AN ASPECT OF LOCAL PROPERTY OF \(|R, \log n, 1|\)
SUMMABILITY OF FOURIER SERIES

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1. 1. DEFINITION. Let \(S_n\) denote the \(n\)-th partial sum of the series \(\sum a_n\). We write
\[
R_n = \left\{ S_1 + \frac{1}{2} S_2 + \ldots + \frac{1}{n} S_n \right\} \log n.
\]
Then the series \(\sum a_n\) is said to be absolutely summable \((R, \log n, 1)\) or summable \(|R, \log n, 1|\) if the sequence \(|R_n|\) is of bounded variation, that is to say, the infinite series
\[
\sum |R_n - R_{n+1}|
\]
is convergent.

It has been pointed out by Bosanquet* that for the case \(\lambda_n = \log n\), this definition is equivalent to the definition of the summability \(|R, \lambda_n, 1|\) used by Mohanty [5], \(\lambda_n\) being a monotonic increasing sequence tending to infinity with \(n\).

1. 2. Let \(f(t)\) be a periodic function with period \(2\pi\) and integrable \((L)\) over \((-\pi, \pi)\). Without any loss of generality the constant term in the Fourier series of \(f(t)\) can be taken to be zero, so that
\[
(1.2.1) \quad f(t) \sim \sum (a_n \cos nt + b_n \sin nt) = \sum A_n(t),
\]
and
\[
(1.2.2) \quad \int_{-\pi}^{\pi} f(t) \, dt = 0.
\]
We write
\[
\varphi(t) = \frac{1}{2} \{ f(x + t) + f(x - t) \}.
\]

1. 3. It has been proved independently by Izumi [3] and Mohanty [5] that summability \(|R, \log n, 1|\) of a Fourier series is not a local property of the generating function. The question, naturally arises as to what conditions

* L. S. Bosanquet, Mathematical Review, 12 (1951), 254, see review of the paper of Izumi [3].
should be satisfied by the general terms of a Fourier series at a point such that its summability \(|R, \log n, 1|\) may depend only upon the behaviour of the generating function in the immediate neighbourhood of the point considered. The first answer to a question of this character is due to Izumi [3] who proved that if

\[ A_n(x) = O((\log n)^{-\varepsilon}), \]

then the summability \(|R, \log n, 1|\) of the Fourier series \(\Sigma A_n(t)\), at \(t = x\), is a local property. More recently Mohanty and Izumi [6] have improved upon this result and established the following theorem:

**THEOREM A.** If

\[ \sum_n \frac{|A_n(x)|}{\log \log n} < \infty, \]

then the summability \(|R, \log n, 1|\) of \(\Sigma A_n(x)\) depends only upon a local condition.

It is known [5, 7] that if \(\Sigma a_n\) is summable \(|R, \lambda_n, k|, k > 0\), then \(\Sigma a_n/\lambda_n^k\) is summable \(|R, e^{\lambda_k}, k|\). Hence it follows that if \(\Sigma a_n\) is summable \(|R, \log n, 1|\), then \(\Sigma a_n/\log n\) is summable \(|R, n, 1|\) i.e. summable \(|C, 1|\), [2]. Therefore, by a well-known result of Kogbetliantz [4], it follows that

\[ \Sigma |a_n|/\{n \log n\} < \infty. \]

Thus it follows that the summability \(|R, \log n, 1|\) of the Fourier series necessarily implies that

\[ \Sigma |A_n(x)|/\{n \log n\} < \infty. \] (1.3.2)

In this paper we establish a theorem, more general than theorem A, inasmuch as we assume, instead of the condition (1.3.1) the less stringent condition (1.3.2) which is seen to be also the necessary condition of the \(|R, \log n, 1|\) summability of the corresponding Fourier series.

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2.1. We prove the following theorem.

