On Dominated Ergodic Theorems in $L_p \, (p \geq 1)$

by

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1. This note contains two theorems, which will be the complements to the dominated ergodic theorem in $L_1$, discovered by N. Wiener. The essential part of the proofs is the well-known maximal theorem of Hardy-Littlewood, somewhat differing from the direct proof of Wiener. Hardy-Littlewood’s theorem is this:

Let $a_1, a_2, \ldots, a_n$ be any sequence of non-negative numbers, given except arrangement and $s(x)$ be any monotone non-decreasing function of $x (x \geq 0)$; and let us denote

$$\alpha_j = \max \left( \frac{a_j + a_{j-1}}{2}, \ldots, \frac{a_j + a_{j-1} + \ldots + a_1}{j} \right),$$

$$j = 1, 2, \ldots, n.$$

Then

$$\sum_{j=1}^{n} s(\alpha_j)$$

is a maximum when the sequence $a_1, a_2, \ldots, a_n$ is arranged in descending order.

We will prove:

Theorem I. Let $S$ be a set of finite measure, and $T$ a measurable, measure-preserving transformation of $S$ into itself. If $f(P) \in L_p(S), p > 1$, then the dominant

$$f^*(P) = \text{l.u.b.} \sum_{0 \leq k < \infty} \frac{f(P) + \ldots + f(T^k P)}{k+1}$$
belongs to $L_p$, and $\int |f^*(P)|^p dV_r \leq \left( \frac{p}{p-1} \right)^p \int |f(P)|^p dV_r$.

**Theorem II.** Under the same condition as in Theorem I, if $f(P) \in L_1(S)$, then a dominant $f^*(P)$ belonging to $L_{1-\varepsilon}$ ($\varepsilon > 0$) exists, and $\int |f^*(P)|^{1-\varepsilon} dV_r \leq A \int |f(P)| dV_r + mS$.

The author expresses his hearty thanks to Mr. T. Kawata, whose suggestion of Theorem II, from the analogy of one of his researches (5), has been the motive of this note.

2. Proof of Theorem I. It is sufficient to prove for the case $f(P) \geq 0$. By $W_n$ we denote the set of points $P$ such that $n-1 \leq f(P) < n$ and define

$$f'(P) = n \text{ for } P \in W_n, \quad n = 1, 2, 3, \ldots .$$

We will prove the theorem for $f'(P)$.

Let us put $W_{n_1 \ldots n_k} = W_{n_1} \cdot T^{-1} W_{n_2} \cdot T^{-2} W_{n_3} \ldots \cdot T^{-k} W_{n_k}$, and

$$F_j(P) = \max \left( f'(P), \frac{f'(P) + f'(TP)}{2}, \ldots, \frac{f'(P) + \ldots + f'(T^{j-1}P)}{j} \right) .$$

If $P \in T^{-j} W_{n_1 \ldots n_k}$, $1 \leq j \leq k$, then

$$F_j(P) \leq \max \left( n_j, \frac{n_j + n_{j-1} + \ldots + n_1}{j} \right),$$

and by Hardy-Littlewood's maximal theorem,

$$\frac{1}{k} \sum_{j=0}^{k-1} \int_{T^{-j} W_{n_1 \ldots n_k}} |F_j(P)|^p dV_r \leq \frac{m W_{n_1 \ldots n_k}}{k} \sum_{j=1}^{k} \left( n_j^* + n_{j-1}^* + \ldots + n_1^* \right)^p,$$

where $n_1^*, n_2^*, n_3^*, \ldots, n_k^*$ is the descending rearrangement of $n_1, n_2, \ldots, n_k$. It follows, by the H"{o}lder inequality,

$$\frac{m W_{n_1 \ldots n_k}}{k} \sum_{j=1}^{k} \left( n_j^* + \ldots + n_1^* \right)^p \leq \left( \frac{p}{p-1} \right)^p \frac{m W_{n_1 \ldots n_k}}{k} \sum_{j=1}^{k} n_j^{* p}$$

$$= \left( \frac{p}{p-1} \right)^p \frac{1}{k} \sum_{j=1}^{k} n_j^* \cdot m T^{-j} W_{n_1} \ldots n_k^*.$$ 

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During the correction, the author found a paper of Wiener, The Ergodic Theorem, The Duke Math. Journal, 5-1 (1939), which has just arrived. Theorems in this note were already proved, quite elegantly, in his paper.
Summing up both sides for all $W_{n_1...n_k}$ respectively, we have

$$\int_{S} \frac{1}{k} \sum_{j=0}^{k-1} (F_j(P))^p dV_P \leq \frac{1}{k} \sum_{j=0}^{k-1} \int_{S} (F_j(P))^p dV_P \leq \left(\frac{p}{p-1}\right)^p \int_{S} (f'(P))^p dV_P.$$ 

Since $F_j(P)$ is monotone non-decreasing, it follows that

$$\frac{1}{k} \sum_{j=0}^{k-1} (F_j(P))^p \leq F^*(P), \text{ for } k=1, 2, \ldots,$$

$$|F_j(P)|^p \leq F^*_{j}(P), \text{ for } j=0, 1, 2, \ldots,$$

and

$$\int_{S} \frac{F^*(P)}{F^*_{j}(P)} dV_P \leq \left(\frac{p}{p-1}\right)^p \int_{S} |f'(P)|^p dV_P.$$ 

Thus the theorem is proved.

