On the Circle Circumscribed about and the Circle Inscribed in an Oval, and the Sphere Circumscribed about and the Sphere Inscribed in an Ovaloid,

by

Takashi Minoda, Sendai.

The purpose of this paper is to give a relation between the "Durchmesser" and the "Dicke" of an oval and the radii of its circumscribed circle and its inscribed circle, and to derive certain theorems on the curve of constant breadth as special cases, and to prove similar results concerning the ovaloid and the surface of constant breadth parallel to them.

Theorem 1. The sum of the radii of the circle circumscribed about and the circle inscribed in the oval lies between the "Durchmesser" and the "Dicke" of the oval.

Cor. If the sum of the radii of the circle circumscribed about and the circle inscribed in the oval is equal to the "Durchmesser" or the "Dicke", these two circles are concentric.

Lemma 1. If a triangle is circumscribed about the oval whose "Durchmesser" and "Dicke" are respectively not equal to the diameters of its circumscribed circle and its inscribed circle and the triangle includes the oval entirely, the maximum and the minimum of the circle inscribed in the triangle are respectively the circumscribed circle and the inscribed circle of this oval (implying the position, too).

(In proving the Theorem 1 it suffices to consider only the case where the assumption of this lemma is satisfied.)

For the proof of Theorem 1 it suffices to apply the Lemma 1 to the triangle composed of three supporting lines of the oval intersecting at right angles to the productions, to the directions of the centre, of the radii of the circumscribed circle through the three points of contact which do not lie on the semi-circumference of the circumscribed circle, among the points of contact with the oval and the
circumscribed circle, | inscribed circle, and to pay attention to the fact that the
maximum | minimum
of the distances from the centre of the
circumscribed circle | inscribed circle
to the three sides of that triangle is not
smaller | greater
than the radii of the
inscribed circle. | circumscribed circle.

Moreover if we prove by interchanging the situation of the
circumscribed circle and the inscribed circle, for example, it suffices
to prove in the following Lemma 2 by taking $P$ as the centre of the
inscribed circle $I$. | circumscribed circle $C$.

**Lemma 2.** Let the "Durchmesser", the "Dicke" and the
radii of the circumscribed circle $C$ and the inscribed circle $I$ of an
oval $U$ be respectively $D$, $\Delta$, $R$ and $r$, and let $P$ and $P'$ be the points
which give respectively the maximum and the minimum of the
distances from a point $P$ within $U$ to the points on the perimeter
of $U$, then $D \geq PP' + PP' \geq \Delta$. $PP'$ and $PP'$, when $P$ coincides re-
spectively with the centres of $C$ and $I$—these centres are within $U$—,
it takes the maximum value $R$ and the minimum value $r(1)$.

The corollary follows from the fact that the "Durchmesser" or the "Dicke" is a double normal of the oval.

If we put $D = \Delta$ in Theorem 1 and the corollary, we obtain the
following Theorem 2.

**Theorem 2.** The circle circumscribed about and the circle
inscribed in the curve of constant breadth are concentric, and the
sum of their radii is equal to the breadth of the curve(2).

The former proposition is equivalent to the latter.

**Cor.** The two polygons formed by joining the points, at which
the circumscribed circle and the inscribed circle of the curve of con-
stant breadth touch respectively the curve, are similar and similarly
situated with the internal centre of similitude and the absolute ratio
of similarity is equal to that of the radii of the circumscribed circle

(1) "$D \geq PP' + PP' \geq \Delta$, $PP' \leq R$, $PP' \leq r$" has been inserted as a problem at the end
of "Modern Geometry" by Prof. S. Morimoto.

(2) In January, this year (1987), Mr. J. Hirakawa remarked me that "$D = R + r$" had been proved in T. Bonnesen und W. Fenchel, *Theorie der konvexen
Körper*, p. 127 (1935). Here I express my gratitude for his kind remark.
and the inscribed circle, and the centre of similarity coincides with
the common centre of the circumscribed circle and the inscribed
circle.

The above results, also, hold good for the circumscribed sphere
and the inscribed sphere of an ovaloid and a surface of constant
breadth. Namely,

Theorem 3. The sum of the radii of the sphere circumscribed
about and the sphere inscribed in an ovaloid lies between the
"Durchmesser" and the "Dicke" of the ovaloid.

Cor. If the sum of the radii of the sphere circumscribed about
and the sphere inscribed in an ovaloid is equal to the "Durchmesser"
or the "Dicke", these two spheres are concentric.

