ON THE DISTRIBUTION OF VALUES OF THE TYPE $\sum f(q^k t)$

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1. Let $f(t)$ be a measurable function satisfying the conditions;

(1. 1) $f(t + 1) = f(t), \quad \int_0^1 f(t) dt = 0$ and $\int_0^1 f^2(t) dt < + \infty$.

In [1] M.Kac proved that if $f(t)$ is a function of Lip $\alpha, \alpha > 1/2,$ or of bounded variation, then it is seen that, for $-\infty < \omega < + \infty$,

(1. 2) $\lim_{n \to \infty} \left\{ \int_0^1 \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(2^k t) \leq \omega \right\} = \frac{1}{\sqrt{2\pi}} \int_0^\omega e^{-u^2/2} du,

provided that the following limit is positive;

$\sigma^2 = \lim_{n \to \infty} \int_0^1 \left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(2^k t) \right)^2 dt$.

At the end of that paper he proposed the problem to replace the sequence $\{2^k\}$ in (1. 2) by a sequence of real numbers satisfying the Hadamard’s gap condition.

In this direction R.Salem and A.Zygmund proved the central limit theorem of lacunary trigonometric series (c.f. [2]). Also they showed that if $f(t) = \cos 2\pi t + \cos 4\pi t$ and $n_k = 2^k - 1, \quad k = 1, 2, \ldots$, then

$\lim_{n \to \infty} \left\{ \int_0^1 \frac{1}{n} \sum_{k=1}^{n-1} f(n_k t) \leq \omega \right\} = \frac{1}{\sqrt{\pi}} \int_0^1 dx \int_{-\infty}^{\omega e^{x/2}} e^{-u^2/2} du$.

In this note we consider the sequence $\{f(q^k t)\}$, where $q$ is any real number greater than 1. To state our result we need some definitions. For any measurable set $A$ in $(-\infty, \infty)$, we define its relative measure $\mu_n[A]$ as follows;

$\mu_n[A] = \lim_{T \to \infty} \frac{1}{2T} |A \cap (-T, T)|$,

and for any measurable function $g(t)$ defined on $(-\infty, \infty)$ its relative mean $M[g(t)]$ as follows;

$M[g(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T g(t) dt$,

provided the two limits exist (c.f. [4]). It is easily seen that if $g(t)$ is periodic with period 1 and integrable on the interval $(0, 1)$, then
$M\{g(t)\} = \int_0^ng(t)dt$, and that if $f(t)$ satisfies the condition (1.1), then for each $n$

the set \{ $t; \sum_{k=0}^{n} f(q^k t) \leq \omega$ \} has the relative measure for any $q$ and $\omega$.

The purpose of the present note is to prove the following

**THEOREM.** Let $q$ be any real number greater than 1 and $f(t)$ satisfy the condition (1.1) and, for some $\varepsilon > 0$,

\[(1.3) \quad \left[ \int_0^1 |f(t) - S_n(t)|^2 dt \right]^{1/2} = O [(\log n)^{-\alpha - \varepsilon}], \quad \text{as } n \to + \infty,\]

where $S_n(t)$ denotes the $n$-th partial sum of the Fourier series of $f(t)$. Then

the following limit

\[
\sigma^2 = \lim_{n \to \infty} M \left\{ \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \right|^2 \right\}
\]

exists and if $\sigma^2$ is positive, we have for any $\omega$,

\[
\lim_{n \to \infty} \mu_n \left\{ t; \frac{1}{\sigma \sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \leq \omega \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du.
\]

**REMARK 1.** If $q^k$ is an irrational number for any positive integer $k$, then

we have $\sigma^2 = \int_0^1 (f(t))^2 dt$ (cf. the proof of Lemma 1).

**REMARK 2.** If $q = 2$, then we have, for each $n$,

\[
\mu_n \left\{ t; \frac{1}{\sigma \sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \leq \omega \right\} = \left| \left\{ t; 0 \leq t \leq 1, \frac{1}{\sigma \sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \leq \omega \right\} \right|.
\]

Hence if $\sigma^2 > 0$, then (1.2) holds under the condition (1.3) which is weaker than that of M.Kac.

To prove (1.2) Kac approximated $\Sigma f(2^k t)$ by sums of independent functions using the system of Rademacher functions. To prove our theorem we approximate $\Sigma f(q^k t)$ by sums of gap sequences with infinite gaps (cf. [3]).

