Some Inequalities in the Theory of Linear Differential Equations,

by

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Notations. (i) $x(t)$ is a vector with $n$ components $x_i(t)$ ($i=1, 2, \ldots, n$), where $x_i(t)$ are real continuous functions in the interval $t_0 \leq t \leq t_1$.

(ii) $A(t)$ is a square matrix with $n^2$ components $a_{ik}(t)$, which are real continuous functions of $t$.

(iii) $|x(t)| = \left( \sum_{i=1}^{n} |x_i(t)|^p \right)^{\frac{1}{p}}$ ($p > 1$).

(iv) $|A(t)| = \max_{|x| > 0} \left( \frac{|A(t)x|}{|x|} \right)$, where $A(t)x$ is the product of the matrix $A(t)$ and the matrix $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, that is, the "norm" of the matrix $A(t)$.

Inequalities. The object of the present paper is to prove the following theorem.

Theorem. If $x(t)$ is a system of solutions of linear differential equations

$$\frac{dx_i(t)}{dt} = \sum_{k=1}^{n} a_{ik}(t)x_k(t) \quad (i=1, 2, \ldots, n) \quad (1)$$

for $t_0 \leq t \leq t_1$, then

$$e^{\int_{t_0}^{t} a(t) \, dt} \leq \frac{|x(t)|}{|x(t_0)|} \leq e^{-\int_{t_0}^{t} a(t) \, dt},$$

in which the equal signs occur when and only when $x(t) = e^{\pm \int_{t_0}^{t} a(t) \, dt} x(t_0)$.

Proof of the Theorem. First we shall prove the inequalities without any mention to cases of equalities, which are somewhat troublesome.

Lemma 1. If $f(t)$ is a continuous function of $t$, then

$$\prod_{i=1}^{m} (1 \pm f(t_0 + l\Delta t) \Delta t) \rightarrow e^{\int_{t_0}^{t} f(t) \, dt} \quad \text{as } m \rightarrow \infty,$$
where \[ \Delta t = \frac{t_1 - t_0}{m}. \]

**Proof.** Since for sufficiently small positive \( \varepsilon \)

\[
\log (1 + \varepsilon) = \varepsilon + O(\varepsilon^2),
\]

\[
\prod_{i=1}^{m} (1 \pm f(t_0 + l \Delta t) \Delta t) = e^{\sum_{i=1}^{m} \log (1 \pm f(t_0 + l \Delta t) \Delta t)}
\]

\[
= e^{\sum_{i=1}^{m} f(t_0 + l \Delta t) \Delta t + \sum_{l=0}^{m} O \{ (t_0 + l \Delta t) \Delta t \}^2}
\]

\[
\rightarrow e^{\int_{0}^{2\pi} f(t) dt}, \quad \text{as } m \to \infty.
\]

**Lemma 2.** If \( B_1, B_2, \ldots, B_s \) are matrices, such that

\[ |B_i| < 1 \quad (i = 1, 2, \ldots, s), \]

then

\[
\prod_{i=1}^{s} (1 + |B_i|) \leq \frac{|y|}{|x|} \leq \prod_{i=1}^{s} (1 - |B_i|),
\]

where

\[ y = \prod_{i=1}^{s} (E + B_i)x. \]

**Proof.** Put

\[ x^{(r-1)} = (E + B_2)x_1, \]

\[ x^{(r-2)} = (E + B_{r-1})x^{(r-1)}, \]

\[ \ldots \ldots \ldots \ldots \]

\[ y = (E + B_1)x^{(1)}. \]

