Let $\mathfrak{G}$ be a finite group, $k$ be an algebraic field of finite degree over the field of rationals $\mathbb{Q}$. In a representation space $V$ over $k$ we consider a $\Gamma = \mathfrak{G}[\mathbb{Q}]$-lattice (Gitter) $M$ in $V$ which is a regular $\mathfrak{G}$-right module and $\mathfrak{G}$-left module where $\mathfrak{o}$ is the ring of integers in $k$. The set of all $\Gamma$-lattices which we denotes by $\{M; k/\mathfrak{o}\}$ can be classified into $\Gamma$-isomorphic $\Gamma$-lattices in the following way:

$$\{M; k/\mathfrak{o}\} = \{M_1; \mathfrak{o}/\mathfrak{o}\} + \ldots \ldots + \{M_r; \mathfrak{o}/\mathfrak{o}\}.$$ 

If $k = \mathbb{Q}$ is the field of rationals and $V$ is irreducible, this class number is always finite and was proved by C. Jordan [13].

In the book of Speiser [20] this theorem was proved only in two special cases, namely, $\mathfrak{G}$ is a cyclic group or $V$ is absolutely irreducible. The reason for this may be explained by the following considerations.

Let $p$ be a finite or infinite prime. We can consider $p$-extension $M_p$ of the $\Gamma$-lattice $M$ and put

$$\{M_p; k_p/\mathfrak{o}_p\} = \{M_p^{(1)}; \mathfrak{o}_p/\mathfrak{o}_p\} + \ldots \ldots + \{M_p^{(r)}; \mathfrak{o}_p/\mathfrak{o}_p\}.$$ 

The local class number $j = j(p)$ is always finite and $= 1$ if $p$ does not divide the order $g = \# \mathfrak{G}$ of the group $\mathfrak{G}$.

If we define genus of $M$ as

$$\{M; \overline{\mathfrak{o}}/\mathfrak{o}\} = \bigcap_p \{M; \mathfrak{o}_p/\mathfrak{o}_p\}$$ 

then the number of genera in all $\Gamma$-lattices in $V$ is

$$j = \prod_{p \mid \mathfrak{o}} j(p)$$ 

and is finite ($\S 7$). If $M$ is absolutely irreducible we have

$$c = j \quad (\S 10).$$

On the other hand, number of classes in a genus is expressible as a kind of class number of a suitable algebraic group ($\S 9$), which was considered by T. Ono [17] and its finiteness was proved for commutative case by him. Simple considerations show that if $\mathfrak{G}$ is cyclic and $k = \mathbb{Q}$

1) Number in the bracket refers to the bibliography at the end of this paper.
where $h$ is the class number of the field of $g$-th roots of unity. General cases are somewhat complicated but relate with class number of a suitable algebraic extension $K/k$ (§11).

After this investigation was almost completed, the author found papers by Maranda [15], [16]. He introduced the concept of genus and its product formula (§§7-8), but his definition is a global one and its locality and hence equality with my definition was not proved by him.

Finally, I must express my hearty thanks to Prof. Tannaka for his kind advices and encouragement during the preparation of this paper.

CONTENTS

1. Preliminaries on lattices (Gitter).  
2. Representations by lattices.  
3. Reducibility of representations.  
4. Some cohomology groups.  
5. Maschke pair.  
6. Representations in $p$-adic fields.  
7. Equivalence theory of $\Gamma$-lattices.  
8. Genus of representations.  
9. Class number in a genus.  
10. Absolutely irreducible representations.  
11. Irreducible representations.  
12. Some examples.

NOTATIONS

$\emptyset$: finite group.  
$k$: algebraic number field of finite degree over the rational field $\mathbb{Q}$.  
o: ring of integers in $k$.  
$\Gamma = o[\emptyset]$; group ring of $\emptyset$ over $o$.  
$V$: vector space of dimension $m$ over $k$; mostly $\Gamma$-space.  
$A(x)$: representation of $\emptyset$ by $GL(V; k)$.  
$M$: lattice in $V$; mostly $\Gamma$-lattice.

1. Preliminaries on lattices (Gitter). By a lattice in an algebraic field $k$ we mean an $o$-module $M$ contained in a definite vector space $V$ over $k$ such that

1) $M$ is a finitely generated $o$-module,  
2) $M$ generates over $k$ the vector space $V$ i.e. $Mk = V$.  

After this investigation was almost completed, the author found papers by Maranda [15], [16]. He introduced the concept of genus and its product formula (§§7-8), but his definition is a global one and its locality and hence equality with my definition was not proved by him.

Finally, I must express my hearty thanks to Prof. Tannaka for his kind advices and encouragement during the preparation of this paper.
Or, equivalently, a lattice is a regular $\mathfrak{o}$-module i.e.

1') $M'$ is a finitely generated $\mathfrak{o}$-module,
2') $u \in M, \alpha \in \mathfrak{o}, u\alpha = 0$ imply $u = 0$ or $\alpha = 0$.

Namely, a lattice $M$ in former sense is of course a regular $\mathfrak{o}$-module and regular $\mathfrak{o}$-module $M'$ is a lattice contained in the vector space $M'k = V'$ of $k$-extension of $M'$.

Let $\mathfrak{p}$ be a prime in $k$. Assume first $\mathfrak{p}$ is finite. $k_{\mathfrak{p}}, \mathfrak{o}_{\mathfrak{p}}$ denote respectively $\mathfrak{p}$-adic completion of $k$ and $\mathfrak{p}$-adic integers in $k_{\mathfrak{p}}$. If $M$ is a lattice in $k$, then its $\mathfrak{p}$-adic extension

$$M_{\mathfrak{p}} = M\mathfrak{o}_{\mathfrak{p}}$$

is a lattice contained in the vector space $V_{\mathfrak{p}} = V_{k_{\mathfrak{p}}}$. For infinite prime $\mathfrak{p}_{\infty}$, we simply put

$$M_{\mathfrak{p}_{\infty}} = V_{\mathfrak{p}_{\infty}}$$

in accordance with the convention $\mathfrak{o}_{\mathfrak{p}_{\infty}} = k_{\mathfrak{p}_{\infty}}$.

**Proposition 1.1.** If $M$ is a lattice contained in $V$, then

$$M = \bigcap_{\mathfrak{p}} (V \cap M_{\mathfrak{p}})$$

where the intersection extends over all finite and infinite primes in $k$.

A proof is found in Eichler\(^2\) [10] and almost clear if we assume Stenitz’s basis theorem\(^3\).

**Proposition 1.2.** Let $v_1, \ldots, v_m$ be an arbitrary $k$-basis of $V$. Then for any lattice $M$ in $V$ we have

$$M_{\mathfrak{p}} = v_1\mathfrak{o}_{\mathfrak{p}} \oplus \ldots \oplus v_m\mathfrak{o}_{\mathfrak{p}}$$

except for a finite number of primes in $k$.

For, by Steinitz’s basis theorem

$$M = u_1\mathfrak{o} \oplus \ldots \oplus u_{m-1}\mathfrak{o} \oplus u_m\mathfrak{a}$$

with an ideal $\mathfrak{a}$ in $k$. For a prime not in $\mathfrak{a}$ we have

$$M_{\mathfrak{p}} = u_1\mathfrak{o}_{\mathfrak{p}} \oplus \ldots \oplus u_m\mathfrak{o}_{\mathfrak{p}}.$$
ARITHMETIC OF GROUP REPRESENTATIONS

\[ M^{(q)} = v_1 v_p \bigoplus \ldots \bigoplus v_m v_p \]

where \( v_1, \ldots, v_m \) is a \( k \)-basis of \( V \). Then the intersection

\[ M = \bigcap_p (V \cap M^{(q)}) \]

over all primes in \( k \), is a lattice in \( V \) such that

\[ M = M^{(q)} \]

for all primes in \( k \).

**Proof.** Put \( M' = v_1 v_0 \bigoplus \ldots \bigoplus v_m v_0 \). Since \( M'_q = M^{(q)} \) except for a finite number of primes. We can find \( \gamma, \gamma' \in v \) such that

\[ M^{(q)} \gamma \subseteq M'_q \subseteq M^{(q)} \gamma' \]

for all primes in \( k \). From \( M \subseteq M' \gamma^{-1}, M \) is a finite \( v \)-module. On the other hand, \( M' \subseteq M \gamma \) implies \( M k = V \). Therefore \( M \) is a lattice in \( V \). Next, \( M \subseteq M^{(q)} \) implies \( M_q \subseteq M^{(q)} \) for all primes in \( k \). Take \( u \in M^{(q)} \) arbitrarily, put \( u_1, \ldots, u_n (n \geq m) \) for an \( v \)-generator of \( M \), secured by first part of the proof. We have

\[ u = u_1 \alpha_1 + \ldots + u_n \alpha_n \]

with \( \alpha_i \in k_v \).

From approximation theorem on valuations, we can take \( \beta_i \in k \) such that

\[ \beta_i \equiv \alpha_i (v_v) \]

\[ \beta_i \equiv 0 (v_v) \] for all primes \( v' \neq v \) in \( k \).

Then

\[ v = u_1 \beta_1 + \ldots + u_n \beta_n \]

is a vector in \( V \) such that it is contained in \( M^{(q)} \) and \( M^{(q')} \) for any prime \( v' \neq v \), i.e.

\[ v \in \bigcap_v (V \cap M^{(q)}) = M. \]

On the other hand, we have

\[ u = v + \sum_{i=1}^n u_i (\alpha_i - \beta_i) \]

with \( v \in M, \alpha_i - \beta_i \in v_v \). This means \( \sum_{i=1}^n u_i (\alpha_i - \beta_i) \in M_q \) and finally \( u \in M_q \), q.e.d.

