - 1)}{\Pi(-m - 1)} p^{m-\mu}. \] (4)

Especially when \( \mu \) becomes a positive integer \( n \), (4) becomes

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(1) Read before the meeting of the Physico-math. Soc. of Japan, April 1, 1938.


(4) If we eliminate \( p \) in both sides of (3) with the aid of the operational equation

\[ \frac{1}{p^{\mu+1}} \cdot 0 = \frac{x^\mu}{\Pi(\nu)}, \]

we find both sides of (3) equal.
\[ x^n p^m = \left( -\frac{d}{dp} \right)^n p^m \quad (m \neq 0) \quad (4.1) \]

and

\[ x^n p^0 = \left[ \frac{(n-1)!}{p^n} \right] \quad (4.2) \]

Similarly we have

\[ x p^m = -\frac{d}{dp} \left\{ (1 + \omega p) p^m \right\} \quad (5) \]

We know also the relation

\[ \frac{d^h y}{dx^h} = \{ p^n u(p) \} \cdot 0 \quad (6) \]

and

\[ \Delta^h z = \{ p^n v(p) \} \cdot 0 \quad (7) \]

Now we shall evaluate the operational solutions expressed in the power series of Heaviside's operator or Milne-Thomson's operator with the aid of the equations (4), (4.1), (4.2), (5), (6) and (7).


Let us consider the linear differential equation of the \( n \)th order

\[ y^{(n)} + \frac{Q_1(x)}{x} y^{(n-1)} + \ldots + \frac{Q_{n-1}(x)}{x^{n-1}} y^{(n-k)} + \ldots + \frac{Q_n(x)}{x^n} y = 0, \quad (8) \]

where we denote by \( y^{(m)} \) the \( m \)th derivative of \( y(x) \).

In this paper we shall only treat the case where the functions \( Q_1(x), Q_2(x), \ldots, Q_n(x) \) in (8) are all analytic at \( x=0 \).

If we make use of \( u(p) \) defined by (1), (8) is written in the form

\[ \left\{ p^n + \frac{Q_1(x)}{x} p^{n-1} + \ldots + \frac{Q_{n-1}(x)}{x^{n-1}} p^{n-k} + \ldots + \frac{Q_n(x)}{x^n} \right\} u(p) \cdot 0 = 0 \quad (9) \]

on account of (6), which we write, for convenience, in the following form

\[ (8) \quad \text{Cf. Proc. Physico-math. Soc. of Japan 20 (1938), p. 213.} \]
\[ (9) \quad \text{H. Jeffreys, Operational Methods in Math. Phys., p. 17.} \]
We now assume that
\[ u(p) = \sum_{r=0}^{\infty} c_r p^{n-r} \]  
(11)
and substitute (11) in (10) and eliminate the functions of \( x \) by the relation (4) we got a new power series of \( p \).

Now we shall investigate for a while how the highest exponent of this new series is obtained by the substitution of (11) in (10).

Since the highest exponent of the series (11) is \( p^m \), it is evident that the highest exponent of the new series comes from \( p^m \). Hence to the above end we have only to discuss the result of putting
\[ u(p) = p^m \]  
(12)
in (10).

Take now the \( k+1 \) th term of the left-hand expression of (10), namely,
\[ \frac{Q_k(x)}{x^{k-1}} p^{n-k} u \]  
(13)
and substitute (12) in (13), then (13) becomes
\[ \frac{Q_k(x)}{x^{k}} p^{n+m-k} \]  
(14)

If \( Q_k(x) \) is expressed in the power series (8)
\[ Q_k(x) = \sum_{r=0}^{\infty} a_r^{(k)} x^r, \]  
(15)
(14) becomes
\[ \sum_{r=0}^{\infty} \frac{(-1)^{k-r} a_r^{(k)} p^{n+m-r}}{(n+m-r)(n+m-r-1) \ldots (n+m-k-1)} \]  
(16)

The highest exponent of (16) is \( p^{n+m} \) which results from the constant term of (15). Since this exponent is independent of \( k \), the highest exponents of
\[ \frac{Q_k(x)}{x^{k}} p^{n-k} u \quad (k=0, 1, 2, \ldots, n) \]  
(17)

\( \text{(8)} \) This is always possible since we have assumed that \( Q_k(x) \) are analytic at \( x=0 \).