Then

\[
\frac{|x^{(r-1)}|}{|x^{(r)}|} = \frac{|(E + B_r)x^{(r)}|}{|x^{(r)}|} = \frac{|x^{(r)} + B_rx^{(r)}|}{|x^{(r)}|} \leq \frac{|x^{(r)}| + |B_rx^{(r)}|}{|x^{(r)}|} = 1 + |B_r|,
\]

\[
\frac{|x^{(r-2)}|}{|x^{(r)}|} = \frac{|(E + B_{r-1})x^{(r)}|}{|x^{(r)}|} = \frac{|(E + B_{r-1})x^{(r)}|}{|x^{(r)}|} \leq \frac{|x^{(r)}| + |B_{r-1}x^{(r)}|}{|x^{(r)}|} = 1 + |B_{r-1}|,
\]

\[
(r = 1, 2, \ldots, s);
\]

and by multiplication, we got
Thus the lemma is proved.

If we construct an approximate solution of (1) by the so-called Cauchy's polygon method, dividing the interval \((t_0, t_1)\) in \(m\) equal parts, then we see that the end point \(x^{(m)}(t_1)\) of the curve is expressed as follows:

\[
x^{(m)}(t_1) = \prod_{i=1}^{m} (E + A(t_0 + \frac{t_1 - t_0}{m} - i\Delta t)\Delta t)x(t_0),
\]

\(\Delta t = \frac{t_1 - t_0}{m}\).

Since \(x^{(m)}(t_1)\) converges to \(x(t_1)\) uniformly, when \(m \to \infty\), we have by lemmas 1 and 2,

\[
\frac{|x(t_1)|}{|x(t_0)|} = \lim_{m \to \infty} \frac{|x^{(m)}(t_1)|}{|x(t_0)|} \leq \lim_{m \to \infty} \prod_{i=1}^{m} (1 + |A(t_0 + \frac{t_1 - t_0}{m} - i\Delta t)|\Delta t) = e^{\int_{t_0}^{t_1} |A(t)| dt}.
\]

Lemma 3.

\[
\lim_{m \to \infty} \prod_{i=1}^{m} |x(t_0 + i\Delta t) + \frac{d}{dt} x(t_0 + i\Delta t)\Delta t| = 1.
\]

Proof. Denoting by \(\dot{x}\) the derivative of \(x\),

\[
L = \log \frac{|x(t_0 + i\Delta t) + \dot{x}(t_0 + i\Delta t)\Delta t|}{|x(t_0 + \dot{x} + 1\Delta t)|} = \log \left(1 + \frac{|x(t_0 + i\Delta t) + \dot{x}(t_0 + i\Delta t)\Delta t|}{|x(t_0 + \dot{x} + 1\Delta t)|} - |x(t_0 + \dot{x} + 1\Delta t)| \right).
\]

Since \(|\log(1 + \varepsilon)| \leq C|\varepsilon|\) \((C > 0)\), when \(\varepsilon\) is under certain limit we have
\[ L \leq C \left( \frac{|x(t_0 + \xi + 1\Delta t) - x(t_0 + i\Delta t) - \dot{x}(t_0 + i\Delta t)\Delta t|}{|x(t_0 + i + 1\Delta t)|} \right) \]

Since \(|x(t)| \neq 0\), there exists \(k\) such that \(|x(t)| \geq k > 0\).

So

\[ L \leq \frac{C}{k} \left( |x(t_0 + i + 1\Delta t) - x(t_0 + i\Delta t) - \dot{x}(t_0 + i\Delta t)\Delta t| \right) \]

\[ \leq \frac{C}{k} \left| \int_{t_0 + i\Delta t}^{t_0 + i + 1\Delta t} \dot{x}(t) dt - \dot{x}(t_0 + i\Delta t)\Delta t \right| \]

\[ = \frac{C}{k} |\dot{x}(t_0 + i\Delta t + h\Delta t) - \dot{x}(t_0 + i\Delta t)| \quad (0 \leq h \leq 1) \]

\[ = \frac{C}{k} \Delta t |\dot{x}(t_0 + i\Delta t + h\Delta t) - \dot{x}(t_0 + i\Delta t)|. \]