2. **Representations by lattices.** Let \( \mathcal{G} \) be a finite group and \( \Gamma = \mathcal{G}[\mathcal{G}] \) be the group ring over \( \mathcal{G} \). Assume now \( V \) is a \( \Gamma \)-left space over \( k \). Any element \( x \in \mathcal{G} \) is represented by an automorphism
of the vector space $V$. Symbolically $xV = VA(x)$.

By a $\Gamma$-lattice in $V$, we mean a lattice $M$ such that

$$MA(x) \subseteq M$$

for all $x \in \mathfrak{g}$.

To a $\Gamma$-lattice $M$ we can associate a finite set of matrix representations in the following way. Let $v_1, \ldots, v_m$ be a $k$-basis of $V$, since $M$ is a lattice in $V$ by Prop. 1.2, except for a finite system of primes $\mathfrak{p}_i, \ldots, \mathfrak{p}_r$ we have

$$M_v = v_1 \mathfrak{p}_1 \oplus \cdots \oplus v_m \mathfrak{p}_r.$$ 

For exceptional $\mathfrak{p}_i (i = 1, \ldots, r)$ we can put

$$M_{\mathfrak{p}_i} = v_1 \mathfrak{p}_i \oplus \cdots \oplus v_m \mathfrak{p}_i i = 1, \ldots, r$$

since $\mathfrak{p}_i$ are principal ideal domains.

Put

$$xv_i = \sum_{j=1}^{m} v_j a_{ji}^0(x) \quad a_{ji}^0(x) \in k$$

$$xv_{ij} = \sum_{i=1}^{m} v_i a_{ij}^0(x) \quad a_{ij}^0(x) \in \mathfrak{p}_i$$

then matrices

$$A_i(x) = (a_{ij}^0(x)) \quad i = 0, 1, \ldots, r$$

are $(r+1)$-matrix representations of the group $\mathfrak{g}$ such that $A_i(x) (i = 1, \ldots, r)$ are $k_{\mathfrak{p}_i}$-equivalent to $A_0(x)$. Notice that the elements $a_{ij}^0(x) \in k$ are integral for all prime $\mathfrak{p} \neq \mathfrak{p}_i (i = 1, \ldots, r)$.

Conversely given a matrix representation $A_0(x)$ in $k$ and $\mathfrak{p}_r$-adic integral matrix representations $A_i(x) (i = 1, \ldots, r)$ which are $k_{\mathfrak{p}_i}$-equivalent to $A_0(x)$ for any prime $\mathfrak{p}_i$ for which $A_0(x)$ is not necessarily $\mathfrak{p}_i$-integral. Then we can fined a $\Gamma$-lattice $M$ whose associated matrix representations are given $A_i(x) (i = 0, 1, \ldots, r)$. Namely, if $v_1, \ldots, v_m$ be a $k$-basis of the vector space $V$, we put

$$M^{(x)} = v_1 \mathfrak{p}_1 \oplus \cdots \oplus v_m \mathfrak{p}_r \quad \mathfrak{p} \neq \mathfrak{p}_i (i = 1, \ldots, r)$$

with $\mathfrak{g}$-left operation:

$$xv_i = \sum_{i=1}^{m} v_j a_{ji}^0(x)$$

where $(a_{ji}^0(x)) = A_0(x)$. For an exceptional prime $\mathfrak{p}_i$ let $R_i$ be a regular matrix in $k_{\mathfrak{p}_i}$ such that

$$A_i(x) = R_i^{-1} A_0(x) R_i$$

and put
\[ M^{(n)} = v_{ij}^p \oplus \ldots \oplus v_{im}^p, \]

where

\[ (v_{i1}, \ldots, v_{im}) = (v_1, \ldots, v_m) R_i \]

is a \( k_n \)-basis of \( V_m \).

Then by Prop. 1.3

\[ M = \bigcap_{\nu} (V \cap M^{(\nu)}) \]

is a desired \( \Gamma \)-lattice in \( V \).

3. Reducibility of representations. We consider now reducibility of a \( \Gamma \)-lattice \( M \) in connection with reducibility of matrix representation by the vector space \( V = Mk \).

**Lemma 1.** Let \( M, N \) be two regular \( \nu \)-modules. Then we have

\[ (M \cap N)k = Mk \cap Nk. \]

**Proof.** From \( M \cap N \subseteq M \) and \( M \cap N \subseteq N \), it is obvious that

\[ (M \cap N)k \subseteq Mk \cap Nk. \]

Let \( a\alpha = b\beta \in Mk \cap Nk \) with \( a \in M, b \in N, \alpha, \beta \in k \) be given. Take \( \gamma \in \nu \) such that \( \alpha\gamma \in \nu, \beta\gamma \in \nu \), then \( a\alpha\gamma = b\beta\gamma \in M \cap N \) and \( a\alpha = (a\alpha\gamma)\gamma^{-1} \in (A \cap B)k \).

We say that a submodule \( N \) of a regular \( \nu \)-module \( M \) is primitive in \( M \) if one of the following, equivalent, condition is satisfied:

1) \( Nk \cap M = N \),
2) Quotient module \( M/N \) also is a regular \( \nu \)-module,
3) \( a \in M, a\alpha \in N \) with \( \alpha \in k \). \( \alpha \neq 0 \) imply \( a \in N \).

**Lemma 2.** If \( N \) is a primitive submodule of \( \Lambda \), then naturally

\[ (M/N)k \simeq Mk/Nk, \]

**Proof.** The map \( \varphi : M/N \rightarrow Mk/Nk \) defined naturally by \( \varphi(a) = a \) for \( a \in M \) is into isomorphic by the primitivity of \( N \) in \( M \) (e.g. by 3)). Therefore it remains to show that \( M/N \) contains as many linearly independent elements as that of \( Mk/Nk \). But this is obvious since any elements \( a_1, \ldots, a_r \) of \( M \) that are linearly independent mod \( Nk \) are a priori linearly independent mod \( N \).

We now define reducibility of a \( \Gamma \)-lattice \( M \) as follows:

\( M \) is reducible if it contains a primitive submodule \( N \) neither 0 nor \( M \) such that \( N \) itself is also a \( \Gamma \)-lattice in \( Nk = W \).
PROPOSITION 3.1. A Γ-lattice $M$ is reducible if and only if the matrix representation defined by $V = Mk$ is reducible.

PROOF. Assume first $M$ is reducible, then there exists a primitive submodule $N$. $N^k$ is a subspace of $Mk = V$ neither 0 nor $V$ by primitivity of $N$ in $M$. Of course $N^k$ is a $Γ$-space and therefore $V$ is reducible.

Next, let $Mk = V$ be reducible, then there exists a $Γ$-subspace $W ⊂ V$ different from 0 or $V$. Put $N = W \cap M$. As a submodule of $M$, $N$ is a regular $v$-module. By lemma 1 $N^k = W$, it follows that $N$ is a primitive submodule of $M$. Since $N$ is a $Γ$-module, $M$ is reducible. q. e. d.

4. Some cohomology groups. Let $A_1(x), A_2(x)$ be two representations of the group $Γ$ by matrices of degree $r, s$ respectively with elements in a commutative ring $R$ with unity element. We now define cohomology groups $H^n(Γ; A_1, A_2)$ as follows:

$n$-cochains are functions $E(x_1, \ldots, x_n)$ from $Γ \times \ldots \times Γ (n$-factors) to $R_{rs}$, where $R_{rs}$ denotes the set of all matrices consist of $r$-rows and $s$-columns with elements in $R$.

Coboundary operations are defined by

\[
\delta E(x_1, \ldots, x_{n+1}) = A_1(x_1)E(x_2, \ldots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i E(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) + (-1)^{n+1} E(x_1, \ldots, x_n) A_2(x_{n+1})
\]

$n = 0, 1, 2, \ldots$

From these, cohomology groups are defined as usual

\[H^n(Γ; A_1, A_2) = n\text{-cocycle}/n\text{-coboundary} \quad n = 0, 1, 2, \ldots\]

Obviously,

PROPOSITION 4.1. The set $H^n(Γ; A_1, A_2)$ consist of all intertwining matrices $E$ between $A_1, A_2$, namely,

\[A_1(x)E = EA_2(x)\]

for all $x \in Γ$.

If $R = k$ is a field then

\[\dim_k H_n(Γ; A_1, A_2) = I(A_1, A_2)\]

is called intertwining number.

The “norm” of a matrix $T \in R_{rs}$ defined by

\[\sum_{y \in Γ} A_1(y)TA_2(y^{-1})\]
is a 0-cocycle.

**Proposition 4.2.**\( H^1(\emptyset; A_1, A_2) \) and matrix representations of type
\[
\begin{pmatrix}
A_1(x) & E(x) \\
0 & A_2(x)
\end{pmatrix}
\]
classified by
\[
\begin{pmatrix}
1 & T \\
0 & 1
\end{pmatrix}
\]
are in one to one correspondences.

**Proof.** From
\[
\begin{pmatrix}
A_1(x) & E(x) \\
0 & A_2(x)
\end{pmatrix}
\begin{pmatrix}
A_1(y) & E(y) \\
0 & A_2(y)
\end{pmatrix}
= \begin{pmatrix}
A_1(x)A_1(y) & A_1(x)E(y) + E(x)A_2(y) \\
0 & A_2(x)A_2(y)
\end{pmatrix}
\]
it follows that this is a representation of \( \emptyset \) if and only if
\[
A_1(x)A_1(y) = A_1(xy) \quad i = 1, 2
\]
\[
E(xy) = A_1(x)E(y) + E(x)A_2(y)
\]
i. e. \( E(x) \) is a 1-cocycle. The rest follows from direct computations. q. e. d.