\( \text{(9)} \) When \( h=0 \), we assume
\[ \frac{Q_d(x)}{x^d} p^d u = p^n u. \]
are all \( p^{n+m} \) when we substitute (12) in (17) and those exponents always result from the constant terms of the Taylor expansion of \( Q_k(x) \), we can conclude:

The highest exponent of the series which is got by substituting (11) in (10) is \( p^{n+m} \), and this is obtained by substituting (12) in (10) and gathering the terms which result from the constant terms of the Taylor expansions of all \( Q_k(x) \).

Now we shall assume that the constant term of the Taylor expansion of \( Q_k(x) \) is not zero and the constant terms of the Taylor expansions of \( Q_{k+1}(x) \), \( Q_{k+2}(x) \), \ldots, \( Q_n(x) \) are all zero.

Owing to the above assumption the highest exponent of the power series of \( p \) which is obtained by substituting (11) in (10) is by (16)

\[
p^{n+m} \sum_{k=0}^{h-k} \frac{(-1)^k a^{(n)}_k}{(n+m)(n+m-1) \ldots (n-m-h-1)}. \quad (18)
\]

Since the power series of \( p \) obtained by the substitution of (11) in (10) must satisfy (10) the coefficient of \( p^{n+m} \) must be zero. Thus we have

\[
\sum_{k=0}^{h-k} \frac{(-1)^k a^{(n)}_k}{(n+m)(n+m-1) \ldots (n-m-h-1)} = 0.
\]

This equation can be arranged in the following algebraic equation of the \( k \) th degree

\[
m^k + A_1 m^{k-1} + \ldots + A_{k-1} m + A_k = 0, \quad (19)
\]

which we shall call the indicial equation of (10).

The roots of the equation (19) are the values of \( m \) of the first term of (11). Since the equation (19) yields \( k \) roots, there are \( k \) kinds of the series (11) corresponding to these \( k \) roots, and the results of operating with these \( k \) series on zero are \( k \) fundamental solutions of the differential equation (8).

According to the theory of differential equations there are \( n \) fundamental solutions of the equation (8). Therefore the above result seems to contradict the known fact of the theory of differential equation. This apparent contradiction will be resolved by the fact that the series (11) can sometimes be the operational solution of (8) when \( m \) becomes a negative integer.

Now we shall investigate the operational solutions of (8) whose highest exponents are negative integers.

Assuming that \( m \) is a negative integer we substitute (12) in the
left-hand expression of (10). If we fix our attention to the $h+1$ th term of (10), the result of putting (12) in (10) becomes

$$Q_h(x)\frac{p^{n+m-h}}{x^h} \quad (h=0, 1, 2, \ldots, n). \quad (20)$$

By the same reasoning as before the highest exponent of (16) results from the constant term of (15). Hence if we drop all the terms of (15) except the constant term, (20) becomes

$$\frac{a_{h}^{(0)}}{x^h} p^{n+m-h}, \quad (21)$$

where $h \leq k$ owing to the assumption already related.

If we evaluate (21) by (4), we have

$$\frac{a_{h}^{(0)}}{x^h} p^{n+m-h} = (-1)^h a_{0}^{(0)} \frac{(n+m-h)!}{(n+m)!} p^{n+m}, \quad (21)$$

when

$$m = -1, -2, \ldots, -(n-h), \quad (h \leq k). \quad (22.1)$$

But (21) becomes

$$\frac{a_{h}^{(0)}}{x^h} p^{n+m-h} = \left[ (-1)^{n+m} a_{0}^{(0)} p^{n+m} \right] \frac{a_{h}^{(0)}}{(n+m)! (h-n-m)!} \quad (23)(16)$$

when $m$ is a negative integer less than $-(n-h)$. It is notable that the right-hand members of both (22) and (23) contain $p^{n+m}$.