2. From now on let $f(t)$ and $q$ satisfy the conditions of the theorem. Further without loss of generality we may assume that the Fourier series of $f(t)$ contains cosine terms only. This assumption is introduced solely for the purpose of shortening the formulas. Let us put

\[f(t) \sim \sum_{k=1}^{\infty} a_k \cos 2\pi kt, \quad \text{and} \quad S_n(t) = \sum_{k=1}^{n} a_k \cos 2\pi kt.\]

From (1.3) it is seen that

*) $\sigma$ denotes a non-negative number.
ON THE DISTRIBUTION OF THE TYPE $zf(q^t)$

(2.1) $\left[ \int_0^1 |f(t) - S_n(t)|^2 dt \right]^{1/2} = \left( \frac{1}{2} \sum_{k>n} a_k^2 \right)^{1/2} \leq A(\log n)^{-1+\epsilon}$

Further let us put, for $k = 0, 1, \ldots, n$ and $n = 1, 2, \ldots$,

(2.2) $N_k, n = k[n^\beta], N_{k,n} = N_{k+1,n} - [\log^2 n],$

(2.3) $T_{k,n}(t) = \sum_{l=\ell_{k,n}}^{\ell_{k+1,n}} g_n(q^l t), \quad \text{and} \quad R_{k,n}(t) = \sum_{N'_{k,n}<l<N_{k,n}} g_n(q^l t),$

where

(2.4) $g_n(t) = S_{l_0\beta^2/n^\beta} (t),$ \hspace{1cm} \text{and} \hspace{1cm} \beta \text{ is a constant such that}$

(2.5) $0 < \beta < 1/3.$

Then we have

(2.6) $|g_n(t)| \leq \sum_{k=1}^{n^{\beta/2}} |a_k| \leq \left( \sum_{k=1}^{n} a_k^n \right)^{1/2} \leq An^{\beta/4}.$

**LEMMA 1.** The following limit exists;

$$\sigma^2 = \lim_{n \to \infty} M \left\{ \frac{1}{n} \sum_{k=0}^{n-1} f(q^k t) \right\}^2.$$  

**PROOF.** We have

$$M \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \right\}^2 = M[f^n(t)] + \frac{2}{n} \sum_{r=1}^{n-1} \sum_{k=0}^{n-1-r} M[f(q^r t)f(q^{r+t} t)]$$

$$= \int_0^1 f^2(t) dt + 2 \sum_{r=1}^{n-1} \left( 1 - \frac{r}{n} \right) M[f(t)f(q^r t)].$$

By (2.1), we have

$$|M[f(t)f(q^r t)]| = \frac{1}{2} \left| \sum_{m=k}^{m=n} a_m a_k \right| \leq \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} \left( \sum_{m=\ell_{k,n}}^{\ell_{k+1,n}} a_m^2 \right)^{1/2} \leq Ar^{-1+\epsilon}.$$  

Hence $\sum_r |M[f(t)f(q^r t)]| < +\infty$, and this proves the lemma.

**LEMMA 2.** We have

$$\lim_{n \to \infty} M \left\{ \frac{1}{\sqrt{N_{k,n}}} \sum_{k=0}^{N_{k,n}-1} f(q^k t) - \frac{1}{\sqrt{N_{k,n}}} \sum_{k=0}^{n-1} T_{k,n}(t) \right\}^2 = 0,$$  

and

*) Now and later A will denote a constant not necessarily the same.
\[
\lim_{n \to \infty} M \left[ \frac{1}{N_{n,n}} \sum_{k=0}^{N_{n,n}-1} T_{k,n}(t) \right] = \sigma^2.
\]

**Proof.** We have

\[
M \left[ \frac{1}{N_{n,n}} \sum_{k=0}^{N_{n,n}-1} \left( f(q^k t) - g_n(q^k t) \right)^2 \right] 
\leq \int_0^1 |f(t) - g_n(t)|^2 dt + 2 \sum_{r=1}^{N_{n,n}-1} M\{ \left[ f(t) - g_n(t) \right] \left[ f(q^r t) - g_n(q^r t) \right] \}.
\]

By (2.1) and (2.4), we have

\[
\int_0^1 |f(t) - g_n(t)|^2 dt = \frac{1}{2} \sum_{k>n/2} a_k^2,
\]

and

\[
M\{ \left[ f(t) - g_n(t) \right] \left[ f(q^r t) - g_n(q^r t) \right] \} = \left| \frac{1}{2} \sum_{k>n/2} a_m a_k \right| \leq A \left( \sum_{k>n/2} a_k^2 \right)^{1/2} r^{-(1+\epsilon)}.
\]