Concerning the structure of \( R \)-module \( H^n(\emptyset; A_1, A_2) \) we have:

**Proposition 4.3.** Let \( g \neq \emptyset \) be the order of \( \emptyset \). Then for any representations \( A_1, A_2 \),
\[
gH^n(\emptyset; A_1, A_2) = 0, \quad n > 0.
\]
In particular if \( g \) is a unit in \( R \),
\[
H^n(\emptyset; A_1, A_2) = 0, \quad n > 0.
\]
**Proof.** Let \( E(x_1, \ldots, x_n) \) be an \( n \)-cocycle, i.e.
\[
\delta E(x_1, \ldots, x_{n+1}) = A_1(x_1)E(x_2, \ldots, x_{n+1})
\]
\[
+ \sum_{i=1}^{n} (-1)^i E(x_1, \ldots, x_ix_{i+1}, \ldots, x_{n+1})
\]
\[
+ (-1)^{n+1} E(x_1, \ldots, x_n)A_2(x_{n+1}).
\]
Multiply \( A_2(x_{n+1}) \) from right and add over \( x_{n+1} \in \emptyset \) we have
\[
A_1(x_1) \sum_{x \in \emptyset} E(x_2, \ldots, x_n, x)A_2(x^{-1})
\]
\[
+ \sum_{i=1}^{n-1} (-1)^i \sum_{x \in \emptyset} E(x_1, \ldots, x_ix_{i+1}, \ldots, x_n)E(x^{-1})
\]
\[ + (-1)^n \sum_{x \in \mathfrak{g}} E(x_1, \ldots, x_{n-1}, x_n x) A_2(x^{-1}) + (-1)^{n+1} gE(x_1, \ldots, x_n) = 0. \]

If we put
\[
F(x_1, \ldots, x_{n-1}) = \sum_{x \in \mathfrak{g}} E(x_1, \ldots, x_{n-1}, x) A_2(x^{-1})
\]
in this equation, we have
\[
gE(x_1, \ldots, x_n) = (-1)^n \delta F(x_1, \ldots, x_n).
\]

q.e.d.

**Proposition 4.4.** If \( R \) is noetherian and \( R/gR \) is a finite ring, then
\[ \# H^n(\mathfrak{g}; A_1, A_2) < + \infty, \quad n > 0. \]

**Proof.** The \( R \)-module of \( n \)-cochains \( C^n(\mathfrak{g}; A_1, A_2) \) is a finite \( R \)-module. Since \( R \) is noetherian, its submodule of \( n \)-cocycles \( Z^n(\mathfrak{g}; A_1, A_2) \) is also a finite \( R \)-module, hence a priori \( H^n(\mathfrak{g}; A_1, A_2) \) is a finite \( R \)-module. Since by Prop. 4.3 any element \( E \in H^n(\mathfrak{g}; A_1, A_2) \) has finite order \( g E = 0 \). This with the hypothesis \( \# (R/gR) < + \infty \) implies
\[ \# H^n(\mathfrak{g}; A_1, A_2) < + \infty. \]

5. **Maschke pair.** We say that two representations \( A_1(x), A_2(x) \) of the group \( \mathfrak{g} \) in matrices with elements in a commutative ring \( R \) with unity element form a Maschke pair if
\[ H^1(\mathfrak{g}; A_1, A_2) = H^1(\mathfrak{g}; A_2, A_1) = 0, \]

By Prop. 4.3. if \( p \) is a prime which does not divide the order \( g \) of \( \mathfrak{g} \):
\[ g \equiv 0 \pmod{p} \]
and \( R \) is a field of characteristic \( p \) or \( R = \mathbb{Z}_p \) a ring of \( p \)-adic integers with \( p \mid p \), any two representations in \( R \) are Maschke pair.

Another example is:

**Proposition 5.1.** Let \( \Gamma = R[\mathfrak{g}] \) be the group ring of \( \mathfrak{g} \) with coefficients in \( R \). Assume that either representation module of \( A_1 \) be \( \Gamma \)-injective\(^4\) or that of \( A_2 \) be \( \Gamma \)-projective\(^4\), then
\[ H^1(\mathfrak{g}; A_1, A_2) = 0. \]

Notice that if a representation \( A(x) \) is a direct constituent of the regular representation then its representation module is \( \Gamma \)-projective.

\(^4\) These terminologies are those used in Cartan-Eilenberg's "Homological Algebra".
PROOF. We prove only in case that the representation module $A_2$ of the representation $A_2(x)$ is $\Gamma$-projective, since other case is similar.

By Prop. 4.2 to any element $E \in H^1(\mathfrak{g}; A_1, A_2)$ there corresponds an $R$-free $\Gamma$-module $B$ such that

$$0 \to A_1 \to B \to A_2 \to 0$$

is exact. By $\Gamma$-projectivity of $A_2$ there exists a $\Gamma$-homomorphism

$$\varphi : A_2 \to B$$

such that

$$A_2 \to B \to A_2$$

is the identity map.

Let a basis of $B$ be so chosen that

$$x(a_1, \ldots, a_r, b_1, \ldots, b_s) = (a_1, \ldots, a_r, b_1, \ldots, b_s) (A_1(x) \ E(x))$$

with $E(x) \in E$. Since $(a_1, \ldots, a_r, \varphi(b_1), \ldots, \varphi(b_s))$ is a basis of $B$, there exist two matrices $S, T$ with regular $S$ such that

$$(a_1, \ldots, a_r, \varphi(b_1), \ldots, \varphi(b_s)) = (a_1, \ldots, a_r, b_1, \ldots, b_s) (\begin{pmatrix} 1 & T \\ 0 & S \end{pmatrix})$$

Put

$$(a_1, \ldots, a_r, b_1, \ldots, b_s) (\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}) = (a_1, \ldots, a_r, c_1, \ldots, c_s).$$

Then $(a_1, \ldots, a_r, c_1, \ldots, c_s)$ is a basis of $B$ such that

$$x(a_1, \ldots, a_r, c_1, \ldots, c_s) = (a_1, \ldots, a_r, c_1, \ldots, c_s) (A_1(x) \ 0)$$

$$0 \ \ A_2(x)$$

By Prop. 4.2 this means $E = 0$. q. e. d.

6. Representations in $\wp$-adic fields. In this section, $\wp$ is a finite prime in an algebraic number field $k$, $\wp$ the ring of $\wp$-adic integers.

THEOREM 1 (HENSEL LEMMA). Let $A(x)$ be a representation of the group $\mathfrak{g}$ in matrices with elements in $\wp$. $\bar{A}(x)$ be the reduction mod $\wp$ of the representation $A(x)$. Assume in the modular field $\mathfrak{f}_\wp = \wp/\wp$ a direct decomposition:

$$\bar{A}(x) \sim \begin{pmatrix} \mathfrak{f}_\wp A_1(x) & 0 \\ 0 & \mathfrak{f}_\wp A_2(x) \end{pmatrix}$$
in which $\mathfrak{U}_1$, $\mathfrak{U}_2$ form a Maschke pair (§5) i.e.

$$H^1(\mathfrak{G} ; \mathfrak{U}_1, \mathfrak{U}_2) = H^1(\mathfrak{G} ; \mathfrak{U}_2, \mathfrak{U}_1) = 0.$$  

Then there exists a direct decomposition in $\mathfrak{v}_i$:

$$A(x) \sim \begin{pmatrix} A_1(x) & 0 \\ 0 & A_2(x) \end{pmatrix}$$

such that

$$\bar{A}_i(x) = \mathfrak{u}_i(x) \quad i = 1, 2.$$

**Proof.** Without loss of generality, we may assume

$$\bar{A}(x) = \begin{pmatrix} \mathfrak{u}_1(x) & 0 \\ 0 & \mathfrak{u}_2(x) \end{pmatrix}.$$  

Then the representation $A(x)$ has in $\mathfrak{v}_i$ the following form

$$A(x) = \begin{pmatrix} A_{11}(x) & \pi A_{12}(x) \\ \pi A_{21}(x) & A_{22}(x) \end{pmatrix}$$

where $\pi$ is a primitive element for the prime $\mathfrak{v}$, and $A_{ij}(x)$ are matrices with elements in $\mathfrak{v}_i$. We prove by induction that representation of the form:

$$\left( \begin{array}{cc} A_{11}(x) & \pi^n A_{12}(x) \\ \pi^n A_{21}(x) & A_{22}(x) \end{array} \right), \quad n > 0, m > 0$$

with $A_{ij}(x)$ matrices in $\mathfrak{v}_i$, can be transformed by a matrix of type:

$$\begin{pmatrix} 1 & \pi^n T \\ 0 & 1 \end{pmatrix}, \quad T \text{ in } \mathfrak{v}_i$$

into the form

$$\left( \begin{array}{cc} A_{11}(x) & \pi^{n+1} A_{12}(x) \\ \pi^n A_{21}(x) & A_{22}(x) \end{array} \right)$$

with matrices $A_{ij}(x)$ in $\mathfrak{v}_i$ such that

$$A_{ij}(x) \equiv A_{ij}'(x) \quad (\pi^{n+m}) \quad i = 1, 2$$

under the condition

$$H^1(\mathfrak{G} ; \mathfrak{u}_1, \mathfrak{u}_2) = 0.$$  

Similar result holds for $m$.

