On the Inscribed Rectangles of a Closed Convex Curve,

by

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Recently, Mr. A Emch (1) has proved that we can inscribe at least one square in any closed convex curve. This can be extended in a certain direction, and the following theorem can be stated, the demonstration of which is the main aim of the following lines.

Theorem. In every closed convex curve, at least two rectangles of the given shape (similar to a given rectangle) can be inscribed. Only when the given shape is square, the required figure may be determined uniquely.

To prove this, we shall first confine ourselves to such a closed convex curve that it contains no portion of straight segment. Let us call such a normal curve simply the oval. And then we shall extend that result to the general curve.

Though the rigorous expression of the proof may be somewhat complicated, its principle is so elementary that it seems not to be insufficient to show only the outlines of the proof.

1. Let $K$ be the given oval and $R$ be a rectangle of the given shape. We are required to inscribe the rectangles similar to $R$ in the oval $K$.

Let $ABCD$ be a rectangle similar to $R$ and let the direction $\theta$ (the angle made with a fixed line) of its side $AB$ be fixed in a moment. Then it is evident that we can make the three points $A, B$ and $C$ fall on $K$, by a translation and a change of the size of the rectangle; and then the remaining point $D$ is determined uniquely.

If we now make the direction $\theta$ vary from $\theta$ to $\theta+2\pi$, the rectangle $ABCD$ makes a complete revolution and returns to the initial position; and then the locus $L$ of the point $D$ become a closed continuous curve.

Let every two of the points on $L$ which correspond to two directions $\theta$ and $\theta+\pi$ be called the conjugate points. From the property of an oval, we see at once that:

Lemma 1. Any two conjugate points lie at the same time either within, on or without the oval.

Our theorem is proved if we see that there are at least two points of intersection of $K$ and $L$, which are not conjugate to each other.

2. We are now to consider the area of $L$. For that purpose, we shall prove a lemma which seems to have some interest independently from others.

Consider a moving rectangle $ABCD$ which is similar to a given rectangle $R$. Let the four angular points $A, B, C, D$ travel simultaneously round any four closed curves, the areas of which being $a$, $\beta$, $\gamma$, $\delta$ respectively. Suppose that the center $T$ of the rectangle describes a curve, the area of which is $\tau$. It is to be remarked that the area is defined by the integral

$$\int y \, dx,$$

the path of which coincides with that of the moving point.

By the theorem of E. B. Elliott (1), we must have, since the point $T$ always divides the two segments $AC$ and $BD$ in the constant ratio $1:1$,

$$\tau = \frac{a + \gamma}{2} - \frac{1}{4} \varepsilon_1$$

$$= \frac{\beta + \delta}{2} - \frac{1}{4} \varepsilon_2.$$

Here $\varepsilon_1$ and $\varepsilon_2$ denote the closed areas generated by the radii vectors, parallel and equal to $AC$ and $BD$ respectively. So we must have $\varepsilon_1 = \varepsilon_2$, and consequently $a + \gamma = \beta + \delta$.

Lemma 2. If the four vertices of a moving rectangle $ABCD$ having the given shape travel simultaneously round the perimeters of any four areas $a$, $\beta$, $\gamma$, $\delta$ respectively, then we must have

$$a + \gamma = \beta + \delta.$$

In the case of § 1, the three points $A$, $B$ and $C$ describe one and the same curve, viz. the given oval $K$. Hence from the preceding lemma, we see that the point $D$ describes a curve, the area of which is equal to that of $K$.

Lemma 3. The area of the locus $L$ is equal to that of the given oval $K$.

(1) See Williamson, Integral Calculus, p. 209.
3. Since the point $D$ lies always on the same side of the revolving chord $AC$, we get at once.

Lemma 4. If $L$ lie wholly outside of the oval $K$, it should contain the oval.

Moreover we can prove that:

Lemma 5. If $L$ possess any multiple point $M$, it should lie within the oval.

For, if $M$ lie without the oval, then there exist at least two rectangles of the given shape, say $A_1B_1C_1M$, and $A_2B_2C_2M$ lying on the same side of a straight line passing through $M$, for which the six points $A_1, B_1, C_1, A_2, B_2, C_2$ lie on the oval $K$. That it is impossible can easily be proved separately in all possible typical cases of positions. In every case, some one of the six points must fall within either a triangle or a quadrilateral formed by the three or four remaining points.

4. Now we are able to prove the required theorem.

By the lemmas 4 and 5, we see that if $L$ lie wholly outside of $K$, the area of $L$ should be greater than that of $K$. This is impossible in virtue of the lemma 3. Moreover, it is evident that if $L$ lie wholly inside of $K$, the area of $L$ should be smaller than that of $K$, which also contradicts the lemma 3.

Consequently $L$ and $K$ must intersect at least at two points, $D_1$ and $D_2$, say. If there were only the two points of intersection, then the arc $D_1D_2$ of $L$ should lie wholly on one side of $K$ and the remaining arc on the other side of $K$. Hence every point on the arc $D_1D_2$ must be conjugate to a point on the same arc, so that there should exist a point on $D_1D_2$ which is conjugate to itself. This is evidently impossible. Thus we get at least three points of intersection, say $D_1, D_2, D_3$. Since the points conjugate to $D_1, D_2, D_3$ also lie on $K$, we finally get at least four points of intersection $D_1, D_2, D_3, D_4$.

Let $D_1$ and $D_4$ be two points of intersection which are not conjugate, then the corresponding rectangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ inscribed in $K$, which have the given shape, are different to each other, provided that the given shape is not square.

Thus our theorem is proved in the case of oval.

5. If in the preceding discussion, we take an arbitrary closed convex curve instead of an oval, then the uniqueness of the point $D$ corresponding to a direction $\theta$ can not be taken into account.
But the uniqueness of the point $D$ is not absolutely necessary in our proof.

For an arbitrary closed convex curve we also get a continuous locus $L$ of the point $D$; but in this case it may occur that every point $D$ on some portion of $L$ corresponds to one and the same direction $\theta$, which has no different influence on our reasoning.

Thus our theorem is proved completely.

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