Theorems on Nörlund's Method of Summation, I,

by

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1. T. Kojima (1) proved the following theorem.

Theorem. A necessary and sufficient condition that \((R, q_n) \subseteq (R, p_n)\) (1) is that

\[
\frac{1}{P_n} \sum_{r=0}^{n-1} \left| \frac{p_r}{q_r} - \frac{p_{r+1}}{q_{r+1}} \right| q_r + \frac{p_n q_n}{q_n P_n} = O(1),
\]

where summation \((R, p_n)\) means Riesz's mean, i.e.,

\[
\sigma_n = \frac{(p_0 s_0 + p_1 s_1 + \ldots + p_n s_n)}{P_n}, \quad P_n = p_0 + p_1 + \ldots + p_n,
\]
tends to \(\sigma\) as \(n \to \infty\).

In sections 2 and 3 of this paper we prove the analogous theorem for Nörlund's method of summation, which contains the consistency theorem for Cesàro's method of summation.

Messers. Garabedian and Randels (3) have proved the following theorem.

Theorem. If the series \(\sum_{n=0}^{\infty} a_n\) is summable \((R, p_n)\) to the value \(s\), then

\[
s_n = \sum_{r=0}^{n} a_r = o\left(\frac{P_n}{P_n/s}\right) + s.
\]

In section 4 we prove the corresponding theorem for Nörlund's method of summation. Finally in section 5 Tauberian theorems are proved.

Let \(p_0 \neq 0, p_1, p_2, \ldots\) be a given sequence and set \(P_n = p_0 + p_1 + \ldots + p_n\). For a given series \(\sum_{n=0}^{\infty} a_n\), where \(s_n = \sum_{r=0}^{n} a_r\), we form

(1) \[
\sigma_n = \frac{(p_0 s_0 + p_1 s_1 + \ldots + p_n s_n)}{P_n}.
\]

(1) T. Kojima, This Journal, 12(1917). This theorem was recently rediscovered by Messers. H. L. Garabedian and W. C. Randels in the Duke Mathematical Journal, 4 (1938).

(2) \((R, q_n) \subseteq (R, p_n)\) means that the series summable \((R, q_n)\) is also summable \((R, p_n)\), and their sums are equal.

(3) Loc. cit. 1.)
If \( \sigma_n \to \sigma \) as \( n \to \infty \), we say that the series \( \sum_{n=0}^{\infty} a_n \) is summable by the Nörlund's method of summation with respect to the sequence \( \{p_n\} \) or simply summable \((N; p)\) to the value \( \sigma \).

2. If a given series \( \sum_{n=0}^{\infty} a_n \) is summable \((N; q)\), then

\[
\rho_n = \frac{q_0s_0 + q_1s_1 + \ldots + q_ns_n}{Q_n},
\]

where \( Q_n = q_0 + q_1 + \ldots + q_n \) tends to a limit as \( n \to \infty \). From (2) we have

\[
\begin{align*}
\rho_0 Q_0 &= q_0s_0, \\
\rho_1 Q_1 &= q_1s_0 + q_0s_1, \\
& \ldots \ldots\ldots\ldots \ldots \ldots, \\
\rho_n Q_n &= q_0s_0 + q_1s_1 + \ldots + q_ns_n.
\end{align*}
\]

Without loss of generality we can put \( q_0 = 1 \). If we solve this system of equations, we get

\[
\begin{array}{c|ccc}
& Q_0 & 1 & \\
Q_1 & q_1 & 1 & \\
Q_2 & q_2 & q_1 & 1 \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \ldots \ldots \ldots \ldots \ldots & \\
Q_{n-1} & q_{n-1} & q_{n-2} & \ldots & 1 \\
Q_n & q_n & q_{n-1} & \ldots & q_1
\end{array}
\]

i. e., we have

\[
(3) \quad s_n = Q_0\rho_0 \Delta_n + Q_1\rho_1 \Delta_{n-1} + \ldots + Q_n\rho_n \Delta_0,
\]

where

\[
\begin{array}{c|ccc}
& q_1 & 1 & \\
q_2 & q_1 & 1 & \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \ldots \ldots \ldots \ldots \ldots & \\
q_{n-1} & q_{n-1} & q_{n-2} & q_{n-3} & 1 \\
q_n & q_n & q_{n-1} & q_{n-2} & \ldots & q_1
\end{array}
\]