For, from

$$\left( \begin{array}{cc} A_{11}(x) & \pi^n A_{12}(x) \\ \pi^n A_{21}(x) & A_{22}(x) \end{array} \right) \left( \begin{array}{cc} 1 & \pi^n T \\ 0 & 1 \end{array} \right)$$

$$= \left( \begin{array}{cc} A_{11}(x) & \pi^n A_{11}(x) T + \pi^n A_{12}(x) \\ \pi^n A_{21}(x) & \pi^{n+m} A_{21}(x) T + A_{22}(x) \end{array} \right)$$
the condition for the matrix $T$ is
\[ A_{11}(x)T + A_{12}(x) = TA_{12}(x) \quad (\text{P}). \]
Since $A_{12}(x) \in Z'(C_3; \mathcal{U}_1, \mathcal{U}_2)$ is a 1-cocycle, by hypothesis on $\mathcal{U}_1, \mathcal{U}_2$ such matrix $T$ must exist in $\mathcal{U}_3$.

Starting from
\[ A(x) = \begin{pmatrix} A_{11}(x) & \pi A_{12}(x) \\ \pi A_{21}(x) & A_{22}(x) \end{pmatrix} \]
with $n = m = 1$ we arrive at the $\mathcal{U}_3$-equivalence
\[ A(x) \sim \begin{pmatrix} A_1(x) & 0 \\ 0 & A_2(x) \end{pmatrix} \]
with $A_i(x) = a_i(x)$ $i = 1, 2$. q. e. d.

**COROLLARY 5.** Let $\mathcal{U}$ be a directly indecomposable modular representation of the group $G$ contained in the regular representation. Then there exists a representation $U$ in $\mathcal{U}_3$ such that
\[ U(x) = \mathcal{U}(x). \]
For, in the modular field $\mathcal{U}_3$, the regular representation $R(x)$ in $\mathcal{U}_3$ splits as
\[ \bar{R}(x) \sim \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{B} \end{pmatrix} \]
with suitable modular representation $\mathcal{B}$. Thereby $\mathcal{U}, \mathcal{B}$ are represented by $\Gamma$-projective modules therefore form a Maschke pair.

**THEOREM 2.** Let the prime $\mathfrak{p}$ does not divide order $g$ of $G$. Then matrix representation $A(x)$ in $\mathcal{U}_3$ and $\mathcal{U}(x)$ in modular field $\mathfrak{f}_3 = \mathcal{U}_3/\mathfrak{p}$ are in one to one correspondences by reduction mod $\mathfrak{p}$:
\[ A(x) \rightarrow \bar{A}(x) = \mathcal{U}(x). \]
In other words any representation in $\mathcal{U}_3$ is completely reducible and there are as many irreducible representations in $\mathcal{U}_3$ as that in $\mathfrak{f}_3$.

**PROOF.** Complete reducibility follows from Prop. 4.3. If $A(x)$ is an irreducible representation in $\mathcal{U}_3$ then its reduction mod $\mathfrak{p}$: $\bar{A}(x)$ is also ir-

---

5) This result was announced by Brauer [3].
reducible.
For, suppose contrary to our assertion
\[ \overline{A}(x) \sim \begin{pmatrix} \mathcal{H}(x) & 0 \\ 0 & \mathcal{A}(x) \end{pmatrix} \]
then Hensel lemma would yield a decomposition
\[ A(x) \sim \begin{pmatrix} A_1(x) & 0 \\ 0 & A_2(x) \end{pmatrix} \]
in \( \mathfrak{o}_r \). This is a contradiction.
Conversely, assume \( \mathcal{H}(x) \) be an irreducible representation in \( \mathfrak{i}_r \), then the regular representation \( \mathcal{R}(x) \) splits as
\[ \mathcal{R}(x) \sim \begin{pmatrix} \mathcal{H}(x) & 0 \\ 0 & \mathcal{B}(x) \end{pmatrix} \]
Apply Hensel lemma to the regular representation \( R(x) \) in \( \mathfrak{o}_r \) with \( R(x) = \mathcal{R}(x) \) we have
\[ R(x) \sim \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix} \]
with \( \overline{A}(x) = \mathcal{H}(x) \). Of course \( A(x) \) is irreducible in \( \mathfrak{o}_r \). q.e.d.

**COROLLARY.** In case \( g \neq 0 (p) \). If two matrix representations \( A_1(x) \), \( A_2(x) \) are \( k \)-equivalent then they are \( \mathfrak{o}_r \)-equivalent.

**PROOF.** Since \( k_r \) is a field, ordinary theory of representations shows that
\[ A_1(x) \sim \begin{pmatrix} B_1(x) & 0 \\ 0 & B_2(x) \end{pmatrix} \sim A_2(x) \text{ in } k_r, \]
where \( B_1(x), \ldots, B_t(x) \) are irreducible representations in \( k_r \). Since \( \mathfrak{o}_r \) is a principal ideal domain, we may assume without loss of generality that \( B_1(x), \ldots, B_t(x) \) are matrices with elements in \( \mathfrak{o}_r \). From the Theorem 2
\[ A_t(x) \sim \begin{pmatrix} C_1(x) & 0 \\ 0 & C_t(x) \end{pmatrix} \text{ in } \mathfrak{o}_r \]
where \( C_1, \ldots, C_t \) are irreducible representations in \( \mathfrak{o}_r \). Comparing their characters, we see that \( C_1, \ldots, C_t \) are permutations of \( B_1, \ldots, B_t \) (By suitable \( \mathfrak{o}_r \)-transforms if necessary). The same is true for the representation \( A_t(x) \). Therefore
\[ A_t(x) \sim \begin{pmatrix} B_1(x) & 0 \\ 0 & B_t(x) \end{pmatrix} \sim A_t(x) \text{ in } \mathfrak{o}_r, \]
q.e.d.
Thus, the case $\psi$ with $g \equiv 0(\psi)$ are completely studied. We are therefore in a position to investigate the case $g \equiv 0(\psi)$. More precisely take integer $e_0 > 0$ such that

$$g \equiv 0 \ (\psi^{e_0})$$

$$g \not\equiv 0 \ (\psi^{e_0+1}).$$

**PROPOSITION 6.1 (PRINCIPAL GENUS THEOREM).** Assume $e \geq e_0$ and $A_1(x), A_2(x)$ are representations in $\mathfrak{g}_\psi$. If an $n$-cocycle $E \in Z^n(\mathfrak{g}; A_1, A_2)$ satisfies

$$E(x_1, \ldots, x_n) \equiv 0 \ (\psi^e)$$

then there exists an $(n-1)$-cochain $F \in C^{n-1}(\mathfrak{g}; A_1, A_2)$ such that

$$E = \delta F$$

with

$$F(x_1, \ldots, x_{n-1}) \equiv 0 \ (\psi^{e-e_0}).$$

**PROOF.** Since $E$ is an $n$-cocycle, by the proof of Prop. 4.3, if we put

$$F_1(x_1, \ldots, x_{n-1}) = \sum_{x \in \mathfrak{g}} E(x_1, \ldots, x_{n-1}, x) A_2(x^{-1})$$

then

$$gE = (-1)^n \delta F_1.$$  

From the hypothesis $E \equiv 0 (\psi^e)$ it follows that

$$F = (-1)^n \frac{1}{g} F_1$$

is indeed an $(n-1)$-cochain in $\mathfrak{g}_\psi$ satisfying

$$F(x_1, \ldots, x_{n-1}) \equiv 0 (\psi^{e-e_0})$$

$$E = \delta F \quad q. e. d.$$  

**PROPOSITION 6.2.** Let $A_1, A_2$ be two representations in $\mathfrak{g}_\psi$, and $e > e_0$ be an integer. Then equivalences:

$$A_1 \sim A_2 \quad \text{in } \mathfrak{g}_\psi/\psi^e$$

and

$$A_1 \sim A_2 \quad \text{in } \mathfrak{g}_\psi$$

are completely equivalent.

**PROOF.** Equivalence in $\mathfrak{g}_\psi$ implies equivalence in $\mathfrak{g}_\psi/\psi^e$ is trivial. Let us show the converse. Assume

$$A_1 \sim A_2 \quad \text{in } \mathfrak{g}_\psi/\psi^e.$$  

6) This proposition has some analogy to a result of Kuniyoshi-Takahashi [14].
In other words there exists a matrix $T$ in $\mathfrak{o}$ such that

$$A_1T - TA_2 \equiv 0 (\mathfrak{p}^e), \quad \det T \not\equiv 0 (\mathfrak{p}).$$

Then

$$E(x) = A_1(x)T - TA_2(x)$$

is a 1-cocycle $\in \mathbb{Z}^1(\mathfrak{p}; A_1, A_2)$ and

$$E(x) \equiv 0 (\mathfrak{p}^e).$$

Since $e > e_0$, we can apply principal genus theorem (Prop. 6.1) and it yields a matrix $S$ in $\mathfrak{o}$ such that

$$E(x) = A_1(x)S - SA_2(x)$$

$$S \equiv 0 (\mathfrak{p}^{e-e_0}).$$

If we put $T' = T - S$, then $T'$ is a matrix in $\mathfrak{o}$, such that

$$A_1(x)T' = T'A_2(x)$$

$$\det T' \equiv \det T \not\equiv 0 (\mathfrak{p})$$

i.e. $A_1(x)A_2(x)$ are $\mathfrak{o}$-equivalent. q. e. d.

7. Equivalence theory of $\Gamma$-lattices. In this section we use same notations as that of §2. Namely $k$ is an algebraic number field and $\mathfrak{o}$ the ring of integers in $k$. $\Gamma = \mathfrak{o}[\mathfrak{g}]$ is the group ring over $\mathfrak{o}$.

