ON A TENSOR FIELD $\phi_i^h$ SATISFYING $\phi^3 = \pm \mathbf{I}$

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Dedicated to Professor Kentaro Yano on his fiftieth birthday.

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S. Tachibana [3] has recently studied linear connections with respect to which a tensor field $\phi_i^h$ satisfying $\phi^3 = \pm \mathbf{I}$, is parallel, and got some necessary and sufficient conditions for a linear connection to make such a structure parallel.

In the present paper, we shall study the integrability condition of such a structure. In case $p = 2$, $\phi_i^h$ is an almost complex structure, or an almost product structure. In case $p = 3$, $\phi_i^h$ gives a structure closely related to the almost contact structure or the so-called $(F, \xi, \eta)$-structure introduced by S. Sasaki in [2].

The tensor calculus developed in the present paper is quite similar to that given by M. Obata [1].

After giving some preliminaries in §1, we shall study in §2 the linear connection with respect to which a tensor field $\phi_i^h$, such that $\phi^3 = \pm \mathbf{I}$, is parallel. §3 is devoted to the study of relations between linear connections making $\phi_i^h$ parallel and a tensor $L_{ij}^h$ constructed only from $\phi_i^h$. In §4, we shall discuss the properties of a tensor field $\phi_i^h$ such that $\phi^3 = \mathbf{I}$ as the simplest example for our structures and obtain an integrability condition of such a structure. In the last section the integrability condition for the general case will be given without proof.

1. Preliminaries. In an $n$-dimensional manifold, a tensor field $\phi_i^h$ of type $(1,1)$ and a tensor field $T_{ij}^h$ of type $(1,2)$ are sometimes denoted respectively by

$$\phi = (\phi_i^h) \quad \text{and} \quad T = (T_{ij}^h)$$

by making use of matrix notations with respect to the indices $h$ and $i$. Let $\psi = (\psi_i^h)$ be another tensor field of type $(1,1)$. Then we shall use the following notation:

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1) See the Bibliography at the end of the paper.
2) We restrict ourselves to differentiable manifolds of class $C^\infty$ and we suppose for all quantities to be of class $C^\infty$.
3) $a, b, c, h, i, j = 1, 2, \ldots, n$. 

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The identity matrix $I$ denotes obviously the numerical tensor field $S^h_i$ such that $S^i_i = 1$, if $h = i$, and $S^i_j = 0$, if $h \neq i$.

We suppose that on a differentiable manifold there is given a non-trivial tensor field $\phi = (\phi^h_i)$ of type $(1,1)$ satisfying $\phi^h_i \neq \pm I$ and

$$\phi^p = \varepsilon I,$$

for some integer $p \geq 2$, where $\varepsilon$ is a constant $+1$ or $-1$ and $\phi^p$ denotes the $p$-th power of the matrix $\phi$. Such a tensor field is briefly called a $(p,\varepsilon)$-structure. Because this $\phi$ is non-singular, it has the inverse tensor $\phi^{-1}$, which we denote by $\psi = (\psi^h_i)$. Denoting $\phi^r$ and $\psi^r$ respectively by

$$\phi^r = (\phi^r_i),$$

$$\psi^r = (\psi^r_i),$$

we have easily from the definition

$$\phi^{-1} = \psi^r, \quad \phi^r = \psi = \varepsilon I,$$

where $r$ is an arbitrary integer.

We shall now define a correspondence $\Phi_1$ which associates a tensor field $\Phi_1 T$ of type (1,2) to any tensor field $T$ of the same type by the following formula:

$$\Phi_1 T = \frac{1}{p} \sum_{r=0}^{p-1} \phi^r \cdot T \cdot \psi^r.$$

The components of $\Phi_1 T$ are sometimes denoted by

$$\Phi_1 T = (\Phi_1 T^h_i).$$

Then, taking account of (1.2), we see easily

$$\Phi_1 \Phi_1 = \Phi_1.$$

Next, defining another correspondence $\Phi_2$ by

$$\Phi_2 T = T - \Phi_1 T,$$

we obtain directly from (1.4)

$$\Phi_2 \Phi_2 = \Phi_2, \quad \Phi_2 \Phi_1 = \Phi_2 \Phi_1 = 0,$$

where 0 means the zero correspondence assigning the zero tensor field to any tensor of type (1.1). Taking account of (1.4) and (1.5), we have

\textbf{Lemma 1}. A tensor field $T$ of type (1,2) satisfies the equation $\Phi_2 T = 0$.
if and only if there exists another tensor field \( S \) of the same type such that \( T = \Phi_1 S \).