If we substitute (3) into (1), then we get

\[
(4) \quad \sigma_n = \frac{1}{P_n} \sum_{n=0}^{\infty} p_{n-\nu} (Q_0\rho_0 \Delta_n + Q_1\rho_1 \Delta_{n-1} + \ldots + Q_n\rho_n \Delta_0).
\]
Therefore, if the transformation matrix from the sequence \( \{ \rho_n \} \) to the sequence \( \{ \sigma_n \} \) defined by (4) is regular, then from \( (N; q) \) summability of the series \( \sum a_n \) we can conclude \( (N; p) \) summability of this series.

The transformation matrix from \( \{ \rho_n \} \) to \( \{ \sigma_n \} \) is

\[
[a_{nr}] = \begin{bmatrix}
\frac{1}{P_n} Q_r \rho_0 + p_n \Delta_1 + \ldots + p_{n-r} \Delta_r
\end{bmatrix},
\]

and we can write the general term of this matrix as follows:

\[
a_{nr} = (-1)^n \begin{bmatrix}
p_0 & 1 \\
p_1 & q_1 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
p_{n-r-1} & q_{n-r-1} & q_{n-r-2} & \ldots & 1 \\
p_n & q_n & q_{n-1} & \ldots & q_1
\end{bmatrix} \times \frac{Q_r}{P_n}
\]

By Kojima-Toeplitz's theorem the condition of the regularity\(^{(4)}\) of the matrix \( [a_{nr}] \) are

1° \( a_{nr} \to 0 \), as \( n \to \infty \)

and

2° \( \sum_{r=0}^{n} |a_{nr}| \leq M, \) for \( n = 0, 1, 2, \ldots \)

Thus we get the following theorem.

**Theorem 1.** The necessary and sufficient conditions that \( (N; q) \subseteq (N; p) \) are

1° \( D_{n-r}/P_n \to 0 \), as \( n \to \infty \)

and

2° \( \sum_{r=0}^{n} |D_{n-r}|/|Q_r|/|P_n| \leq M, \) for \( n = 0, 1, 2, \ldots \),

where

\(^{(4)}\) Notice that the definition of the regularity in this section is somewhat different from the ordinary sense and the third condition of Kojima-Toeplitz's theorem was omitted. But in the case of Nörlund's method of summation the third condition of Kojima-Toeplitz's theorem is satisfied which will be seen also by (8) and we do not enter into the discussion.
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3. Let us put \( \Delta q_v = q_{v+1} - q_v \) for \( v \geq 1 \) and \( \Delta q_0 = q_0 = 1 \), and suppose that the sequence \( \{\Delta q_n\} \) is positive decreasing and \( \{\Delta P_n\} \) is positive increasing. Moreover suppose that \( P_n \) and \( Q_n \) are positive.

We require an inequality due to Bloch-Pólya\(^{(5)}\) which runs as follows.

If the elements of the determinant \( D = |a_{p,v}| \) are real, then

\[
|D| \leq \prod_{v=1}^{n} \frac{1}{2} |a_{p,v}| + |a_{p,v} - a_{q,v}| + \ldots + |a_{p,v,n-1} - a_{q,v,n} | + |a_{p,v,n} |.
\]

Applying this inequality to the determinant

\[
D_{n,v} = \begin{vmatrix} p_0 & 1 \\ p_1 & \Delta q_1 \\ \vdots & \vdots \\ p_{n-v} & \Delta q_{n-v} \end{vmatrix}
\]

we get

\[
|D_{n,v}| \leq \Delta p_{n,v}.
\]

Therefore

\[
|a_{n,v}| = |D_{n,v}| Q_v/P_n \leq \Delta p_{n,v} Q_v/P_n
\]

and

\[
\sum_{v=0}^{n} |a_{n,v}| \leq \sum_{v=0}^{n} \Delta p_{n,v} Q_v/P_n \leq Q_n P_n/P_n,
\]

since \( \{Q_n\} \) is an increasing sequence; or

\[
\sum_{v=0}^{n} |a_{n,v}| \leq \sum_{v=0}^{n} p_{n,v} Q_v/P_n.
\]

Thus we get the following theorem.