**THEOREM.** If

\[ \Sigma |A_n(x)|/\{n \log n\} < \infty, \]

then the \(|R, \log n, 1|\) summability of \(\Sigma A_n(t)\) depends only on the behaviour of the generating function \(f(t)\) in the immediate neighbourhood of the point \(t = x\).
2.2. We require the following lemma for the proof of the theorem.

**Lemma.** If the series

\[ \sum_{n=1}^{\infty} \frac{|S_n|}{n \log (n + 1)} \]

is convergent then the sequence \( \{S_n\} \) is summable \( |R, \log n, 1| \).

**Proof.**

\[
R_n - R_{n+1} = \frac{1}{\log n} \sum_{\nu = 1}^{n} \frac{S_{\nu}}{\nu} - \frac{1}{\log (n + 1)} \sum_{\nu = 1}^{n+1} \frac{S_{\nu}}{\nu}
\]

\[
= \Delta \left( \frac{1}{\log n} \right) \sum_{\nu = 1}^{n} \frac{S_{\nu}}{\nu} - \frac{1}{\log (n + 1)} \frac{S_{n+1}}{n + 1},
\]

where

\[
\Delta \lambda_n = \lambda_n - \lambda_{n+1}.
\]

Therefore

\[
\sum_{n=2}^{m} |R_n - R_{n+1}|
\]

\[
\leq A + \sum_{n=2}^{m} \Delta \left( \frac{1}{\log n} \right) \sum_{\nu = 2}^{n} |S_{\nu}/\nu| + \sum_{n=2}^{m} \frac{|S_{n+1}|}{(n + 1) \log (n + 1)}
\]

\[
= A + \sum_{\nu = 2}^{m} \frac{|S_{\nu}|}{\nu} \sum_{n=2}^{\nu} \Delta \left( \frac{1}{\log n} \right) + \sum_{n=2}^{m} \frac{|S_{n+1}|}{(n + 1) \log (n + 1)}
\]

\[
= A + O \left( \sum_{\nu = 2}^{m} \frac{|S_{\nu}|}{\nu \log \nu} \right)
\]

\[
= A + O \left( \sum_{\nu = 2}^{m} \frac{|S_{\nu}|}{\nu \log (\nu + 1)} \right).
\]

This completes the proof of the lemma.

2.3. **Proof of the theorem.** We have

\[ S_n(x) = \sum_{\nu=1}^{n} A_\nu(x) \]

\[
= \frac{1}{2\pi} \int_0^x \varphi(u) \sin \left( n + \frac{1}{2} \right) u \left( -\sin u/2 \right) \, du
\]

\[
= \frac{1}{2\pi} \int_0^n \varphi(u) \frac{\sin u/2}{\sin^2 \frac{n}{2}} \sin \left( n + \frac{1}{2} \right) u \, du
\]
\[ \int_{\eta}^{\pi} \phi(u) \frac{\sin (n + \frac{1}{2})u}{\sin u/2} \, du \]

\[ + \frac{1}{2\pi} \int_{0}^{\pi} \phi(u) \left[ 1 - \left( \frac{\sin u/2}{\sin \eta/2} \right)^2 \right] \frac{\sin (n + \frac{1}{2})u}{\sin u/2} \, du \]

(2.3.1) \[ = \frac{1}{2\pi} [P_n + Q_n], \text{ say.} \]

The sequence \( \{S_n(x)\} \) will be summable \( |R, \log n, 1| \) if the sequences \( \{P_n\} \) and \( \{Q_n\} \) are summable \( |R, \log n, 1| \). We observe that, for positive \( \eta \), however small but fixed, the summability \( |R, \log n, 1| \) of the sequence \( \{Q_n\} \) depends only upon the behaviour of the generating function \( f(t) \) in the immediate neighbourhood of the point \( x \), defined by \( (x - \eta, x + \eta) \). Hence to prove the theorem it is sufficient to show that the sequence \( \{P_n\} \) is summable \( |R, \log n, 1| \) under the hypothesis of the theorem. By virtue of the lemma, this will be satisfied if we prove that