**Lemma 2.** Let \( A \) be a given tensor field of type \((1,2)\) and

\[
\Phi_2 T = A
\]

be a linear equation with unknown tensor field \( T \) of the same type. Then

\((1.6)\) has at least one solution if and only if \( \Phi_1 A = 0 \) (or equivalently \( \Phi_2 A = A \)). If this is the case, the general solution of \((1.6)\) is given by

\[
T = A + U,
\]

where \( U \) is an arbitrary tensor field of type \((1,2)\) satisfying \( \Phi_2 U = 0 \).

We now give two identities for the later use. For any tensor field \( T \) of type \((1,2)\) we have identities:

\[
\Phi_2 T = \sum_{r=0}^{p-1} \sum_{s=1}^{q-1} \phi(T - \Phi \cdot T \cdot \psi) \cdot \phi,
\]

\[
T - \phi \cdot T \cdot \psi = \Phi_2 T - \phi \cdot (\Phi_2 T) \cdot \psi.
\]

Let \( h_\mu \) be a positive definite Riemannian metric. Then, as was proved in [3], it is easily verified that a tensor field \( g_\mu \) defined by

\[
g_\mu = \sum_{r=0}^{p-1} \phi h_{i} \phi^b
\]

is a positive definite Riemannian metric satisfying

\[
\phi^b g_\mu \phi^b = g_\mu.
\]

2. **\( \phi \)-connections.** Let \( \Gamma \) be a linear connection with respect to which the covariant derivative \( \nabla_i \phi^a \) of a contravariant vector field \( \phi^a \) is given by

\[
\nabla_i \phi^a = \partial_i \phi^a + \Gamma^a_{\mu} \phi^a,
\]

where \( \Gamma^a_{\mu} \) are coefficients of the connection \( \Gamma \). A linear connection \( \Gamma \) is called a \( \phi \)-connection if it makes a \((p,\phi)\)-structure \( \phi \) parallel, i.e. if \( \nabla_i \phi^a = 0 \). We define a correspondence \( \Phi \) associating a linear connection \( \Gamma \) to any linear connection \( \Gamma \) by the formula

\[
\Phi \Gamma^a_{\mu} = \Gamma^a_{\mu} + \frac{1}{p} \sum_{r=1}^{p-1} (\nabla_i \phi^a) \phi^r \phi^a,
\]

where \( \Phi \Gamma^a_{\mu} \) denote the coefficients of the new connection \( \Phi \Gamma \). Let \( T = (T^h_{\mu}) \) be a tensor field of type \((1,2)\). Then \( \Gamma + T \) denotes a linear connection with coefficients \( \Gamma^a_{\mu} + T^h_{\mu} \). Now, by making use of the definition \((2.1)\) of \( \Phi \) and the definition \((1.3)\) of \( \Phi_1 \), we have directly
LEMMA 3. We have
\[ 
\Phi(\Gamma + T) = \Phi\Gamma + \Phi _1 T \]
for any linear connection \( \Gamma \) and any tensor field \( T \) of type \((1,2)\).

S. Tachibana [3], basing on the notion of the infinitesimal connection defined in the principal tangent bundle, has proved

THEOREM 1. A linear connection \( \Gamma \) is a \( \Phi \)-connection if and only if there exists another linear connection \( \Gamma^* \) such that
\[ 
\Gamma = \Phi \Gamma^*. 
\]

Theorem 1 shows that there exists always a \( \Phi \)-connection in any manifold admitting a \((p,\varepsilon)\)-structure. This theorem together with (2.1) implies that the correspondence \( \Phi \) satisfies \( \Phi \Phi = \Phi \) and that, for any \( \Phi \)-connection \( \Gamma, \Phi \Gamma = \Gamma \) holds good.