Putting the above results together we can conclude that the highest exponent of the power series of $p$ which is got by substituting (11)(11) in (10) is

$$p^{n+m} \left\{ \sum_{h=0}^{h=k} (-1)^h a_{0}^{(0)} \frac{(n+m-h)!}{(n+m)!} \right\} \quad (24)(12)$$

when

$$m = -1, -2, \ldots, -(n-h), \quad (24.1)$$

or

$$p^{n+m} \left\{ \sum_{h=0}^{h=j} (-1)^h a_{0}^{(0)} \frac{(n+m-h)!}{(n+m)!} + \sum_{n=m+1}^{h=k} (-1)^{n+m} a_{0}^{(0)} p^{n+m} \right\} \frac{a_{h}^{(0)}}{(n+m)! (h-n-m)!} \quad (25)(13)(16)$$

\footnote{Of course here we assume that $m$ is a negative integer.}

\footnote{Since the value of $m$ given by (24.1) is included in the value of $m$ in (22.1) on account of the condition $h \leq k$, the terms of the type (23) does not make its appearance in the sum (24).}

\footnote{The terms of the type (23) exist always in (25), while the terms of the type (22) need not be included in (25).}
when $m$ is a negative integer less than $-(n-k)$ and where $j$ is an integer limited by $0 \leq j < k$. It is remarkable that the function of the infinitely large space (14) makes its appearance in (25).

Since only the highest exponent of the power series of $p$ has relation to the discussion of the value of $m$ of the first term in the series (11), if we omit all the terms of the power series of $p$ which is got by substituting (11) in (10) except for the highest exponent of this series, the result of the substitution of (11) in (10) is written in the form

$$p^n u + \frac{Q_1(x)}{x} p^{n-1} u + \ldots + \frac{Q_k(x)}{x^k} p^{n-k} u + \ldots + \frac{Q_n(x)}{x^n} u = B p^{n+m},$$

where $B$ stands for the sum in the braces of (24) or (25). When $B$ represents the sum in the braces of (24) $B$ is a constant in the common space, but when $B$ represents the sum in the braces of (25) $B$ is a constant in the infinitely large space.

Now making use of the relation

$$p^{n+m} \cdot 0 = \left[ \frac{(1)^{n+m}(n+m)}{x^{n+m+1}} \right]_{\infty}$$

we operate both sides of (26) on zero, and we have by (6)

$$y^{(n)} + \frac{Q_1(x)}{x} y^{(n-1)} + \ldots + \frac{Q_k(x)}{x^k} y^{(n-k)} + \ldots + \frac{Q_n(x)}{x^n} y = \frac{C}{x^{n+m+1}},$$

where

$$C = B \cdot \left[ \frac{(1)^{n+m}(n+m)}{x^{n+m+1}} \right]_{\infty}.$$  

Hence reducing the measure of the infinitely large space (15) with that of infinitely small space (17) we can conclude that $C$ given by (28.1) is a constant in the infinitely small space when $B$ represents the sum in the braces of (24), i.e., when $m$ assumes any value given by (24.1) and $C$ is a constant in the usual space when $B$ represents the sum in the braces of (25).

(16) The measure of the infinitely large space is $\pi/\sin \pi$, and we write $(\pi/\sin \pi) f(x) = \lceil f(x) \rceil$. See for details Proc. Physico-math. Soc. Japan 19 (1937), p. 906.
(17) The measure of the infinitely small space is $(\sin \pi)/\pi$, and we write $(\sin \pi /\pi) f(x) = \lfloor f(x) \rfloor$. See for details Proc. Physico-math. Soc. Japan 19 (1937), p. 907.
When \( C \) is a constant in the usual space the differential equation (28) which the highest exponent of the series satisfies clearly differs from (8), and consequently in this case we can not adopt (11) as the operational solution of (8).

On the other hand if \( C \) is a constant in the infinitely small space, the right-hand member of (28) is equal to zero\(^{(18)}\) except for the pole at \( x=0 \), and consequently in this case we can adopt (11) as the operational solution of (8).

Owing to the discussions already related \( C \) becomes a constant in the infinitely small space when

\[
m = -1, -2, \ldots, -(n-k). \tag{24.1}
\]

Hence the series (11) can be the operational solution of (8) if we give \( m \) any value given by (24.1).