Theorem 2. If \( P_n \) and \( Q_n \) are positive and \( \{\Delta q_n\} \) is positive

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decreasing and moreover \( \{\Delta p_n\} \) is positive increasing such that

1° \[ \frac{\Delta p_n}{P_n} \rightarrow 0 \text{ as } n \rightarrow \infty \]

and

2°

(a) \[ p_n Q_n / P_n \leq M, \text{ for } n=0, 1, 2, \ldots, \]

or

(b) \[ \sum_{\nu=0}^{n} p_{n-\nu} q_{\nu} / P_n \leq M, \text{ for } n=0, 1, 2, \ldots, \]

then \( (N; q) \subset (N; p) \).

If the summation is regular, the hypothesis 1° is satisfied. Further if \( (N; q) = (C, 1) \), Theorem 2 implies that

\( (C, 1) \subset (C, \alpha) \) for \( \alpha \geq 2 \).

Analogously we get the following theorem.

**Theorem 3.** If \( P_n \) and \( Q_n \) are positive and \( \{\Delta q_n\} \), \( \{\Delta p_n\} \) are positive increasing such that

1° \[ \Delta p_n \Delta q_n \Delta q_{n-1} \ldots \Delta q_1 / P_n \rightarrow 0 \text{ as } n \rightarrow \infty \]

and

2° \[ \sum_{\nu=0}^{n} \Delta p_{n-\nu} \Delta q_{n-\nu} \Delta q_{n-1-\nu} \ldots \Delta q_1 Q_{\nu} / P_n \leq M, \]

for \( n=0, 1, 2, \ldots, \)

then \( (N; q) \subset (N; p) \).

This theorem implies the theorems

\( (C, 1) \subset (C, \alpha) \) for \( \alpha \geq 1 \),

and

\( (C, 2) \subset (C, \beta) \) for \( \beta \geq 2 \).

If we use \( m \)-th difference, we can derive the theorem containing

\( (C, m) \subset (C, j) \) for \( j \geq m \)

for any integer \( m \).

4. From (3) we can prove the following theorem.

**Theorem 4.** Suppose that the series \( \sum_{n=0}^{\infty} a_n \) is summable \( (N; q) \), then

\[ |s_n| \leq c \sum_{\nu=0}^{n} Q_{\nu}, \quad \text{if } \{q_n\} \text{ decreases; } \]

\[ |s_n| \leq c \sum_{\nu=0}^{n} Q_{\nu} q_{1} \ldots q_{n-\nu}, \quad \text{if } \{q_n\} \text{ increases; } \]
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where \( c \) is a constant.

(3) may be derived by the generating function

\[
g(z) = \sum_{n=0}^\infty q_n z^n
\]

in the following manner. Let us put

\[
\frac{1}{g(z)} = \sum_{n=0}^\infty U_n z^n, \quad T_n = \sum_{n=0}^n q_{n-\delta_0}.
\]

We have formally

\[
\sum_{n=0}^\infty T_n z^n = \sum_{n=0}^\infty s_n z^n \sum_{n=0}^\infty q_n z^n,
\]

\[
\sum_{n=0}^\infty s_n z^n = \sum_{n=0}^\infty T_n z^n \sum_{n=0}^\infty U_n z^n.
\]

Comparing the coefficient of \( z^n \) on both sides

\[
s_n = \sum_{v=0}^n U_{n-v} T_v = \sum_{v=0}^n T_v Q_v U_{n-v} = \sum_{v=0}^n \rho_v Q_v U_{n-v},
\]

From (5) and (6) we have

\[q_n U_0 + q_{n-1} U_1 + \ldots + q_0 U_n = 0, \quad n \geq 1,
\]

\[q_0 U_0 = 1,
\]

therefore we get

\[
Q_n U_0 + Q_{n-1} U_1 + \ldots + Q_0 U_n = 1,
\]

for all \( n \); thus we have for \( Q_0 = q_0 = 1
\]

If we substitute this expression into (7), then (7) equals to (3).

Wiegert's theorem can be stated as follows:

If \( f(z) = \sum_{n=0}^\infty a_n z^n \) has unit circle as the convergence circle and \( z = 1 \) is
the only singularity on \(|z|=1\) which is a pole with order \(m\), then
\[ a_n = O(n^{m-1}) \quad (n=1, 2, \ldots). \]
If we use this theorem for \(f(z) = \frac{1}{g(z)}\), then we get.