Because of Lemmas 1 and 3, Theorem 1 implies

THEOREM 2. Let \( \Gamma^* \) be a \( \Phi \)-connection. Then a necessary and sufficient condition for a linear connection \( \Gamma \) to be a \( \Phi \)-connection is that there exists a tensor field \( U \) of type \((1,2)\) such as
\[ 
\Gamma = \Phi \Gamma^* + U, \quad \Phi _2 U = 0. 
\]

Next, we shall give a proof of Theorem 1 other than that given in [3].

PROOF OF THEOREM 1. Let \( \Gamma^* \) be an arbitrary linear connection. Then a linear connection \( \Gamma \) is a \( \Phi \)-connection if and only if

\[ 
T^a _b = \Phi ^a _b T^a _b = (\nabla \Phi _b ^a)\Psi _a ^b, \quad \Phi ^a _b = \Gamma _b ^a - \Gamma ^* _b. 
\]

Taking account of the identities (1.7) and (1.8), we see easily that the equation (2.2) is equivalent to

\[ 
\Phi _2 T = A, 
\]

where the unknown tensor field \( T^a _b \) is denoted by \( T \) and \( A = (A^a _b) \) is the tensor field given by

\[ 
A^a _b = \frac{1}{p} \sum _{r=1} ^{p-1} (\nabla \Phi _b ^a)\Psi _a ^b. 
\]

Here, if we take account of (1.7), we have \( \Phi _2 A = A. \) This means that \( T = A \) is a solution of (2.3). Therefore, Lemma 2 implies that the general
solution of (2.3) is given by
\[ T = A + \Phi U, \]
where \( U \) is a certain tensor field of type \((1,2)\). Thus, a linear connection \( \Gamma \) is a \( \Phi \)-connection if and only if
\[ \Gamma = \hat{\Gamma} + A + \Phi U, \]
\[ = \Phi \Gamma + \Phi U, \]
\[ = \Phi (\hat{\Gamma} + U). \]

This proves Theorem 1.

3. The tensor \( L^a_{\mu b} \). Let \( \hat{\Gamma} \) be a symmetric linear connection and put \( \Gamma = \Phi \hat{\Gamma} \), which is a \( \Phi \)-connection. Denoting by \( S = (S^a_{\mu b}) \) the torsion tensor of the \( \Phi \)-connection \( \Gamma \), we have
\[ S^a_{\mu b} = \frac{1}{2} \sum_{r=1}^{p-1} (\hat{\nabla}_{\mu} \Phi_{r b})_{a}^r \Psi^b_a, \]
where \( \hat{\nabla} \) means the covariant differentiation with respect to \( \hat{\Gamma} \). Since \( \hat{\Gamma} \) is symmetric, it follows from (3.1)
\[ S^a_{\mu b} = \frac{1}{p} \left\{ \sum_{r=1}^{p-1} (\partial_{\mu} \Phi_{r b})_{a}^r \Psi^b_a - \sum_{r=1}^{p-1} \phi_{r b} \hat{\Gamma}^r_{a b} \right\}, \]
where \( \hat{\Gamma}^r_{a b} \) denote the coefficients of \( \hat{\Gamma} \). Now, taking account of (3.2) and the symmetry of \( \hat{\Gamma} \), we see that the tensor field
\[ L^a_{\mu b} = S^a_{\mu b} - (\Phi S^a_{\mu b} - \Phi S_{\mu b}), \]
is independent of the connection \( \hat{\Gamma} \), i.e. \( L^a_{\mu b} \) is a tensor field completely determined by the given structure \( \Phi \). The tensor \( L^a_{\mu b} \) can be explicitly written down as
\[ p^a L^a_{\mu b} = (p - 1) \sum_{r=1}^{p-1} (\partial_{\mu} \Phi_{r b})_{a}^r \Psi^b_a \]
\[ - \sum_{s=1}^{p-1} \sum_{r=1}^{p-1} \phi_{s b} (\partial_{\mu} \Phi_{r b})_{a}^r \Psi^b_a. \]

Now, we consider a linear connection \( \hat{\Gamma} \) defined by
\[ \hat{\Gamma} = \Gamma - 2 \Phi S. \]
Then, by Theorem 2, \( \hat{\Gamma} \) is a \( \Phi \)-connection. Its torsion tensor is obviously equal to \( L^a_{\mu b} \). Thus, we have
LEMMA 4. In any manifold admitting a $(p, \varepsilon)$-structure $\phi$, there always exists a $\phi$-connection whose torsion tensor is equal to $L_{\mu}^{h}$.