(2.3.2) \[ \sum |P_n(x)| / |n \log (n + 1)| < \infty. \]

We now proceed to prove (2.3.2). Let us define a function \( \xi(u) \), as follows.

\[ \xi(u) = \begin{cases} 
\left( \frac{\sin \eta/2}{2} \right)^2 \sin \frac{u}{2} & (0 \leq u \leq \eta) \\
\left( \frac{\sin u/2}{2} \right)^{-1} & (\eta \leq u \leq \pi).
\end{cases} \]

Then, for \( 0 \leq u \leq \pi \), \( \xi(u) \) is of bounded variation and continuous, with \( \xi(0) = 0 \). Also \( \xi'(u) \) is bounded and \( \xi'(u) \) is integrable \( (L) \). Now, since \( \xi(u) \) is of bounded variation in \( (0, \pi) \), by a well-known result* we have, setting

\[ A_{-\nu}(x) = A_{\nu}(x) = A_{\nu}, \]

\[ P_n = \frac{1}{2} A_0 \int_{0}^{\pi} \xi(u) \sin \left( n + \frac{1}{2} \right) u \, du \]

\[ + \sum_{\nu=1}^{\infty} A_{\nu} \int_{0}^{\pi} \xi(u) \cos \nu u \sin \left( n + \frac{1}{2} \right) u \, du \]

\[ = \frac{1}{2} \sum_{\nu=-\infty}^{\infty} A_{\nu} \int_{0}^{\pi} \xi(u) \sin \left( n - \nu + \frac{1}{2} \right) u \, du \]

* See Hobson [1], page 567.
\[ \sum_{n=-\infty}^{\infty} A_r \left[ \xi'(u) \frac{\cos\left(\frac{n-\nu + \frac{1}{2}}{2}\right)u}{n-\nu + \frac{1}{2}} \right]_{0}^{\infty} \]

\[ + \frac{1}{2} \sum_{n=-\infty}^{\infty} A_r \int_{0}^{\infty} \xi'(u) \frac{\cos\left(\frac{n-\nu + \frac{1}{2}}{2}\right)u}{n-\nu + \frac{1}{2}} \, du \]

\[ = \frac{1}{2} \sum' A_r \int_{0}^{\infty} \xi'(u) \frac{\cos\left(\frac{n-\nu + \frac{1}{2}}{2}\right)u}{n-\nu + \frac{1}{2}} \, du + O(|A_n|), \]

where \( \Sigma' \) denotes summation extending over \(-\infty < \nu \leq n - 1 \) and \((n + 1) \leq \nu < \infty \). Let

\[ \mu = \min\{|n-\nu|^{-1}, \eta\}. \]

Then we have

\[ P_n = \frac{1}{2} \sum' A_r \left( \int_{0}^{\mu} + \int_{\mu}^{\infty} \right) \xi'(u) \frac{\cos\left(\frac{n-\nu + \frac{1}{2}}{2}\right)u}{n-\nu + \frac{1}{2}} \, du + O(|A_n|) \]

\[ = P_1 + P_2 + O(|A_n|), \text{ say.} \]

Thus we have

\[ P_1 = O(1) \sum' \frac{|A_r|}{(n-\nu)^2}; \]

and

\[ P_2 = \frac{1}{2} \sum' A_r \left[ \xi'(u) \frac{\sin\left(\frac{n-\nu + \frac{1}{2}}{2}\right)u}{(n-\nu + \frac{1}{2})^2} \right]_{\mu, \eta-0}, \tau \]

\[ - \frac{1}{2} \sum' A_r \int_{\tau}^{\pi} \xi'(u) \frac{\sin\left(\frac{n-\nu + \frac{1}{2}}{2}\right)u}{(n-\nu + \frac{1}{2})^2} \, du, \]

where integration by parts is taken separately over the ranges \((\mu, \eta - 0)\) and \((\eta + 0, \pi)\). Thus we have
\[ P_\nu = O(1) \sum' \frac{|A_\nu|}{(n - \nu)^{\delta}}. \]