Since \( \sum_{k>n/2} a_k^2 \to 0 \) as \( n \to +\infty \), it follows that

\[
(2.7) \quad \lim_{n \to \infty} M \left[ \frac{1}{N_{n,n}} \sum_{k=0}^{N_{n,n}-1} \left( f(q^k t) - g_n(q^k t) \right)^2 \right] = 0.
\]

On the other hand from (2.3), we have

\[
(2.8) \quad \sum_{k=0}^{N_{n,n}-1} g_n(q^k t) - \sum_{k=0}^{n-1} T_{k,n}(t) = \sum_{k=0}^{n-1} R_{k,n}(t).
\]

The maximum frequency of cosine terms of \( R_{k,n}(t) \) is \( q^{N_{n+1,n}} [n^{\theta/2}] \) and the minimum frequency of terms of \( R_{k+1,n}(t) \) is \( q^{N_{n+1,n}^{-1}} [n^{\theta/2}] \), and by (2.2), \( q^{N_{n+1,n}^{-1}} [n^{\theta/2}] \) if \( n > n_0 \). Therefore the sequence \( \{R_{k,n}(t)\} \), \( k = 0, 1, \ldots, n-1 \), is orthogonal on \( (-\infty, +\infty) \) with respect to the relative measure if \( n > n_0. \)

Further we have, by (2.3), (2.6) and (2.2),

\[
R_{k,n}(t) \leq A(N_{k+1,n} - N_{k,n})^{1/2} n^{\theta/2} \leq A n^{\theta/2} \log^4 n.
\]

Hence we have, by (2.5),

\[
(2.9) \quad M \left[ \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} R_{k,n}(t) \right]^2 \leq \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} M \{R_{k,n}(t)\}.
\]

*) We say that \( f(t) \) and \( g(t) \) are orthogonal on \( (-\infty, +\infty) \) with respect to the relative measure if \( M[g(t)f(t)] = 0 \).
ON THE DISTRIBUTION OF VALUES OF THE TYPE $zf(q^t)$

$\leq A_n^{n^1+n^2 \beta} n^{n^1+n^2 \beta} (\log n) = o(1), \quad \text{as } n \to +\infty.$

By (2.7), (2.8), (2.9) and the Minkowski's inequality, we can prove the first part of the lemma. By Lemma 1 and the relation just proved it is seen that

$$\lim_{n \to +\infty} M \left\{ \frac{1}{\sqrt{N_{n^1+n^2}}} \sum_{k=0}^{n-1} T_{k,n}(t) \right\} = \sigma^2.$$ 

In the same way as $\{R_{k,n}(t)\}$ we can see that $\{T_{k,n}(t)\}$, $k = 0, 1, \ldots, n-1$, is orthogonal on the interval $(-\infty, +\infty)$ with respect to the relative measure if $n > n_0$. Hence we have

$$\lim_{n \to +\infty} M \left\{ \frac{1}{\sqrt{N_{n^1+n^2}}} \sum_{k=0}^{n-1} T_{k,n}(t) \right\} = \lim_{n \to +\infty} \frac{1}{N_{n^1+n^2}} \sum_{k=0}^{n-1} M[T_{k,n}(t)] = \sigma^2.$$ 

This is the second part of the lemma.

3. Lemma 3. We have

$$\lim_{n \to +\infty} M \left\{ \frac{1}{N_{n^1+n^2}} \sum_{k=0}^{n-1} T_{k,n}(t)^2 - \sigma^2 \right\} = 0.$$ 

PROOF. We have, by (2.3) and (2.4),

$$T_{k,n}(t) = \sum_{l=N_{n^1+n^2}}^{N_{n^1+n^2}} \sum_{r=1}^{N_{n^1+n^2}} g_r(q^t) g_r(q^t),$$

$$g_r(q^t) = \frac{1}{2 \pi} \sum_{s=1}^{\beta/2} a_s \{1 + \cos \theta q^t\}$$

$$+ \sum_{0 < q < m, s < n/\beta/2} a_m a_s \{\cos 2\pi q(m-s) + \cos 2\pi q(m+s)\},$$

and

$$g_r(q^t) g_r(q^t) = \frac{1}{2 \pi} \sum_{0 < q < m, s < n/\beta/2} a_m a_s \{1 + \cos 4\pi q m t\}$$

$$+ \frac{1}{2 \pi} \sum_{0 < q < m, s < n/\beta/2} a_m a_s \{\cos 2\pi q(m-s) + \cos 2\pi q(m+s)\},$$