Taking account of the definition (3.3) of the tensor field $L_{\mu}^{h}$, we see that $L_{\mu}^{h}$ vanishes if the manifold admits a symmetric $\phi$-connection. Then Lemma 4 implies

THEOREM 3. In a manifold admitting a $(p, \varepsilon)$-structure, a necessary and sufficient condition for the corresponding tensor field $L_{\mu}^{h}$ to vanish identically is that there exists a symmetric $\phi$-connection.

We shall next give simple forms of $L_{\mu}^{h}$ for some smaller values of $p$.

When $p = 2$, (3.4) is reduced to

$$4\varepsilon L_{\mu}^{h} = (\partial_{\mu} \phi_{l})^{a} \phi_{a}^{h} - \phi_{l}^{h} \partial_{\mu} \phi_{l}^{a}, \quad (p = 2),$$

which is nothing but the Nijenhuis tensor of the almost complex structure $\phi$ (if $\varepsilon = -1$) or that of the almost product structure $\phi$ (if $\varepsilon = +1$).

When $p = 3$, (3.4) is reduced to

$$9\varepsilon L_{\mu}^{h} = 2 \{ (\partial_{\mu} \phi_{l})^{a} \phi_{a}^{h} + (\partial_{\mu} \phi_{l})^{a} \phi_{a}^{h} \}
- \{ (\phi_{l}^{h} \partial_{\mu} \phi_{l})^{a} \phi_{a}^{h} + (\phi_{l}^{h} \partial_{\mu} \phi_{l})^{a} \phi_{a}^{h} \}
- \{ \phi_{l}^{h} \partial_{\mu} \phi_{l}^{a} + \phi_{l}^{h} \partial_{\mu} \phi_{l}^{a} \} \quad (p = 3).$$

4. $(3, +1)$-structures. Let $\phi$ be a $(p, -1)$-structure. If $p$ is odd, $-\phi$ is obviously a $(p, +1)$-structure. Then, in the case where $p$ is odd, it is sufficient for us to consider only $(p, +1)$-structures.

Let $\phi$ be a $(3, +1)$-structure. First, putting

$$Q = \frac{1}{3} (I + \phi + \phi^{*}),$$

we have

$$Q^{*} = Q.$$

Hence, defining $P$ by

$$P = I - Q,$$

we see easily that

$$P^{*} = P, \quad P \cdot Q = Q \cdot P = 0,$$

i.e. that the pair $(P, Q)$ defines an almost product structure if $Q \neq 0$. When $Q = 0$, if we put

$$F = \frac{1}{\sqrt{3}} (I + 2\phi),$$
we have

\[ F^3 = -I. \]

This implies that the manifold admits an almost complex structure if \( Q = 0 \).

Next, putting in general case

\[ F = \frac{1}{\sqrt{3}} (\phi - \phi^3), \]

we obtain easily

\[(4.1) \quad F^3 = -P, \quad F \cdot P = P \cdot F = F, \]

which implies

\[(4.2) \quad F \cdot Q = Q \cdot F = 0, \]
\[(4.3) \quad F^3 = -F. \]

Conversely, we assume the existence of a non-zero tensor field \( F \) of type (1,1) which satisfies (4.3). On putting

\[ \phi = \frac{3}{2} F^2 + \sqrt{3} F + I, \]

it is easily verified

\[ \phi^3 = I. \]

Summing up, we obtain

**Lemma 5.** A necessary and sufficient condition for a manifold to admit a (3, +1)-structure \( \phi \) is that it admits a non-zero tensor field \( F \) of type (1,1) satisfying \( F^3 = -F \).