**Proposition 7.1.** There exists at least one $\Gamma$-lattice $M$ in $V$, if $V$ is a $\Gamma$-space.

**Proof.** If $V$ is written by a $k$-basis as

$$V = v_1k + \ldots + v_mk,$$

then the following finite $\mathfrak{o}$-module

$$M = \sum_{x \in \mathfrak{o}} \sum_{i=1}^m xv_i$$

is a $\Gamma$-lattice in $V$. q. e. d.

If $R \supseteq \mathfrak{o}$ is a ring over $\mathfrak{o}$, we put for a $\Gamma$-lattice $M$;

$$\{M; R/\mathfrak{o}\} = \{N \in \Gamma$-

lattices in $V$ $| NR \simeq MR$ as $\Gamma R$-modules$\}$.

In particular

$$\{|M; k/\mathfrak{o}|$$

is the set of all $\Gamma$-lattices in $V$, for any $\Gamma$-lattice $M$ in $V$.

Since $M_1, M_2 \in \{M; R/\mathfrak{o}\}$ lie in the same class $\{M; k/\mathfrak{o}|$, we can write

$$\{|M; k/\mathfrak{o}| = \{|M_1; R/\mathfrak{o}| + \ldots + \{|M_2; R/\mathfrak{o}|$$

as a disjoint union of finite or infinite number of subclasses. We put
and call it the class number of ℓ-lattices with respect to \( R \).

If \( K/k \) is an extension field with a maximal order \( \mathcal{O} \supseteq \mathfrak{o} \), we can define \( \Gamma\mathcal{O} \)-lattices in \( VK \) and the symbol
\[
| M ; R / \mathcal{O} |
\]
with a ring \( R \supseteq \mathfrak{B} \). There exists always a map
\[
| M ; R / \mathfrak{o} | \ni M_1 \to M / \mathfrak{O} \in | M ; R / \mathcal{O} |
\]
called injection.

Main examples of \( R \) and \( \mathcal{O} \) are:
\( K = k_{\mathfrak{p}} : \mathfrak{p} \)-adic completion of the field \( k \), \( \mathcal{O} = \mathfrak{o}_{\mathfrak{p}} : \mathfrak{p} \)-adic integers in \( k_{\mathfrak{p}} \),
\( R = \mathfrak{o}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r) = \prod_{i=1}^r (k / \mathfrak{o}_{\mathfrak{p}_i}) \supseteq \mathfrak{o} \) where \( \mathfrak{p}_1, \ldots, \mathfrak{p}_r \) are finite primes in \( k \).

**Proposition 7.2.** The injection
\[
| M ; k / \mathfrak{o} | \ni M \to | M_{\mathfrak{p}} ; k_{\mathfrak{p}} / \mathfrak{o}_{\mathfrak{p}} |
\]
is an onto map with same class number
\[
c(\mathfrak{o}_{\mathfrak{p}} / \mathfrak{o}) = c(\mathfrak{o}_{\mathfrak{p}} / \mathfrak{o}_{\mathfrak{p}}).
\]

**Proof.** Take an \( M_{\mathfrak{p}} \in | M_{\mathfrak{p}} ; k_{\mathfrak{p}} / \mathfrak{o}_{\mathfrak{p}} | \), we can define a \( \Gamma \)-lattice \( M_1 \in | M ; k / \mathfrak{o} | \) such that \( M_1 \mathfrak{p} = M_{\mathfrak{p}} \). Namely, let \( M \) be a \( \Gamma \)-lattice in \( V \). Put
\[
M_1 \mathfrak{p} = M_{\mathfrak{p}} \quad \text{for prime } \mathfrak{p} \nmid \mathfrak{p}.
\]
Then
\[
M_1 = \bigcap_{\mathfrak{p} \nmid \mathfrak{p}} (M_{\mathfrak{p}} \cap V)
\]
is a desired \( \Gamma \)-lattice with \( M_1 \mathfrak{p} = M_{\mathfrak{p}} \) by Prop. 1.3.
As to class numbers \( c(\mathfrak{o}_{\mathfrak{p}} / \mathfrak{o}) \), \( c(\mathfrak{o}_{\mathfrak{p}} / \mathfrak{o}_{\mathfrak{p}}) \),
\[
M_1, M_{\mathfrak{p}} \in | M_{\mathfrak{p}} ; \mathfrak{o}_{\mathfrak{p}} |
\]
imply \( M_1 \mathfrak{o}_{\mathfrak{p}} \simeq M_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}} \) as \( \Gamma \mathfrak{o}_{\mathfrak{p}} \)-modules.
Therefore
\[
M_1 \mathfrak{o}_{\mathfrak{p}}, M_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}} \in | M_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}, \mathfrak{o}_{\mathfrak{p}} / \mathfrak{o}_{\mathfrak{p}} |
\]
and conversely.

**Proposition 7.3.** For any \( \Gamma \)-lattice \( M \)
\[
| M ; \mathfrak{o}(\mathfrak{p}) / \mathfrak{o} | = | M ; \mathfrak{o}_{\mathfrak{p}} / \mathfrak{o} |.
\]

**Proof.** Since \( M_1 \mathfrak{o}(\mathfrak{p}) \simeq M_{\mathfrak{p}} \mathfrak{o}(\mathfrak{p}) \) as \( \Gamma \mathfrak{o}(\mathfrak{p}) \)-modules implies \( M_1 \mathfrak{o}_{\mathfrak{p}} \simeq M_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}} \) as

---

7) This and following Prop. 7.3 give a proof for locality of Maranda [16]'s concepts of \( \mathfrak{p} \)-equivalence and genus, noticed in the introduction.
\( \Gamma_\nu \)-modules, it is trivial that
\[
[M; \varnothing(\nu)/\varnothing] \subseteq [M; \varnothing/\varnothing].
\]
Conversely, suppose \( M_1, M_2 \in [M; \varnothing/\varnothing] \).
Since \( \varnothing(\nu) \) is a principal ideal domain, we can write
\[
\begin{align*}
M_1\varnothing(\nu) &= u_1\varnothing(\nu) \oplus \ldots \oplus u_m\varnothing(\nu) \\
M_2\varnothing(\nu) &= v_1\varnothing(\nu) \oplus \ldots \oplus v_m\varnothing(\nu)
\end{align*}
\]
with matrix representations with elements in \( \varnothing(\nu) \):
\[
\begin{align*}
xu &= uA_1(x) \\
xv &= vA_2(x).
\end{align*}
\]
The \( \Gamma_\nu \)-isomorphism \( \varphi : M_1\varnothing_1 \rightarrow M_2\varnothing_2 \) can be written as
\[
\varphi(\nu) = u \cdot T
\]
with matrix \( T \) in \( \varnothing \) such that \( \det T \not\equiv 0 (\nu) \).
In terms of matrix representations \( A_1(x), A_2(x) \) we have
\[
A_1(x)T = TA_2(x).
\]
Take an exponent \( e > e_0 \) with \( g = \prod v_i \equiv 0 (\varnothing^e) \) but \( g \not\equiv 0 (\varnothing^{e+1}) \), there exists a matrix \( T \) in \( \varnothing \) such that
\[
T_1 \equiv T (\varnothing^e).
\]
Consider a 1-cocycle
\[
E(x) = A_1(x)T_1 - T_1A_2(x) \equiv 0 (\varnothing^e)
\]
in \( \varnothing(\nu) \). By the principal genus theorem\(^8\) (Prop. 6.1) we can find a matrix \( S \) in \( \varnothing(\nu) \) such that
\[
E(x) = A_1(x)S - SA_2(x)
\]
with \( S \equiv 0 (\varnothing^{e-e_0}) \) and hence \( S \equiv 0 (\varnothing) \).
Then \( T_2 = T_1 - S \) is a matrix in \( \varnothing(\nu) \) intertwines \( A_1(x), A_2(x) \):
\[
A_1(x)T_2 = T_2A_2(x)
\]
such that
\[
\det T_2 \equiv \det T_1 \equiv \det T \not\equiv 0 (\varnothing).
\]
Therefore the new map
\[
\varphi(\nu) = u \cdot T_2
\]
is a \( \Gamma\varnothing(\nu) \)-isomorphism \( M_1\varnothing(\nu) \cong M_2\varnothing(\nu) \) i.e.
\[
M_1, M_2 \in [M; \varnothing(\nu)/\varnothing].
\]
q. e. d.

\textbf{Proposition 7.4.} If \( \nu_1, \ldots, \nu_r \) are finite primes in \( k \),

\(^8\) This holds for the ring \( \varnothing(\nu) \) instead of \( \varnothing_2 \) if we consider its proof.
\[ |M; \mathfrak{o}(v_1, \ldots, v_r)/\mathfrak{o}| = \bigcap_{i=1}^{r} |M; \mathfrak{o}(v_i)/\mathfrak{o}|. \]

**Proof.** From preceding Prop. 7.3 we have only to prove

\[ |M; \mathfrak{o}(v_1, \ldots, v_r)/\mathfrak{o}| = \bigcap_{i=1}^{r} |M; \mathfrak{o}(v_i)/\mathfrak{o}|. \]

Since \( \mathfrak{o}(v_1, \ldots, v_r) \subseteq \mathfrak{o}(v_i) \), it is clear that

\[ |M; \mathfrak{o}(v_1, \ldots, v_r)/\mathfrak{o}| \subseteq \bigcap_{i=1}^{r} |M; \mathfrak{o}(v_i)/\mathfrak{o}|. \]

Take an \( M_i \in \bigcap_{i=1}^{r} |M; \mathfrak{o}(v_i)/\mathfrak{o}| \) and put

\[ \mathfrak{o}' = \mathfrak{o}(v_1, \ldots, v_r). \]

Since \( \mathfrak{o}' \) is a principal ideal domain, we can express the proposition, if we take suitable \( \mathfrak{o}' \)-basis of \( \Gamma \)-lattices in consideration, by words of matrix representations. Namely, if \( A_1(x), A_2(x) \) be two matrix representations in \( \mathfrak{o}' \), such that there exist matrices \( T_i \) in \( \mathfrak{o}(v_i) \) (\( i = 1, \ldots, r \)) with \( \det T_i \neq 0 \) (\( v_i \)) and

\[ A_1(x)T_i = T_iA_2(x) \quad i = 1, \ldots, r, \]

we can find a matrix \( T \) in \( \mathfrak{o}' \) with \( T^{-1} \) in \( \mathfrak{o}' \) and

\[ A_1(x)T = TA_2(x). \]

Take elements \( \omega_i \in \mathfrak{o}' \) such that

\[ \omega_i \equiv 0 (v_i), \quad \omega_i \equiv 0 (v_j) \quad j \neq i, 1 \leq i, j \leq r, \]

whose exponents \( e_j > 0 \) are taken as

\[ \pi_j^{e_j}T_i \equiv 0 (v_j) \]

with primitive element \( \pi_j \) of \( v_j \).