Since there are \((n-k)\) values of \( m \) given by (24.1), there are \((n-k)\) kinds of the series (11) corresponding to these values of \( m \).

If we add \( k \) kinds of the operational solution of (8) to \((n-k)\) kinds of the operational solutions just related we get the \( n \) kinds of operational solutions of (8), and this fact is in accord with the fact that there are \( n \) fundamental solutions of the equation (8).

The operational solution expressed in the form (11) is the descending powers of \( p \), and the solution of the equation (8) which is got by operating with this series on zero is a series of ascending powers of \( x \). Proceeding exactly in the same way as in the case of the solution of the form (11) the operational solution expressed in the series of ascending powers of \( p \) is obtained, and if this operates on zero, this yields the solution of (8) represented in the series of descending powers of \( x \).

The unknown coefficients \( c_r \)'s is determined by the comparison of the coefficients of the same powers of \( p \) of the series which is obtained by substituting (11) in (10). But since the method of the determination of \( c_r \) differs little from the familiar method\(^{(19)}\) which we usually use to obtain the unknown coefficients of the power series of \( x \) satisfying the ordinary linear differential equation, I will not refer specially to the subject. But the special point by which the method of determining \( c_r \) of the series (11) distinguishes the familiar


\(^{(19)}\) See, for example, Forsyth, A Treatise on Differential Equation, pp. 148-159.
one will be discussed as a part of the examples of the general theory
which we shall see in the next section.

3. The Examples of the Operational Solutions.

Let us take the differential equation

\[
x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\} \frac{dy}{dx} - \alpha \beta y = 0. \tag{29}
\]

We shall now find the operational solution of (29) in the series
of the form (11).

We first deduce from (29) the equation of the following form

\[
x(1-x)p^2u + \{\gamma - (\alpha + \beta + 1)x\} pu - \alpha \beta u = 0. \tag{29.1}
\]

If we substitute (12) in (29.1), we get

\[
\{\gamma - (m+2)\} p^{m+1} + \{(m+1)(\alpha + \beta - m - 1) - \alpha \beta\} p^m
\]

with the aid of (4.1). Hence the indicial equation is

\[
-(m+2) + \gamma = 0.
\]

Thus

\[
m = \gamma - 2. \tag{30}
\]

If we put (11) into (29.1), and eliminate \(x\) using the formula
(4.1) and arrange the power series of \(p\) thus obtained in the des-
cending powers of \(p\), we have

\[
\sum_{r=0}^{\infty} \left[ c_{r+1}(m-r+1-\gamma) + c_r(m-r+1-\alpha)(m-r+1-\beta) \right] p^{m-r},
\tag{31}
\]

where we regard \(c_{-1}=0\).

Hence \(c_0\) is arbitrary, and the relation between successive \(c_r\)'s is

\[
c_{r+1}(m-r+1-\gamma) + c_r(m-r+1-\alpha)(m-r+1-\beta) = 0, \tag{32}
\]

so that

\[
c_{r+1} = \frac{(\alpha + \gamma - m - 1)(\beta + \gamma - m - 1)}{\gamma + \gamma - m - 1} c_r. \tag{33}
\]

Hence the series (11) becomes by (30) and (33)

\[
\sum_{r=0}^{\infty} \frac{(\alpha - \gamma)(\alpha + 1 - \gamma) \ldots (\alpha + r - \gamma)(\beta - \gamma)(\beta + 1 - \gamma) \ldots (\beta + r - \gamma)}{r!} c_r \frac{1}{p^{\gamma + r - \gamma}}
\tag{34}
\]

This is the operational solution of (29), and the result of operating
(34) on zero is
which is a solution of (29).

Another operational solution of (29) happens when the highest exponent of the series (11) is equal to a negative integer.

If we substitute

\[ \frac{1}{p} \]  

in (29.1), we get

\[ (\gamma - 1)p^r - \alpha \beta \frac{1}{p}. \]  

Since the coefficient of the highest exponent of (37) is a constant in the usual space, (36) represent the first term of the operational solution of (29) owing to the argument concerning \( C \) in (28) already related.

The relation between successive \( c_r \)'s is again given by (32).