Hence
\[ P_n = O(1) \sum' \frac{|A_\nu|}{(n - \nu)^{\delta}} + O(|A_n|) \]
\[ = O(1) \left( \sum_{\nu=-\infty}^{0} \frac{n-1}{\nu} + \sum_{\nu=n+1}^{m} \frac{n+m}{\nu} \right) \frac{|A_\nu|}{(n - \nu)^{\delta}} + O(|A_n|) \]
\[ = O(1) [M_1 + M_2 + M_3 + M_4 + |A_n|], \text{ say.} \]

Now in order to prove (2.3.2), it is sufficient to show that
\[ (2.3.3) \Sigma M_r/\{n \log (n + 1)\} < \infty, \ r = 1, 2, 3, 4, \]
since, by hypothesis
\[ \Sigma |A_n|/\{n \log n\} < \infty. \]

Let \(0 < \delta < 1\), then we have
\[ \sum_{n=1}^{m} \frac{n \log (n + 1)}{n} M_1 \leq \sum_{n=1}^{m} n^{-2+\delta} \sum_{\nu=0}^{\infty} \frac{|A_\nu|}{|\nu + 1|^{1+\delta}} \]
\[ = O(1) \sum_{n=1}^{m} n^{-2+\delta} = O(1), \]
as \(m \to \infty\).

Again
\[ \sum_{n=2}^{m} \frac{n \log (n + 1)}{n} M_2 = \sum_{\nu=1}^{m-1} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|A_{n-\nu}|}{n \log (n + 1)} \]
\[ \leq \sum_{\nu=1}^{m-1} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|A_{n-\nu}|}{(n - \nu) \log (n - \nu + 1)} \]
\[ = O(1); \]

and
\[ \sum_{n=1}^{m} \frac{n \log (n + 1)}{n} M_3 = \sigma(1) \left\{ \sum_{n=1}^{m} \frac{n \log (n + 1)}{n} \sum_{\nu=n+1}^{m} \frac{1}{(n - \nu)^{-2}} \right\} \]
\[ = \sigma(1) \sum_{n=1}^{m} \frac{1}{m + 1} \frac{1}{n \log (n + 1)} = \sigma(1); \]
as \(m \to \infty\).

Lastly
\[ \sum_{n=1}^{m} \frac{n \log (n + 1)}{n} M_4 = \sum_{n=1}^{m} \frac{n \log (n + 1)}{n} \sum_{\nu=1}^{m} \nu^{-2} |A_{\nu+n}|. \]
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\[
= \sum_{v=1}^{\infty} v^{-2} \sum_{n=1}^{m} \frac{n \log (n+1)}{1} |A_{v+n}|
\]

\[
= \sum_{v=1}^{\infty} v^{-2} \left[ \sum_{n=1}^{v} + \sum_{n=v+1}^{m} \right]
\]

\[
= \Sigma_1 + \Sigma_2, \text{ say.}
\]

Now

\[
\Sigma_1 = O\left( \sum_{v=1}^{m} v^{-2} \log v \right)
\]

\[
= O(1);
\]

and

\[
\Sigma_2 = \sum_{v=1}^{\infty} \sum_{n=v+1}^{m} \frac{|A_{v+n}|}{(v+n) \log (v+n)} \left\{ \frac{(v+n) \log (v+n)}{n \log (n+1)} \right\}
\]

\[
= O(1),
\]

as $m \to \infty$, since

\[
\frac{(n+v) \log (n+v)}{n \log (n+1)} = \left(1 + \frac{v}{n}\right) \left\{ 1 + \frac{\log \left(1 + \frac{v-1}{n+1}\right)}{\log (n+1)} \right\}
\]

\[
= O(1)
\]

for $n \geq v + 1$.

Thus we have established (2.3.3) and thereby (2.3.2). This completes the proof of the theorem.

REFERENCES


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