Then the matrix

\[ T = \sum_{i=1}^{r} \omega_i T_i \]

is a desired matrix in \( \mathfrak{o}' \). Since

\[ \det T = \det \omega_j T_j = \omega_j^{e_j} \det T_j \neq 0 (v_j) \quad j = 1, \ldots, r. \]

**Proposition 7.5.** If a finite prime \( v_r \) is different from \( v_1, \ldots, v_{r-1} \), then

\[ |M_i; \mathfrak{o}(v_1, \ldots, v_{r-1})/\mathfrak{o}| \cap |M_2; \mathfrak{o}(v_r)/\mathfrak{o}| \neq \emptyset \]
for any $\Gamma$-lattices $M_1, M_2$ in $V$.

PROOF. Put $\mathfrak{o}' = \mathfrak{o}(p_1, \ldots, p_r)$. This is a principal ideal domain and each ideal in $\mathfrak{o}'$ is of the form:

$$
\left( \prod_{i=1}^{r} \pi_i^{e_i} \right)
$$

with primitive elements $\pi_i$ of $\mathfrak{o}$ with $\pi_j \neq 0 (p_i)$ for $i \neq j$. We can also prove the proposition by words of matrix representations. Since two matrix representations $A_i(x), A_3(x)$ in $\mathfrak{o}'$ are $k$-equivalent, there exists a non-singular matrix $T$ such that

$$
A_i(x)T = TA_3(x)
$$

with elements in $\mathfrak{o}$ if we multiply $T$ by an element in $\mathfrak{o}$ if necessary.

By elementary divisor theory in $\mathfrak{o}'$ we can find "unimodular" matrices $R, S$ in $\mathfrak{o}'$ such that

$$
RTS = \left( \prod_{i=1}^{r} \pi_i^{e_{i1}} \right) \left( \prod_{i=1}^{r} \pi_i^{e_{im}} \right)
$$

with exponents

$$
e_{i1} \leq \ldots \leq e_{im}, \quad i = 1, \ldots, r.
$$

Put $RTS = T_1T_2$ with

$$
T_1 = \left( \begin{array}{cc}
\pi_r^{e_{r1}} & 0 \\
0 & \pi_r^{e_{rm}}
\end{array} \right), \quad T_2 = \left( \begin{array}{cc}
\prod_{i=1}^{r-1} \pi_i^{e_{i1}} & 0 \\
0 & \prod_{i=1}^{r-1} \pi_i^{e_{im}}
\end{array} \right)
$$

then these are matrices in $\mathfrak{o}'$ such that

$$
\det T_1 \neq 0 (p_i) \quad 1 \leq i \leq r - 1; \quad \det T_2 \neq 0 (p_r)
$$

From the computations:

$$
RA_1(x)R^{-1} \cdot RTS = RTS \cdot S^{-1}A_3(x)S
$$

$$
T_1^{-1}RA_1(x)R^{-1} \cdot T_1 = T_2S^{-1}A_3(x)S \cdot T_2^{-1} = A_3(x)
$$

we see that $A_i(x)$ and $A_3(x)$ are $\mathfrak{o}(p_1, \ldots, p_{r-1})$-equivalent while $A_3(x)$ and $A_3(x)$ are $\mathfrak{o}(p_r)$-equivant.

If we write $M_3$ for a $\Gamma$-lattice which represents $\emptyset$ by matrices $A_3(x)$, we have

$$
M_3 \in \{ M_1; \mathfrak{o}(p_1, \ldots, p_{r-1}) \}$$
THEOREM 3. If \( p_1, \ldots, p_r \) are mutually different finite primes in \( k \), then we have for class numbers:

\[
c(0(p_1, \ldots, p_r)/r) = \prod_{i=1}^{r} c(0(p_i)/r).
\]

PROOF. It will be sufficient to prove

\[
c(0(p_1, \ldots, p_r)/r) = \prod_{i=1}^{r} c(0(p_i)/r).
\]

We prove this by induction on \( r \). For \( r = 1 \) this is trivial. Let \( r > 1 \), we have by definition:

\[
\{M; k/o\} = \{M_1; 0(p_1, \ldots, p_{r-1})/o\}
+ \ldots + \{M_r; 0(p_1, \ldots, p_{r-1})/o\}
= \{N_1; 0(p_r)/o\} + \ldots + \{N_r; 0(p_r)/o\},
= \sum_{i,j} [\{M_i; 0(p_1, \ldots, p_{r-1})/o\} \cap \{N_j; 0(p_r)/o\}]
\]

with \( c = c(0(p_1, \ldots, p_{r-1})/o) \) and \( d = c(0(p_r)/o) \).

From the preceding Prop. 7.5 we have

\[
\{M_i; 0(p_1, \ldots, p_{r-1})/o\} \cap \{N_j; 0(p_r)/o\} \neq \phi.
\]

If we take a \( \Gamma \)-lattice \( M_{ij} \) in this intersection we have

\[
\{M_i; 0(p_1, \ldots, p_{r-1})/o\} \cap \{N_j; 0(p_r)/o\}
= \{M_{ij}; 0(p_1, \ldots, p_r)/o\}
\]

by Prop. 7.4.

Since

\[
\{M; k/o\} = \sum_{i,j} \{M_{ij}; 0(p_1, \ldots, p_r)/o\}
\]

is disjoint, we have finally

\[
c(0(p_1, \ldots, p_r)/o) = c(0(p_1, \ldots, p_{r-1})/o) \cdot c(0(p_r)/o).
\]

q. e. d.

8. **Genus of representations.** Let \( \tilde{k} \) be the adèle ring (or ring of valuation vectors) of \( k \). \( \tilde{0} \) denotes subring of \( \tilde{k} \) consists of all integral elements of \( \tilde{k} \) i.e. a direct sum

\[
\tilde{0} = \sum_p 0_p
\]

of all \( p \)-adic integers \( 0_p \) for finite primes \( p \) and \( 0_v = k_v \) for infinite primes
As in the preceding §7, we define
\[ \langle M; \tilde{o}/0 \rangle \]
and call Γ-lattices in them as belonging to the same genus. The class number \( j = c (\tilde{o}/0) \) defined by
\[ |M; k/0| = |M_1; \tilde{o}/0| + \cdots + |M_j; \tilde{o}/0| \]
is called the genus number of Γ-lattices in \( V \).

**Theorem 4.** Let \( g = \# \mathcal{O} \) be the order of \( \mathcal{O} \), then for any Γ-lattice \( M \) in \( V \)
\[ |M; \tilde{o}/0| = \bigcap_{p \in \mathcal{O}} |M; o_p/0|. \]
From this we have
\[ j = \prod_{p \in \mathcal{O}} c(o_p/0) < + \infty. \]

**Proof.** \( M_1, M_2 \in |M; \tilde{o}/0| \) imply by definition
\[ M_1 \tilde{o} \simeq M_2 \tilde{o} \]
as Γ\( \mathcal{O} \)-modules. Since \( \tilde{o} = \sum_p o_p \) is a direct sum, we have for all primes \( p \)
\[ M_p o_p \simeq M_p o_p \]
as Γ\( \mathcal{O} \)-modules. Since this is trivially verified for infinite primes \( p = p_{\infty} \), it is
sufficient to prove that if \( p \nmid g \)
\[ |M; k/0| = |M; o_p/0|. \]
But this follows at once from Coroll. to Theorem 2. The formula for \( j \) follows from
\[ |M; \tilde{o}/0| = \bigcap_{p \in \mathcal{O}} |M; o_p/0| = |M; o(v_1, \ldots, v_r)/0|. \]
if we write \( v_1, \ldots, v_r \) for all different primes dividing \( g \).
Finally finiteness of \( c(o_p/0) \) follows from Prop. 6.2.

**9. Class number in a genus.** Let \( V \) be a vector space over \( k \), which
has as in preceding sections \( \mathcal{O} \) as left operators and induces a representation
\[ \mathcal{O} \ni x \rightarrow A(x) \in GL(V; k) \]
by automorphism of \( V \).
Similarly, for any prime \( p \), the \( p \)-extension \( V_p = Vk_p \) induces a representation
which we write by the same symbol
\[ A(x) \in GL(V_p; k_p). \]
Moreover, the vector space $\widetilde{V} = V\mathcal{k}$ over adèle ring $\mathcal{k}$ of $k$ induces a representation which will be also written by $$A(x) \in GL(\widetilde{V}; \mathcal{k}).$$ There group $GL(\widetilde{V}; \mathcal{k})$ consists of elements $$S = (S_v), S_v \in GL(V_v; k_v)$$ such that except for a finite set of primes, $S_v$ being $v$-unimodular. Now, $$G = v(A(\emptyset)) = \{ S \in GL(V; k) | A(x)S = SA(x) \text{ for all } x \in \emptyset \}$$ is an algebraic group of automorphisms of $V$. Its idèle group is given by $$\widetilde{G} = v(A(\emptyset)) = \{ \tilde{S} \in GL(\widetilde{V}; \mathcal{k}) | A(x)\tilde{S} = \tilde{S}A(x) \text{ for all } x \in \emptyset \}.$$ $\widetilde{G}$ contains $G$ as a discrete subgroup with its natural topology.