Now, let \( g_{ij} \) be a positive definite Riemann metric satisfying (1.9). Then it is easily verified

\[ P_j^c g_{cb} P_i^b + Q_j^c g_{cb} Q_i^b = g_{ij}, \]
\[ F_j^c g_{cb} F_i^b = P_j^c g_{cb} P_i^b. \]

These two relations imply together with Lemma 5

**Theorem 4.** A necessary and sufficient condition for an \( n \)-dimensional manifold to admit a (3, +1)-structure is that the structure group of its tangent bundle is reducible to the group \( O(m) \times U(r) \), where \( m \geq 0 \) and \( r > 0 \) are certain integers such that \( m + 2r = n \).

In this theorem, we have denoted by \( O(m) \) and \( U(r) \) respectively the orthogonal group of the \( n \)-dimensional Euclidean space and the unitary group
of the unitary space of \( r \) complex dimensions.

In Theorem 4, if \( m = 1 \), the structure \( \Phi \) is closely related to the almost contact structure and the so-called \( (F, \xi, \eta) \)-structure introduced by S. Sasaki [2]. That is, any orientable manifold with a \((3, +1)\)-structure admits a \((F, \xi, \eta)\)-structure (or equivalently an almost contact structure) if the tensor \( F \) corresponding to \( \Phi \) is of rank \( n - 1 \). In fact, at any point \( x \) of the manifold the set of all vectors \( X^a \) satisfying \( X^a F_a^b = 0 \) forms a 1-dimensional subspace \( L_x \) in each tangent space \( T_x \). The set of all \( L_x \) forms obviously a differentiable distribution of 1 dimension throughout the manifold. Let \( g_{ij} \) be a positive definite Riemann metric satisfying (1.9). Then, it is easily verified that the tensor \( F_{ij} = F_j^a g_{ai} \) is skew-symmetric.

We now assume the manifold to be orientable. There exists obviously the skew-symmetric tensor field

\[
\sigma^i = \frac{1}{\sqrt{g}} \varepsilon^{i_{i_2} \ldots i_n}
\]

of type \((n, 0)\), where \( g = |g_{ij}| \) and \( \varepsilon^{i_{i_2} \ldots i_n} \) is equal to \(+1\) if \((i_1, i_2, \ldots, i_n)\) is an even permutation; \(-1\) if \((i_1, i_2, \ldots, i_n)\) is an odd permutation; zero otherwise. Because the rank of \( F_{ij} \) is \( 2r \) \((= n - 1)\), the vector

\[
\omega^a = F_{i_1 i_2 \ldots i_{n-1} a} \xi^{i_1 i_2 \ldots i_{n-1} a}
\]

is everywhere non-zero and \( \omega^a F_a^b = 0 \). This means that the vector field \( \omega^a \) is everywhere non-zero and lying on \( L_x \). On putting

\[
\xi^a = \omega^a / \sqrt{g_{ij} \omega^i \omega^j},
\]

\( \xi^a \) is a field of unit vectors lying on \( L_x \) at each point. Then if we put \( \eta_i = \xi^a g_{ai} \), we see

\[
Q_i^a = \eta_i \xi^a.
\]

This implies together with (4.1)

\[
F_i^a F_i^b = - \delta^b_a + \eta_i \xi^b.
\]

A triple \((F_i^a, \xi^a, \eta_i)\) satisfying this relation is called a \((F, \xi, \eta)\)-structure introduced by S. Sasaki [2].

We suppose now that for the given \((3, +1)\)-structure \( \Phi \) the tensor \( L^a_{ij} \) vanishes identically. Then, by virtue of Theorem 3, the manifold admits a symmetric \( \Phi \)-connection \( \Gamma \). Keeping the notations for tensor fields \( P = (P_i^a) \), \( Q = (Q_i^a) \) and \( F = (F_i^a) \) as above, we obtain

\[
\nabla_j P_i^a = 0, \quad \nabla_j Q_i^a = 0, \quad \nabla_j F_i^a = 0
\]

as a consequence of \( \nabla_j \phi_i^a = 0 \). The first two equations show that the almost
product structure \((P, Q)\) is integrable, i.e. for any point of the manifold there exists a coordinate neighbourhood \((U, x^k)\) of this point in which \(P\) and \(Q\) have respectively the following numerical components:

\[
\begin{pmatrix}
\delta^s_eta \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

where we have assumed that \(P\) is of rank \(n-m\) (\(0 \leq m < n\)).