As a direct consequence of this theorem we get that von Neumann's mean ergodic theorem is derived from Birkhoff's ergodic theorem. This is one of the remark of N. Wiener (5), their details are not yet published. In fact, if $f \in L_2$, then $f \in L_1$, and Birkhoff's theorem asserts that

$$\varphi_n(P) = \frac{1}{n} \sum_{j=0}^{n-1} f(T^jP) \rightarrow f^*(P) \text{ almost everywhere.}$$

Since, by the above theorem, $\{\varphi_n\}$ is dominated by a function of $L_2$, it is converges weakly. By means of the relation

$$\int_{S} \left(\frac{1}{n} \sum_{j=0}^{n-1} f(T^jP) \right)^{1/2} dV_P \leq \frac{1}{n} \sum_{j=0}^{n-1} \left(\int_{S} |f(T^jP)|^2 dV_P\right)^{1/2} = \left(\int_{S} |f(P)|^2 dV_P\right)^{1/2},$$

$\varphi_n(P)$ converges strongly, q.e.d.

3. Proof of Theorem II. $f \geq 0$, $f'$ and $W_{n_1...n_k}$ are the same as in § 2. We put $V_{(N)} = \sum_{n_1+\ldots+n_k = N} W_{n_1...n_k}$, $V^{(N)} = \sum_{N} V_{(N)}$. Then

$$\int_{S} f'(P) dV_P \geq \int_{S} f'(P) + \ldots + f'(T^{k-1}P) dV_P \int_{S} \sum_{n_1+\ldots+n_k = N} \frac{n_1+\ldots+n_k}{k} dV_P$$

(*) Wiener, loc. cit., p. 907.
therefore $mV^{(n)} \leq \frac{k}{N} \int_{S} f' dV_{p}$. In the same way as in §2, we have

$$\int_{S} \frac{1}{k} \sum_{0}^{k-1} (F_{j}(P))^{1-\varepsilon} dV_{p} \leq mS + \frac{1}{k} \sum_{N-k+1}^{N} \sum_{n_{k}+n_{j}+\ldots+n_{k}+n_{j}+\ldots+n_{k}} \left( \sum_{j=1}^{k} |a_{j}|^{1-\varepsilon} \right) mW_{n_{1}n_{k}},$$

where

$$a_{j} = \max \left( \frac{n_{j} + n_{j-1}}{2}, \ldots, \frac{n_{j} + n_{j-1} + \ldots + n_{j}}{j} \right), \quad j = 1, \ldots, k.$$  

Now, by Hardy-Littlewood's theorem (loc. cit.),

$$\max_{n_{1}+\ldots+n_{k}=N} \sum_{j=1}^{k} (a_{j})^{1-\varepsilon} = (N-k+1)^{1-\varepsilon} + \left( \frac{N-k+2}{2} \right)^{1-\varepsilon} + \ldots + \left( \frac{N}{k} \right)^{1-\varepsilon} = A_{N},$$

and therefore

$$\int_{S} \frac{1}{k} \sum_{0}^{k-1} (F_{j}(P))^{1-\varepsilon} dV_{p} \leq mS + \frac{1}{k} \sum_{N-k}^{N} A_{N} mV^{(N)},$$

by means of Abel's transformations,

$$= mS + \frac{1}{k} \left\{ \sum_{N-k}^{N} (A_{N+1} - A_{N}) mV^{(N+1)} + A_{N} mV^{(N)} \right\} = I, \quad \text{say.}$$

Since

$$A_{k} = 1^{1-\varepsilon} + 1^{1-\varepsilon} + \ldots + 1^{1-\varepsilon} = k,$$

$$A_{N+1} - A_{N} = (1-\varepsilon) \sum_{i=1}^{k} \frac{1}{(N-k+i)^{1-\varepsilon}} = \frac{(1-\varepsilon)k^{\varepsilon}}{(N-k)^{\varepsilon}},$$

we have

$$I \leq mS + \frac{1-\varepsilon}{k} \sum_{i=1}^{k} \frac{k^{\varepsilon}}{(N-k)^{\varepsilon}} \frac{k}{N} \int_{S} f'(P) dV_{p} + \int_{S} f'(P) dV_{p}$$

$$\leq mS + (1 + \text{const.} \sum_{i=1}^{k} \frac{1}{k^{1-\varepsilon}} + \text{const.} \sum_{j=1}^{\infty} \frac{k^{\varepsilon}}{j^{1+\varepsilon}}) \int_{S} f'(P) dV_{p}$$

$$\leq mS + \text{const.} \int_{S} f'(P) dV_{p}.$$

Reasoning as in §2, we obtain the result.

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