Let $M$ be a lattice in $V$. We define $M \cdot \tilde{S}$ with $\tilde{S} \in GL(\widetilde{V}; \mathcal{k})$ by $$M \cdot \tilde{S} = \bigcap_v (V \cap M_v S_v) \text{ if } \tilde{S} = (S_v).$$ It is readily seen that $M \cdot \tilde{S}$ is a lattice. Moreover if $M$ is a $\Gamma$-lattice and $\tilde{S} \in \widetilde{G}$ then $M \cdot \tilde{S}$ is also a $\Gamma$-lattice.

**PROPOSITION 9.1.** Let $M$ be a $\Gamma$-lattice in $V$, then $$|M; \emptyset/\emptyset| = |M \cdot \tilde{S}; \emptyset/\emptyset| \tilde{S} \in \widetilde{G}.$$  

**PROOF.** "The fact that $M \cdot \tilde{S}$ is a also a $\Gamma$-attice" is already remarked. $M \cdot \tilde{S}$ is contained in $|M; \emptyset/\emptyset|$. For if we fix a prime $v$, then $$(M \tilde{S})_v = M_v S_v$$ $$\varphi_v; M_v \to M_v S_v$$ is a $\Gamma_{0,v}$-isomorphism by virtue of $$A(x)S_v = S_v A(x)$$ for all $x \in \emptyset$. Conversely, take an $M_1 \in |M; \emptyset/\emptyset|$ arbitrarily. For any prime $v$, we have by definition: $$M_{1v} \cong M_v \text{ as } \Gamma_{0,v}\text{-modules.}$$ Since these are $0_v$-free modules, we can find $S_v \in GL(V_v; k)$ such that $$M_{1v} = M_v S_v.$$ From the fact that $M, M_1$ are lattices in $V$ it follows that $S_v$ are $v$-unimodul-

9) Idèle group of an algebraic group was considered by Ono [17], Tamagawa and Weil.
lar except for a finite number of primes, i.e.
\[ \widetilde{S} = (S_{i}) \in GL(V; \overline{k}). \]

Now, for any prime \( \wp \) we have
\[ xM_{1\wp} = M_{1\wp}A(x) \]
\[ xM_{\wp} = M_{\wp}A(x) \]
hence \( A(x)S_{i} = S_{i}A(x) \). This shows that \( \widetilde{S} \in \widetilde{G} \) and
\[ M_{i} = M \cdot \widetilde{S}. \]
q. e. d.

**Proposition 9.2.** Let \( M \) be a \( \Gamma \)-lattice in \( V \), then
\[ \{M; \, 0/0\} = \{MS \mid S \in G\}. \]

**Proof.** If \( S \in G \), then the fact \( M \to M \cdot S \) is a \( \Gamma \)-isomorphism is trivial. Take an \( M_{i} \in \{M; \, 0/0\} \) arbitrarily, there exists a \( \Gamma \)-isomorphism
\[ \varphi : M \to M_{i}. \]
Since lattices in \( V \) generate \( V \) over \( k \) and are regular \( \wp \)-modules, we can generate \( V \) extend \( \varphi \) uniquely to a \( \Gamma \)-isomorphism\( ^{10} \)
\[ \varphi : M_{k} = V \to M_{i}k = V. \]
Therefore there exists \( S \in GL(V; k) \) such that
\[ M_{i} = MS. \]
Finally \( \Gamma \)-isomorphism of \( \varphi \) implies \( S \in G \).
q. e. d.

**Theorem 5.** Let \( M \) be a \( \Gamma \)-lattice in \( V \). Put
\[ \overline{U} = \{\overline{T} \in \overline{G} \mid M\overline{T} = M\} \]
for a subgroup which fixes \( M \). Then classes in a genus
\[ \{M; \, 0/0\} = \{M_{1}; \, 0/0\} + \ldots + \{M_{i}; \, 0/0\} \]
are in one to one correspondences with double cosets
\[ \overline{U}\backslash \overline{G}/\overline{G} \]
of \( \overline{G} \) with respect to two subgroups \( \overline{U} \) and \( G \). Explicitly, its correspondences are given by
\[ \overline{G} \ni \overline{S} \to M \cdot \overline{S} \in \{M; \, 0/0\} \]
\[ M\overline{S}_{1} \simeq M\overline{S}_{2} \text{ as } \Gamma \text{-lattices}, \]
if and only if
\[ \overline{S}_{1} = T\overline{S}_{2} \cdot S \]
with suitable \( \overline{T} \in \overline{U}, S \in G \).

\(^{10}\) The proof is straightforward e. g. Chevalley [6].
PROOF. That the mapping
\[ \tilde{G} \ni \tilde{s} \mapsto M\tilde{s} \in \left\{ M; \tilde{o}/\tilde{o} \right\} \]
is onto was already given by Prop. 9.1.
From
\[ M\tilde{s}_1 \simeq M\tilde{s}_2 \text{ as } \Gamma\text{-lattices,} \]
we can find by Prop. 9.2 and \( S \in G \) such that
\[ M\tilde{s}_1 = M\tilde{s}_2 \cdot S. \]
This finally means an existence of \( \tilde{T} \in \tilde{U} \) with
\[ \tilde{s}_1 = \tilde{T} \cdot \tilde{s}_2 \cdot S. \]
q. e. d.

Notice that in a recent paper by Ono [17] it was proved that the number of double cosets \( \# \tilde{U}\backslash \tilde{G}/G \) is always finite if \( G \) is a commutative algebraic group.

10. Absolutely irreducible representations. In the preceding §9, we have seen that class number in a genus is expressible as the number of double cosets
\[ \tilde{U} \backslash \tilde{G}/G \]
of a suitable algebraic group \( G \) of automorphisms.
In this and following sections we shall consider more closely this double cosets.

PROPOSITION 10.1. If \( M \) is a lattice in \( V \), then the ring
\[ R = \{ \alpha \in k \mid M\alpha \subseteq M \} \]
coincides with \( \hat{o} \).

PROOF. Since \( M \) is an \( \hat{o} \)-module, \( M\hat{o} \subseteq M \), therefore
\[ R \ni \hat{o}. \]
Take an \( \alpha \in k \) such that \( M\alpha \subseteq M \). We have to show for any finite prime \( v \) that
\[ \alpha \in \hat{o}. \]
Since \( \hat{o} \) is a principal ideal domain we can write
\[ M_v = u_1\hat{o}_v \oplus \ldots \oplus u_m\hat{o}_v \]
as a direct sum. \( M_v\alpha \subseteq M_v \) implies in particular
\[ u_1\alpha = u_1\beta_1 + \ldots + u_m\beta_m \]
with \( \beta_i \in \hat{o}_v \). Take \( \gamma \neq 0, \gamma \in \hat{o}_v \) such that \( \alpha\gamma \in \hat{o}_v \), then
\[ u_1\alpha\gamma = u_1\beta_1\gamma + \ldots + u_m\beta_m\gamma \]
THEOREM 6. If $V$ is an absolutely irreducible space and $M$ is a $\Gamma$-lattice in $V$, then

\[ G = \tilde{a}I \text{ with } \tilde{a} \in J = J(k) \]

\[ U = \tilde{e}I \text{ with } \tilde{e} \in U = U(k) \]

where, $J(k)$ is the group of ideles of $k$ with principal ideles $k^\times$ and units ideles $U(k)$. Therefore

\[ U \cong \text{absolute ideal class group of } k. \]

PROOF. Since $V$ is absolutely irreducible, so also is $V_\nu$ for any prime $\nu$. Therefore the structures of $G$ and $G$ are as in the theorem. For the structure of $U = \tilde{a}I$, $\tilde{a} \in U(k)$, we have to notice Prop 10.1 or more precisely its proof, since by definition

\[ U = \{ aI | \alpha \in J, M\alpha = M \}. \quad \text{q.e.d.} \]

COROLLARY. If $V$ is absolutely irreducible and $M$ is a $\Gamma$-lattice in $V$, then the class number $c = c(\nu/0)$:

\[ \{ M; k/0 \} = \{ M_1; \nu/0 \} + \ldots + \{ M_c; 0/0 \} \]

can be expressed as

\[ c = \prod_{\nu/0} j(\nu) \cdot h \]

where

\[ j(\nu) = c(\nu/\nu) \]

is the local class number and

\[ h = h(k) \]

is the number of absolute classes of ideals in $k$. In particular

\[ c < +\infty. \]

11. Irreducible representations. Let $V$ be an irreducible representation space over $k$. The group $\mathcal{O}$ is represented by automorphisms of $V$ as

\[ \mathcal{O} \ni x \rightarrow A(x) \in GL(V; k). \]

Put the enveloping algebra
and commuting algebra $D$ defined by

$$D = \{ S \mid \forall x \in \emptyset; A(x)S = SA(x) \} \subseteq \mathcal{E}(V; k)$$

where $\mathcal{E}(V; k)$ is the endomorphism algebra of $V$ over $k$. Since $V$ is irreducible, $D$ is a division algebra and $A_k$ is a full matrix algebra over the division algebra $D^*$ inversely isomorphic to $D$.