It is easily seen that in \((U, x^k)\) the \(\phi\)-connection \(\Gamma\) has zero components except \(\Gamma^s_eta\) and \(\Gamma^s_m\). Taking account of (4.1) and (4.2), we see that \(F\) has the components

\[
F = \begin{pmatrix}
F^s_eta & 0 \\
0 & 0
\end{pmatrix}
\]

in \((U, x^k)\), where (4.1) implies

\[
F^s_eta F^s_\gamma \equiv - \delta^s_\beta.
\]

In the neighbourhood \((U, x^k)\) any submanifold \(V\) defined by \(x^a = \text{const}\.\) is an integral manifold of the \((n-m)\)-dimensional distribution determined by \(P\). Then (4.4) and (4.5) mean that \(F\) induces an almost complex structure \(\Gamma = (F^a_b)\) in each \(V\). On the other hand, \(\Gamma^s_\beta\) define a symmetric linear connection \(\tilde{\Gamma}\) in each \(V\). Moreover, \(\nabla_a F^a_b = 0\) implies \(\tilde{\Gamma}_a \tilde{F}^a_b = 0\) and \(\partial_a F^a_b = 0\), where \(\tilde{\nabla}\) denotes the covariant differentiation with respect to \(\tilde{\Gamma}\). Therefore, the almost complex structure \(\tilde{F}\) is integrable in each \(V\). This means that for any point of \(V\) there exists in \(V\) a coordinated neighbourhood \((\tilde{U}, \tilde{x}^a)\) of this point, in which \(\tilde{F} = (\tilde{F}^a_b)\) has the following numerical components

\[
\tilde{F} = (\tilde{F}^a_b) = \begin{pmatrix}
0 & -I_r \\
I_r & 0
\end{pmatrix},
\]

where \(n-m = 2r\) and \(I_r\) is the unit \((r, r)\)-matrix. This fact implies together with \(\partial_a F^a_b = 0\) that for any point of the manifold there exists a coordinated neighbourhood \((U, x^k)\) of this point, in which the tensor field \(F\) has the numerical components

\[a, \beta, \gamma = 1, 2, \ldots, n-m; \lambda, \mu, \nu = n-m + 1, n-m + 2, \ldots, n.\]
Summing up, we have proved that if $L^\mu = 0$, $\phi$ is integrable, i.e. for any point of the manifold there exists a coordinated neighbourhood $(U, x^h)$ of this point, in which the tensor field $\phi$ has the numerical components

$$
\phi = \begin{pmatrix}
-\frac{1}{2} I_r & -\frac{\sqrt{3}}{2} I_r & 0 \\
\frac{\sqrt{3}}{2} I_r & -\frac{1}{2} I_r & 0 \\
0 & 0 & I_m
\end{pmatrix}.
$$

Conversely, it is obvious that when $\phi$ is integrable, the tensor field $L^\mu$ vanishes identically. Then we have

**Theorem 5.** In a manifold admitting a $(3, +1)$-structure $\phi$, a necessary and sufficient condition for $\phi$ to be integrable is that the tensor field $L^\mu$ vanishes identically.

As is proved above, if $\phi$ is integrable, there exist two systems of integral submanifolds in the manifold, corresponding respectively to $P$ and to $Q$, and each integral submanifold corresponding to $P$ admits an integrable almost complex structure defined by $F$.

### 5. The integrability conditions of $(p, \varepsilon)$-structures.

Corresponding to Theorem 5, we shall give without proof a theorem explaining the integrability condition of $(p, \varepsilon)$-structures. The $(p, \varepsilon)$-structure is by definition integrable when for any point of the manifold there exists a coordinated neighbourhood $(U, x^h)$ of this point, in which the structure has numerical components.

**Theorem 6.** A $(p, \varepsilon)$-structure $\phi$ is integrable if and only if the corresponding tensor field $L^\mu$ vanishes identically.