**Theorem 5.** If the function \(\frac{1}{\sum_{n=0}^{\infty} q_n z^n}\) has unit circle as convergence circle and \(z=1\) is the only singularity on \(|z|=1\) which is the pole of order \(m\). Then, under the same condition of Theorem 4,

\[
|s_n| \leq C \sum_{\nu=0}^{n} Q_\nu (n - \nu)^{m-1}.
\]

Two theorems above stated contain the corresponding theorems concernig Cesàro summation.

5. We suppose that \(q_\nu\) are positive. We have by the above notation.

\[
s_n - \rho_n = s_n - \frac{(Q_n a_0 + Q_{n-1} a_1 + \ldots + Q_0 a_n)}{Q_n}
= \frac{1}{Q_n} \sum_{\nu=1}^{n} (Q_n - Q_{n-\nu}) q_\nu Q_\nu a_\nu = \tau_n, \text{ say}.
\]

If we suppose

\[
\frac{Q_\nu a_\nu}{q_\nu} = o(1),
\]
then

\[
\tau_n = \frac{1}{Q_n} \sum_{\nu=1}^{n} \frac{(Q_n - Q_{n-\nu}) q_\nu Q_\nu a_\nu}{Q_\nu}
\]
tends to zero if and only if

\[
\tau' = \frac{1}{Q_n} \sum_{\nu=1}^{n} \frac{(Q_n - Q_{n-\nu}) q_\nu}{Q_\nu}
\]
is bounded. By \(s'_n\) and \(\rho'_n\) we denote \(n\)-th partial sum and \(n\)-th Nörlund's mean of \(\sum_{\nu=1}^{\infty} q_\nu Q_\nu\). Then

\[
\tau'_n = s'_n - \rho'_n.
\]
If the consistency condition \(q_\nu Q_\nu \to 0\) is satisfied, \(\rho'_n\) tends to zero. Hence the boundedness of \(\tau'_n\) and then \(\tau_n\) becomes equivalent to

\[
\sum_{\nu=1}^{\infty} \frac{q_\nu}{Q_\nu} < \infty.
\]
Thus we get the following theorem.

**Theorem 6.** Suppose that \(q_\nu > 0\) and the series
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is convergent. If \( \sum_{n=0}^{\infty} a_n \) is summable \((N; q)\) and

\[
\sum_{n=0}^{\infty} \frac{Q_n}{Q_n^\varepsilon}
\]

is convergent. If \( \sum_{n=0}^{\infty} a_n \) is summable \((N; q)\) and

\[
Q_n a_n / q_n = o(1)
\]

then \( \sum_{n=0}^{\infty} a_n \) converges.

In the case of Cesàro's method of summation (10) becomes \( n a_n = o(1) \), which is the Tauberian condition of this summation. But (9) is not satisfied.

We can prove similarly the following theorem.

**Theorem 7.** Suppose that \( \varepsilon > 0, q > 0 \) and

\[
\sum_{n=0}^{\infty} \frac{Q_n}{Q_n^{1+\varepsilon}}
\]

is convergent. If \( \sum_{n=0}^{\infty} a_n \) is summable \((N; q)\) and

\[
Q_n^{1+\varepsilon} a_n / q_n = o(1),
\]

then \( \sum_{n=0}^{\infty} a_n \) converges.

(12) is stronger than the Tauberian condition of Cesàro's method of summation, but (11) is satisfied in this case.

Finally we add the following theorem:

**Theorem 8.** Suppose that \( q > 0 \) and \( q / Q_n \to 0 \). If \( \sum_{n=0}^{\infty} a_n \) is summable \((N; q)\) and

\[
n a_n = o(1)
\]

then \( \sum_{n=0}^{\infty} a_n \) converges.

This is the immediate consequence of the Tauberian theorem for Abel's methods of summation and the following theorem due to Silverman and Tamarkin:\(^5\):

If \( q > 0 \) and \( q / Q_n \to 0 \), then the series summable \((N; p)\) is also summable in the Abel's methods of summation.

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\(^5\) Silverman-Tamarkin, Mathematische Zeitschrift, 29 (1928).