Hence

\[ \sum_{r=0}^{\infty} \frac{\alpha (+1) \ldots (\alpha + r - 1) \beta (\beta + 1) \ldots (\beta + r - 1)}{\gamma (\gamma + 1) \ldots (\gamma + r - 1)} \frac{1}{p^r+1} . \]  

is another operational solution of (29). The result of operating (38) on zero is

\[ F(\alpha, \beta, \gamma, x). \]  

Let us now try to get the operational solution of (29) of the following form

\[ u(p) = \sum_{r=0}^{\infty} c_r p^{n+r}. \]  

We substitute \( p^m \) in (29.1) and get

\[ \{ \gamma - (m + 2) \} p^{m+1} + \{ (\alpha - m - 1) (\beta - m - 1) \} p^m. \]  

Since (40) is the series in ascending powers of \( p, p^m \) in (41) is the lowest exponent when we substitute (40) in (29.1) and it occurs in only a single term. Therefore the coefficient of the term must disappear, thus

\[ (\alpha - m - 1) (\beta - m - 1) = 0 \]  

\[ F(\alpha, \beta, \gamma, x) \] is the hypergeometric series and defined by

\[ F(\alpha, \beta, \gamma, x) = \sum_{r=0}^{\infty} \frac{\alpha (+1) \ldots (\alpha + r - 1) \beta (\beta + 1) \ldots (\beta + r - 1)}{r! \gamma (\gamma + 1) \ldots (\gamma + r - 1)} x^r. \]
is the indicial equation, and the roots

\[ m = \alpha - 1, \quad m = \beta - 1 \]  

(43)
of the equation (42) is the values of the lowest exponent of the operational solution of the form (40).

The result of substituting (40) in (29.1) is

\[ \sum_{r=0}^{\infty} [c_{r-1}(m+r+1-\gamma) + c_r(m+r+1-\alpha)m+r+1-\beta]p^{m+r}] = 0. \]

Therefore

\[ c_r = \frac{-(m+r+1-\gamma)}{(m+r+1-\alpha)(m+r+1-\beta)}c_{r-1}. \]  

(44)

Hence by (43) and (44) we get two operational solutions of (29) in the form

\[ \sum_{r=0}^{\infty} \frac{(\alpha - \gamma)(\alpha+1-\gamma)\ldots(\alpha+r-\gamma)}{r!(\alpha-\beta)(\alpha+1-\beta)\ldots(\alpha+r-\beta)}(-1)^r p^{\alpha+r}, \]

\[ \sum_{r=0}^{\infty} \frac{(\beta - \gamma)(\beta+1-\gamma)\ldots(\beta+r-\gamma)}{r!(\beta-\alpha)(\beta+1-\alpha)\ldots(\beta+r-\alpha)}(-1)^r p^{\beta+r}. \]

These yield the solutions of (29) when these operate on zero.

We can prove

\[ x^{\mu}[p-a]^m = \frac{II(\mu-m-1)}{II(-m-1)} [(p-a)^{\mu-m} \cdot 0], \]  

(45)

and we write (45) for brevity

\[ x^{\mu}[p-a]^m = \frac{II(\mu-m-1)}{II(-m-1)} (p-a)^{\mu-m}. \]  

(46)

When \( \mu \) becomes a positive integer, this may be written

\[ x^n[p-a]^m = \left(-\frac{d}{dp}\right)^nx^{\mu-m}. \]  

(46.1)

(46) and (46.1) are the general forms of (4) and (4.1) respectively.

Now we shall make use of (46) and (46.1) and obtain the op- erational solutions expressed in the power series of \( p-a \).