On the other hand, from uniform continuity of \(\dot{x}(t)\),

\[ |\dot{x}(t_0 + i\Delta t + h\Delta t) - \dot{x}(t_0 + i\Delta t)| \to 0. \]

Therefore we have

\[ \sum \log \frac{|x(t_0 + i\Delta t) + \dot{x}(t_0 + i\Delta t)\Delta t|}{|x(t_0 + i + 1\Delta t)|} \leq \frac{C}{k} (\sum_{j=1}^{n} \Delta t) \max_{|\dot{x}(t) + \Delta t f|} |\dot{x}(t + \Delta t f) - x(t)| \to 0. \]

**Lemma 4.** Let \(y(t), z(t)\) be two vectors with different directions which are continuous functions of \(t\), and put

\[ \varphi(t, \tau) = |y(t)| + |z(t)| \tau - |y(t) + z(t)| \tau, \]

\[ \psi(t, \tau) = |y(t) + z(t)| \tau - |y(t)| + |z(t)| \tau. \]

Then

(i) \(\varphi(t, \tau)\) is a continuous concave function, and \(\psi(t, \tau)\) a continuous convex function of \(\tau\);

(ii) \(\varphi(t, \tau)\) and \(\psi(t, \tau)\) converge uniformly to continuous functions \(K(t)\) and \(L(t)\), as \(\tau \to 0\), \(K(t)\) and \(L(t)\) being positive.

**Proof.** (i) It suffices to prove the convexity of \(|y(t) + z(t)| \tau|\), because

\(\varphi = \) linear function \(-|y(t) + z(t)| \tau|\),

\(\psi = \) linear function \(+|y(t) + z(t)| \tau|\).

Now

\[ \frac{|y(t) + z(t)| \tau + |y(t) + z(t)| \tau'}{2} - \frac{|y(t) + z(t)| \tau + \tau'}{2} \]
by Minkowski's inequality. Thus the convexity is proved.

(ii) \( \lim_{\tau \to 0} \frac{\varphi(t, \tau)}{\tau} = \frac{\partial \varphi}{\partial \tau}(t, 0), \quad \lim_{\tau \to 0} \frac{\psi(t, \tau)}{\tau} = \frac{\partial \psi}{\partial \tau}(t, 0). \)

And from

\[
\frac{\partial \varphi}{\partial \tau}(t, \tau) = |z| - \frac{1}{p} \left( \sum_{i=1}^{n} |y_i + z_i \tau|^p \right)^{1-p-1} 
\times p \left( \sum_{i=1}^{n} z_i \text{sgn}(y_i + z_i \tau) |y_i + z_i \tau|^p \right),
\]

we get

\[
\frac{\partial \varphi}{\partial \tau}(t, 0) = |z| - \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p-1}} \left( \sum_{i=1}^{n} z_i \text{sgn}(y_i) |y_i|^p \right).
\]

So putting

\[
w_i = \frac{\text{sgn}(y_i) |y_i|^p}{|y|^p - 1}, \quad q = \frac{p}{p-1},
\]

\[
\frac{\partial \varphi}{\partial \ell}(t, 0) = |z| - \sum_{i=1}^{n} z_i w_i = |z| - \left( \sum_{i=1}^{n} w_i \right)^{\frac{1}{q}} - \sum_{i=1}^{n} z_i w_i,
\]

since

\[\sum_{i=1}^{n} |w_i|^q = 1.\]

If for at least one \( i \), \( \text{sgn}(z_i) \neq \text{sgn}(y_i) \), then

\[|z| - \sum_{i=1}^{n} z_i w_i > |z| - \sum_{i=1}^{n} |z_i| |w_i| \geq 0\]

by Hölder's inequality. If for all \( i \), \( \text{sgn}(z_i) = \text{sgn}(y_i) \), then \( |z_i|, |y_i| \) cannot have the same direction, so by Hölder's inequality we get

\[|z| - \sum_{i=1}^{n} |z_i| |w_i| > 0.\]

Hence the function \( K(t) \) is positive, and similarly for the function \( L(t) \). Since \( \varphi(t, \tau), \psi(t, \tau) \) are convex with respect to \( \tau \), \( \frac{\partial \varphi}{\partial \tau}(t, \tau), \frac{\partial \psi}{\partial \tau}(t, \tau) \) converge to \( K(t), L(t) \) monotonously, consequently by the theorem of Dini, they converge uniformly to \( K(t), L(t) \).