Theorem 4. The sphere circumscribed about and the sphere
inscribed in a surface of constant breadth are concentric, and the
sum of their radii is equal to the breadth of the surface (2).

The former proposition is equivalent to the latter.

Cor. The two polyhedrons formed by joining the points, at
which the circumscribed sphere and the inscribed sphere of a surface
of constant breadth touch respectively the surface, are similar and
similarly situated with the internal centre of similitude and the
absolute ratio of similarity is equal to that of the radii of the
circumscribed sphere and the inscribed sphere, and the centre of
similarity coincides with the common centre of these two spheres.

For the proof of these theorems, it suffices to use the same
lemma—in Lemma 1, taking the triangular prism or the tetrahedron
formed by the supporting planes—corresponding to Lemma 1 or 2
in the case of the plane and to use the fact that the "Durchmesser"
or the "Dicke" is a double normale.

By Theorem 2, we can consider that the independent ones of
the six elements of the oval are only three, namely the area and the
radii of the circumscribed circle and the inscribed circle when the
oval is the curve of constant breadth.

Now, let the "Durchmesser", the perimeter, the area, the radius
of the circumscribed circle and that of the inscribed circle of the
curve $U$ of constant breadth be respectively $D$, $L$, $F$, $R$ and $r$, then
it is known that the inequalities

\begin{align*}
LR - F - \pi R^2 &\geq 0, \\
Lr - F - \pi r^2 &\geq 0, \\
L^2 - 4\pi F &\geq \pi^2 (R - r)^2, \\
L^2 - 4\pi F &\geq 4\pi (R - r)^2
\end{align*}
hold between $L, F, R$ and $r$, where the equal sign occurs only when $U$ is a circle.

Among these inequalities, taking any one of the former three inequalities (1), (2) and substituting $L = \pi D = \pi (R + r)$ in $L$, we obtain $F \leq \pi R r$.

Therefore, we get the following theorem.

**Theorem 5.** The area of a curve of constant breadth is not greater than the proportional mean of the areas of the circumscribed circle and the inscribed circle.

From this proof, in the curves of constant breadth, it is seen that any two of the three inequalities (1), (2) are equivalent to each other.

The three sides of the triangle, formed by joining the three points which do not lie on the semi-circumference of $C$ among the points of contact with $U$ and circumscribed circle $C$, intersect the inscribed circle $I$ of $U$. From this, it is seen that the area of $U$ is not smaller than the area of the oval composed of $I$ and the tangents from the two end points of the diameter of $C$ to $I$. Therefore, combining this with Theorem 5 we have:

5.1 *The area of a curve of constant breadth is not greater than the area of the ellipse whose major and minor axes are respectively the diameters of the circumscribed circle and the inscribed circle of the curve, and is not smaller than the area of the oval composed of the circular arcs of the circle whose diameter is the minor axis of this ellipse and the two tangents to this circle from the two end points of the major axis.*

5.2 *The difference between the area of the in-revolvable curve of the square from the area of the rectangle whose two adjacent sides are the diameters of the circumscribed circle and the inscribed circle of a curve is not smaller than the remaining area of the square whose inscribed circle is taken out.*

For this, it suffices to use the last inequality (3) of Bonnesen as above.

As (1) holds always, for the general oval, adding together the corresponding sides of these two inequalities and paying attention to $\pi D \geq L$, $D \geq R + r$, the following 5.3 is obtained.

5.3 *The area of an oval is not greater than the difference between the arithmetic mean of the areas of the circle circumscribed*
about and the circle inscribed in it from the areas of the two circles whose diameters are equal to the "Durchmesser" of the oval.

5.4 For the parallel ovals, both of the left sides of the two inequalities (1) are invariant. (It is already known that the left side of (2) is also invariant for the parallel ovals.)

For the parallel curves of constant breadth $F-\pi Rr$ is invariant.

This is easily proved by using the following Theorem 6 by calculation as well as geometrically.

**Theorem 6.** The circles circumscribed about and the circles inscribed in parallel ovals are all respectively concentric, and the change of the radii is equal to the parallel distance of the corresponding ovals.

This is obvious at a glance, but for the strict proof of this the following lemma is necessary, and if we use this Lemma 8 the proof is very simple.

**Lemma 3.** If a circle including (or is included in) an oval touches the oval at the two end-points of a diameter or at the three points at least which do not lie on the semi-circumference, it is the circle circumscribed about (or the circle inscribed in) the oval."

This theorem holds in the case of the parallel ovaloids, too, which can be seen in the same manner.

Rewritten in December, 1937.

(Received Nov. 28, 1938.)