Let us consider the equation of Laguerre functions

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(21) This relation is proved by eliminating \( p \) from the both members of (45) with the aid of the operational equation

\[ \frac{1}{(p-a)^{\mu+1}} \cdot 0 = \frac{x^\mu p^{\mu}}{II(\nu)}. \]
\[ \frac{d^2 y}{dx^2} + (x + 1) \frac{dy}{dx} + (k + 1) y = 0, \]  
which we write using the relation (6) in the form
\[ x(p+1)^m u + (1-x)(p+1)u + ku = 0. \]  

Now we shall suppose that the operational solution of (47) is of the form
\[ u(p) = \sum_{r=0}^{\infty} c_r (p+1)^{m-r}. \]

Substituting \((p+1)^m\) in (47.1) and putting the coefficient of the lowest exponent of the result thus obtained equal to zero, we get the indicial equation
\[ -(m+2) + 1 = 0, \]
from which we get
\[ m = -1. \]

Thus (48) becomes
\[ u(p) = \sum_{r=0}^{\infty} c_r (p+1)^{-1-r}. \]

If we substitute (50) in (47.1), calculate with the aid of (46.1) and arrange in the descending power of \((p+1)\), we get
\[ \sum_{r=0}^{\infty} \left[ r c_r + (k+1-r)c_{r-1} \right] (p+1)^{-r}. \]

Hence the relation between the successive \(c_r\)'s is
\[ c_r = \frac{k+1-r}{r} c_{r-1} \]
and \(c_0\) is arbitrary, so that
\[ c_r = (-1)^r \frac{c_0 \Gamma(k+1)}{\Gamma(r+1) \Gamma(k+1-r)}, \]
and therefore the series (50) becomes
\[ u(p) = \sum_{r=0}^{\infty} (-1)^r \frac{c_0 \Gamma(k+1)}{\Gamma(r+1) \Gamma(k+1-r)} \frac{1}{(p+1)^{r+1}}. \]

This is the operational solution of (47.1). If this is operated on zero, it results that
\[ y(x) = \sum_{r=0}^{\infty} \left[ (-1)^r \frac{c_0 \Gamma(k+1)}{\Gamma(r+1) \Gamma(k+1-r)} x^r e^{-x} \right]. \]

The value of the highest exponent of the series (48) which is
obtained by the method discussed from (20) to (28.1) is also \(-1\). Therefore the value of \(m\) given by (49) is double roots, and in consequence the lowest exponent of \(x^r\) in (52) is also double roots. Hence

\[
\sum_{r=0}^{\infty} \left[ (-1)^r \frac{c_0 r^k + 1}{\Gamma(r + 1)/\Gamma(k + 1 - r)} x^r e^{-x}\log x 
+ \psi(k + 1 + r) - 2\psi(r + 1) \right].
\] (53)

which are got by differentiating with respect to \(r\) is another solution of (47)\(^{22}\). If we differentiate (51) with respect to \(r\), we have

\[
\sum_{r=0}^{\infty} \left[ (-1)^r \frac{c_0 r^k + 1}{\Gamma(r + 1)/\Gamma(k + 1 - r)} \frac{1}{(p + 1)^{r+1}} \log \frac{1}{p} 
+ \psi(k + 1 - r) - \psi(r + 1) \right].
\] (54)

This is another operational solution of (47), because (54) operates on zero it gives (53).

We now take the differential equation

\[
(x-x^r)\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0.
\] (55)

To get the operational solution of (55), we write

\[
u(p) = \sum_{r=0}^{\infty} c_r p^{m-r},
\] (56)

where we find as usual \(m=2\), and the relation between the successive \(c_r\)'s is

\[
\frac{c_{r+1}}{c_r} = \frac{(r-2)(r-5)}{r+1}
\] (57)

and \(c_0\) is arbitrary.

If we put

\[
c_0 = [1]_x,
\]

the term of (56) is

\[
[p^x]_x.
\] (58.1)

If we put \(r=0\), (57) becomes

\[
c_1 = 10 c_0,
\] (57.1)

thus the second term of (56) is by (57.1) and (58.1)

If we put $r=1$, (57) becomes
$$c_2 = 2c,$$

thus the term of (57) is by (57.2) and (58.2)

$$[20 \rho]'_{∞}.$$  

But if we put $r=2$, (57) becomes
$$c_3 = 0c_2$$
on account of the factor $(r-2)$.

If we consider $c_3$ from the result (57.3), the value of $c_3$ seems to be zero. But if we reflect on the fact that the value of $c_3$ is infinitely large$^{(23)}$, we can not conclude so hastily.

Hence we shall determine the value of $c_3$ by examining the behavior of the right-hand member of (57.3) when $r$ approaches 2.