**Proposition 11.1.** Let $M$ be a $\Gamma$-lattice in $V$, then

$$\mathcal{O} = \mathcal{O}(M) = \{ S \in D \mid MS \subseteq M \}$$

is an order in $D$.

**Proof.**

a) Since $M$ is an $\emptyset$-module, $\mathcal{O}$ contains $\emptyset$. b) Any element $S \in \mathcal{O}$ is integral over $\emptyset$. For, let

$$f(S) = S^n + \alpha_1 S^{n-1} + \ldots + \alpha_n = 0 \quad (\alpha_i \in k)$$

be the irreducible equation in $k$ satisfied by $S$ and $S = S^{(1)}, \ldots, S^{(n)}$ be the conjugates of $S$ over $k$. In the extended vector space $Vk(S^{(1)}, \ldots, S^{(n)})$

we have

$$MS^{(i)} \subseteq M \quad i = 1, \ldots, n.$$ 

Since $\alpha_i$ are symmetric functions of $S^{(j)}$s we have

$$M \alpha_i \subseteq M.$$ 

Therefore $\alpha_i \in \emptyset$ by Prop. 10.1.

c) $\mathcal{O} = D$. For, take an $S \in D$, $S \neq 0$, arbitrarily. Since

$$MS$$

is a $\Gamma$-lattice in $V$, we can find $\alpha \in \emptyset$ such that

$$MS \alpha \subseteq M.$$ 

This shows that $S \alpha \in \mathcal{O}$. q. e. d.

We say that $M$ is maximal if

$$\mathcal{O} = \mathcal{O}(M)$$

is a maximal order in $D$.

Any $\Gamma$-lattice can be embedded in a maximal $\Gamma$-lattice. Namely,

**Proposition 11.2.** If $\mathcal{O}$ is a maximal order containing $\mathcal{O} = \mathcal{O}(M)$, then

$$M^\perp = MS^\perp$$

is a maximal $\Gamma$-lattice in $V$, with
\[ \mathcal{O}(M^-) = \mathcal{O}^- . \]

**Proof.** Since \( \mathcal{O}^- \) is a finite \( \mathfrak{o} \)-module, \( M^- \) is a lattice in \( V \). From
\[ MA(x) = M\mathcal{O}^- A(x) = MA(x)\mathcal{O}^- \subseteq M\mathcal{O}^- = M^- \]
\( M^- \) is a \( \Gamma \)-lattice. And finally
\[ M^\perp \mathcal{O}^- = M\mathcal{O}^- \mathcal{O}^- = M\mathcal{O}^- = M^- \]
implies
\[ \mathcal{O}(M^-) = \mathcal{O}^- . \]
By Prop. 11.1 \( \mathcal{O}(M^-) \) is an order in \( D \) it follows from maximality of \( \mathcal{O}^- \) that
\[ \mathcal{O}(M^-) = \mathcal{O}^- . \] q. e. d.

**Theorem 7.** If \( M \) is a maximal \( \Gamma \)-lattice in an irreducible representation space \( V \) over \( k \), then the double cosets
\[ \widetilde{U} \ G/G \]
of Theorem 5 correspond in one to one way to the \( \mathcal{O} = \mathcal{O}(M) \) left ideal classes in the commuting algebra \( D \) of \( A(x)'s \).

**Proof.** Since \( \mathcal{O} = \mathcal{O}(M) \) is a maximal order in \( D \), \( G \) is the idele group\(^{11} \) of the division algebra \( D \). The correspondences:
\[ \widetilde{G} \ni \widetilde{S} \rightarrow a(\widetilde{S}) = \bigcap_{\mathfrak{p}} (\mathfrak{p}, \mathcal{O}) \cap D \]
are onto \( \mathcal{O} \)-left ideals in \( D \). Its kernel is just
\[ \widetilde{U} = \{ \widetilde{T} | M\widetilde{T} = M \} \text{ i.e. } a(\widetilde{T}\widetilde{S}) = a(\widetilde{S}). \]
Therefore, double cosets
\[ \widetilde{U} \backslash \widetilde{G}/G \]
corresponds in one to one way to \( \mathcal{O} \)-left ideal class i.e.
\[ a(\widetilde{T}\widetilde{S} \cdot S) = a(\widetilde{S}) \cdot S \]
with \( \widetilde{T} \in \widetilde{U} \), \( \widetilde{S} \in \widetilde{G} \), \( S \in G \). q. e. d.

**Corollary.** In addition to the assumptions on the Theorem 7, suppose \( D \) has degree > 2 or ramified infinite primes, then the class number
\[ |M; \mathfrak{o}/\mathfrak{o}| = |M_1; \mathfrak{o}/\mathfrak{o}| + \ldots + |M_c; \mathfrak{o}/\mathfrak{o}| \]
can be expressed as
\[ c = \prod_{\mathfrak{p} | \mathfrak{o}} j(\mathfrak{p}) \cdot h \]
where \( j(\nu) = c(\nu_0/\nu_0) \) are local class numbers and \( h \) is the number of absolute ideal classes of the center \( K \) of \( D \).

**Proof.** This follows from Theorem 7 and a theorem of Eichler\(^{12}\) concerning class number of algebras. q.e.d.

**Theorem 8.** Let \( M \) be an arbitrary \( \Gamma \)-lattice in irreducible \( V \), then the number of double cosets

\[
\tilde{U} \backslash \tilde{G}/G
\]

is always finite.

**Proof.** Let \( M^- \supset M \) be a maximal \( \Gamma \)-lattice in \( V \). Then the number

\[
\# \tilde{U}^- \backslash \tilde{G}/G,
\]

as a class number of \( \mathcal{O}^- = \mathcal{O}(M^-) \)-left ideals of \( D \), is finite.

Since \( \tilde{U}^- \supset \tilde{U} \) it is sufficient to prove

\[
[\tilde{U}^- : \tilde{U}] < +\infty.
\]

Since \( M^- \supset M \) are lattices, except for a finite set of primes we have

\[
M_p^- = M_p
\]

and hence

\[
\]

Take an exceptional prime \( \nu \). \( U_p^- \supset U_p \) are compact and open subgroups in \( D_p^+ \), therefore

\[
[U_p^- : U_p] < +\infty.
\]

q.e.d.

**12. Some examples.** Let \( \mathbb{G} = \mathbb{Z}/(n) \) be a cyclic group of order \( n \). Consider faithful irreducible integral representation in the field of rationals \( \mathbb{Q} \).

Let \( V \) be a representation space of dimension

\[
m = \varphi(n)
\]

\[
A_h^{(n-1)} = \sum_{i=0}^{n-1} A(x^i)\mathbb{Q} = D \cong K = \mathbb{Q}(\xi)
\]

where \( \xi \) is a primitive \( n \)-th roots of unity.

It is readily seen that

\[
A_h \ni A(x) \rightarrow \xi \in K
\]

is an isomorphism over \( \mathbb{Q} \), if \( x \in \mathbb{G} \) is a fixed generator.

**Proposition 12.1.** Any \( \Gamma \)-lattice \( M \) in \( V \) is maximal.

\(^{12}\) Eichler [9], \( n=2 \) and total definite case was also treated by him [8].
PROOF. By definition
\[ \mathcal{O} = \{ S \in D \mid MS \subseteq M \}. \]

As a $\Gamma$-module:
\[ MA(x^i) \subseteq M \]
therefore we have
\[ \mathcal{O} \supseteq \sum_{i=0}^{n-1} A(x^i)Z. \]

Since \( \mathbb{Z}[\zeta] = \sum_{i=0}^{n-1} \zeta^iZ \) is the maximal order of \( K = \mathbb{Q}(\zeta) \) we see that
\[ \mathcal{O} = \sum_{i=0}^{n-1} A(x^i)Z \]
is the maximal order of \( D \).

The class number defined by
\[ [M; \mathbb{Q}/\mathbb{Z}] = [M_i; \mathbb{Z}/\mathbb{Z}] + \ldots + [M_c; \mathbb{Z}/\mathbb{Z}] \]
is therefore given by
\[ c = \prod_{p|n} j(p) \cdot h \]
where
\[ h = h(\mathbb{Q}(\zeta)) \]
is the absolute ideal class number of the field of \( n \)-th roots of unity.

Now consider \( j(p) \). If \( n \) is a prime power and
\[ n \equiv 0 \pmod{p} \]
then
\[ (p - 1, n) = 1 \]
i.e. \( GF(p) \) contains no \( n \)-th roots. Therefore \( p \)-modular representation of \( A(x) \) for \( n \equiv 0 \pmod{p} \) are irreducible. By a theorem of Brauer\(^{13})\)
\[ j(p) = 1 \]

And hence
\[ c = h. \]

As a next example, consider the symmetric group
\[ S_3, \]
of order \( g = 6 \) in the field of rationals \( \mathbb{Q} \). Let \( A(x) \) be the 2-dimensional absolutely irreducible representation with \( \Gamma \)-lattice \( M \).

---

\(^{13}\) Brauer [4], Theorem 10 or Artin-Neshitt-Thrall [1], Lemma 9.8 D.
If $p = 2$, 

$$\frac{6}{2} = 3 \not\equiv 0 \mod 2$$

implies that $A(x)$ is irreducible mod 2, therefore $j(2) = 1$.

If $p = 3$, $A(x)$ is reducible mod 3 and contains two modular irreducible constituents. Therefore by a deep theorem of Brauer $j(3) = 2$.

Finally, since $h(Q) = 1$, we have

$$c = \prod_{p \mid 6} j(p) = j(3) = 2.$$

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