Now we consider the cases of equalities. We suppose that for some \( t, A(t) x(t) \) and \( x(t) \) are not proportional to each other. Then
by the lemma 4, there is an interval \((t_2 - \varepsilon, t_2 + \varepsilon)\) such that \(\dot{x}(t)\), \(A(t)\), \(x(t)\) are not proportional for any \(t\) in the interval and that

\[ K(t) = \frac{K(t_0)}{2}. \]

By lemma 4

\[
\frac{|x(t)| + |A(t)x(t)|}{|x(t)|} \leq \frac{|x(t) + A(t)x(t)|}{|x(t)|} + \left(\frac{K(t_0)}{2} - \delta\right) \leq \frac{|x(t) + A(t)x(t)|}{|x(t)|},
\]

\(\delta\) being independent of the sufficiently small \(\Delta t\),

\[
\left(\Delta t = \frac{2\varepsilon}{m}\right),
\]

so from

\[ 1 + |A(t)| \Delta t = \frac{|x(t)| (1 + |A(t)| \Delta t)}{|x(t)|} \leq \frac{|x(t) + A(t)x(t)|}{|x(t)|}, \]

we get

\[
\prod_{i=1}^{m} \left(1 + |(Ax_0 - \varepsilon) + m - i\Delta t| \right) \Delta t \leq \prod_{i=1}^{m} \left(1 + |x(t_0 - \varepsilon) + m - i\Delta t| \right) \Delta t \leq \prod_{i=1}^{m} \left(1 + |x(t_0 + i\Delta t)| + x(t_0 + i\Delta t)\Delta t| \right) \Delta t,
\]

whence by lemmas 1, 3 we get

\[
e^{-\int_{t_0}^{t_2} |A(t)| dt} \leq \frac{|x(t_2 + \varepsilon)|}{|x(t_2 - \varepsilon)|}.
\]

And also by lemma 2

\[
e^{-\int_{t_0}^{t_2} |A(t)| dt} \geq \frac{|x(t_2 - \varepsilon)|}{|x(t_0)|}, \quad e^{-\int_{t_0}^{t_2} |A(t)| dt} \geq \frac{|x(t_2)|}{|x(t_0)|}.
\]

Therefore it follows that

\[
e^{-\int_{t_0}^{t_2} |A(t)| dt} \geq \frac{|x(t_2)|}{|x(t_0)|}.
\]

Therefore, if the equal sign occurs, \(x(t)\) and \(A(t)\), \(x(t)\) must have the same direction for all \(t, (t_0 \leq t \leq t_2)\), that is,

\[
\lambda(t)x(t) = A(t)x(t) = \frac{d}{dt}x(t).
\]
Hence if we put
\[ \bar{x}(t) = e^{\int_{t_0}^{t} \lambda(s) \, ds} x(t_0), \]
then
\[ \bar{x}(t) = x(t) \]
by the uniqueness of the system of solutions of differential equations. And since \( \lambda(t) = |A(t)| \), we get the final conclusion
\[ x(t) = e^{-\int_{t_0}^{t} |A(s)| \, ds} x(t_0). \]
Similarly we get
\[ x(t) = e^{-\int_{t_0}^{t} |A(s)| \, ds} x(t_0), \]
if
\[ e^{\int_{t_0}^{t} |A(s)| \, ds} = \frac{|x(t_1)|}{|x(t_0)|}. \]

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