Since the expression
$$\left[(-1)^n \rho^n\right]_{∞} (n$ being a positive integer or zero)  

is the value of
$$\frac{\pi}{\sin (\nu+1)\pi} \rho^n,$$

when $\nu$ tends to 0, and $\nu$ in (60) and $r$ in (56) are connected by the relation
$$\nu = 2 - r$$

and in consequence $\nu \to 0$ as $r \to 2$, we can regard $c_1$ as
$$c_1 = 20 \lim_{\nu \to 0} \frac{\pi}{\sin (\nu+1)\pi},$$

and therefore
$$\lim_{r \to 2} (r-2)c_3 = \lim_{r \to 2} \frac{20(r-2)\pi}{\sin (\nu+1)\pi}
= \lim_{r \to 2} \frac{20 \pi}{-\pi \cos (\nu+1)\pi} = 20.$$  

Hence (57.3) is in reality

$^{(23)}$ This comes from the fact that $c_2$ is a constant in the infinitely large space. Cf. foot note (16) in this paper.


$^{(25)}$ Since $\rho^n$ in (60) stands for any exponent in the series (56), we have $\rho^n = \rho^{m-\nu}$. But since we have seen $m = 2$, (61) follows.
We can determine $c_4, c_5, \ldots$ by (57) and (63), and since $c_3$ is a constant in the usual space, $c_4, c_5, \ldots$ are also the constants in the usual space.

If we collect the results from (57) so far obtained, we get the operational solution of (55) in the form

$$u(p) = [p^2 + 10p + 20p^2] - \frac{20}{p} + \frac{10}{p^2} - \frac{4}{p^3}. \quad (64)$$

If this operates on zero, it results that

$$y(x) = \frac{2}{x^3} - \frac{10}{x^2} + \frac{20}{x} - 20 + 10x - 2x^2,$$

which is the solution of the differential equation (55).

To solve the operational solution $v(p)$ of the difference equation, we assume

$$v(p) = \sum_{r=0}^{\infty} c_r p^{m+r}. \quad (65)$$

The method of obtaining the lowest exponent $m$ and the unknown coefficients $c_r$'s are exactly the same as in the case of differential equation. (5) is the most useful relation in this calculation. Hence we shall not go deep into the subject and content ourselves with the following example.

Take

$$(x + \omega) \Delta z + z = 0. \quad (66)$$

This equation is written by (7) as follows

$$(x + \omega) \{Dv(p) \cdot 0\} + \{v(p) \cdot 0\} = 0,$$

which we write for brevity

$$(x + \omega) Dv(p) + v(p) = 0. \quad (67)$$

If we substitute the lowest exponent of (65) in the left-hand expression of (67), we have by (5)

$$\{1 - (m+1)\} p^m + \{a - \omega(m+2)\} p^{m+1}.$$

If we put the coefficient of the lowest exponent of the above result equal to zero, we have

$$\Delta z = \frac{1}{\omega} \{z(x + \omega) - z(x)\}. \quad (29)$$
which is the indicial equation of (67). Since the root of this equation is zero, (65) becomes

$$v(p) = \sum_{r=0}^{\infty} c_r p^r. \quad (68)$$

If we substitute (68) in (67), calculate by the relation (5) and arrange the result thus obtained in the ascending powers of $p$, we have

$$\sum_{r=0}^{\infty} \left\{ - (r+1) c_{r+1} - \omega a (r+2) c_r \right\} p^{r+1} = 0.$$

Hence the relation between successive $c_r$'s is

$$c_{r+1} = - \frac{\omega a (r+2)}{r+1} c_r,$$

and $c_0$ is arbitrary.

If we put

$$c_0 = [1]_\infty,$$

(68) becomes

$$v(p) = \left[ \sum_{r=0}^{\infty} (-1)^r \omega^r a^r (r+1) p^r \right]_\infty, \quad (69)$$

which is the operational solution of (66). If this operates on zero, this yields\(^{27}\)

$$z(x) = \sum_{r=0}^{\infty} \frac{\omega^r a^r (r+1)}{(x+\omega)(x+2\omega) \ldots (x+r+1\omega)},$$

which is the solution of the difference equation (66).

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