Next, corresponding to Theorem 4, we shall state without proof

**Theorem 7.** A necessary and sufficient condition for a manifold to admit a $(p, \varepsilon)$-structure is that the structure group of its tangent bundle is reducible (i) if $p$ is odd, say $p = 2q + 1$, to the group

$$
O(m) \times U(r_1) \times \cdots \times U(r_q),
$$

where $n > m \geq 0$, $r_1, r_2, \ldots, r_q \geq 0$,

$$
m + 2(r_1 + r_2 + \cdots + r_q) = n;
$$
(ii) if \( p \) is even, \( p > 2 \), say \( p = 2q + 2 \), and \( \varepsilon = +1 \), to the group
\[ O(m) \times O(m') \times U(r_1) \times \cdots \times U(r_q), \]
where \( n > m, m' \geq 0, r_1, \ldots, r_q \geq 0, \]
\[ m + m' + 2(r_1 + r_2 + \cdots + r_q) = n; \]
(iii) if \( p \) is even, say \( p = 2q \), and \( \varepsilon = -1 \), to the group
\[ U(r_1) \times U(r_2) \times \cdots \times U(r_q), \]
where \( r_1, r_2, \ldots, r_q \geq 0, 2(r_1 + r_2 + \cdots + r_q) = n. \]

Corresponding to each case given in Theorem 7, we have the following result. For each case, the integers \( m, m', r_1, r_2, \ldots, r_q \) are restricted within the same ranges as in Theorem 7.

In the case (i), where \( p = 2q + 1 \) is odd and \( \varepsilon = +1 \), there exist in the manifold \( q + 1 \) tensor fields \( E, F_1, \ldots, F_q \) of type \((1,1)\) satisfying
\[ E^2 = E, \quad F_u^2 = -F_u, \quad E \cdot F_u = F_u \cdot E = 0, \quad (u = 1, 2, \ldots, q), \]
where the rank of \( E \) is \( r_m \) and the rank of \( F_u \) is \( r_u \). The structure \( \phi \) has the following decomposition:
\[ \phi = E + \sum_{u=1}^{q} \left\{ \left( \frac{2u\pi}{p} \right) E_u + \left( \frac{2u\pi}{p} \right) F_u \right\}, \]
where \( E_u = -F_u^2. \)

In the case (ii), where \( p = 2q + 2 \) is even, \( p > 2 \) and \( \varepsilon = +1 \), there exist in the manifold \( q + 2 \) tensor fields \( E, E', F_1, \ldots, F_q \) of type \((1,1)\) satisfying
\[ E'^2 = E, \quad E'^2 = E', \quad E \cdot E' = E' \cdot E = 0, \]
\[ F_u^2 = -F_u, \quad E \cdot F_u = F_u \cdot E = 0, \quad E' \cdot F_u = F_u \cdot E' = 0, \quad (u = 1, 2, \ldots, q), \]
where the rank of \( E \) is \( m \), the rank of \( E' \) is \( m' \) and the rank of \( F_u \) is \( r_u \). The structure \( \phi \) has the following decomposition:
\[ \phi = E + E' + \sum_{u=1}^{q} \left\{ \left( \frac{2u\pi}{p} \right) E_u + \left( \frac{2u\pi}{p} \right) F_u \right\}, \]
where \( E_u = -F_u^2. \)

In the case (iii), where \( p = 2q \) is even and \( \varepsilon = -1 \), there exist in the
manifold \( q \) tensor fields \( F_1, F_2, \ldots, F_q \) of type \((1,1)\) satisfying

\[
F_u = -F_u, \quad (u = 1, 2, \ldots, q),
\]

\[
F_u \cdot F_v = F_v \cdot F_u = 0, \quad (u = v; \ u, v = 1, 2, \ldots, q),
\]

where the rank of \( F_u \) is \( r_u \). The structure \( \phi \) has the following decomposition:

\[
\phi = \sum_{u=1}^{q} \left[ \left( \cos \frac{2(u + 1)\pi}{p} \right) E_u + \left( \sin \frac{2(u + 1)\pi}{p} \right) F_u \right],
\]

where \( E_u = -F_u^2 \).

When the structure \( \phi \) is integrable, in every case the projection tensor fields \( E, E', E_u \) are all integrable and \( F_u \) determines an integrable almost complex structure in each integral submanifold corresponding to the projection tensor field \( E_u \).

**BIBLIOGRAPHY**


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