and then we can write $T_{k,n}(t)$ in the following form

$$T_{k,n}(t) = M[T_{k,n}(t)] + U_{k,n}(t) + V_{k,n}(t),$$

where
(3. 2) \[ V_{k,n}(t) = \sum_{r=1}^{N_n} \sum_{l=1}^{N_n} \sum_{ \theta < m, \theta \geq m/2 \cap \theta < \text{min} < \beta \cap \theta < 1} a_m a_s \cos 2\pi q(m - sq')t, \]

and \( U_{k,n}(t) \) is the sum of cosine terms whose frequencies are not less than \( q^{n/4} \) and not greater than \( 2q^{n/4} \) \([n^{\beta + \alpha}]\). Therefore \([U_{k,n}(t)] \), \( k = 0, 1, 2, \ldots, n - 1 \), is orthogonal on \((-\infty, +\infty)\) with respect to the relative measure if \( n > n_0 \). On the other hand from the definition of \( U_{k,n}(t) \) and (2.3) (2.4), we have

\[ |U_{k,n}(t)| \leq (N_{n,n} - N_{k,n})^3 \left( \sum_{k=1}^{n} |a_k| \right)^2 \leq n^{6\beta/2} \left( \sum_{k=1}^{n} a_k^2 \right) \]

Since \([U_{k,n}(t)]\) is orthogonal, we have, by (2.5) and the above relation,

(3. 3) \[ M \left\{ \left| \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} U_{k,n}(t) \right|^2 \right\} \leq A \frac{n^{1 + \beta}}{n^{2 + \beta}} = o(1), \]

as \( n \to +\infty \).

In the same way we have, for any fixed \( \theta \) and \( r \) such that \( \theta \neq 0 \) and \( 0 < r < N_{0,n} \),

\[ M \left\{ \left| \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} \sum_{l=1}^{N_n} \cos 2\pi q' \theta t \right|^2 \right\} \leq \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} M \left\{ \left| \sum_{l=1}^{N_n} \cos 2\pi q' \theta t \right|^2 \right\} \leq A \frac{n^{(\alpha + \beta)}}{n^{2 + \beta}} < A \frac{n^{-\alpha}}{n^{-\alpha + \beta}}, \quad \text{if } n > n_0. \]

Changing the order of summation and apply the Minkowski’s inequality to (3. 2), we have, by (2. 1) and the above relation,

(3. 4) \[ M \left\{ \left| \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} V_{k,n}(t) \right|^2 \right\} \leq \sum_{r=0}^{N_n} \sum_{0 < m, \theta \geq m/2 \cap \theta < \text{min} < \beta \cap \theta < 1} \left| a_m a_s \right| M \left\{ \left| \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} \sum_{l=1}^{N_n} \cos 2\pi q(m - sq')t \right|^2 \right\} \leq A n^{-(1 + \beta)2} \sum_{r=1}^{\infty} \sum_{0 < m, \theta \geq m/2 \cap \theta < \text{min} < \beta \cap \theta < 1} \left| a_m a_s \right| \left( \sum_{r=1}^{\infty} \left| a_r \right|^2 \right) \leq A n^{-(1 + \beta)2} \sum_{r=1}^{\infty} r^{-1} = o(1), \]

as \( n \to +\infty \).

From (3. 1), (3. 3) and (3. 4), it is seen that

\[ \lim_{n \to \infty} M \left[ \left| \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} T_k^2(t) - \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} M \{ T_k^2(t) \} \right|^2 \right] = 0. \]
Thus by Lemma 2, we can prove the lemma. Let us put for any real number $\lambda$,

\[ P_n(t, \lambda) = \prod_{k=0}^{n-1} \left( 1 + \lambda \frac{T_{k,n}(t)}{\sqrt{N_{n,n}}} \right). \]

Then we have the following

**LEMMA 4.** There exists an integer $n_0$ depending only on $q$ such that $n > n_0$ implies

\[ M\{P_n(t, \lambda)|^2\} \leq e^{2\lambda A}, \text{ and } M\{P_n(t, \lambda)\} = 1. \]

**PROOF.** By the definition of $T_{k,n}(t)$ and Lemma 2, we have

\[ M\left\{ \frac{T_{k,n}(t)}{\sqrt{n}} \right\} = M\left\{ \frac{1}{\sqrt{N_{n,n}}} \sum_{k=0}^{n-1} T_{k,n}(t) \right\} \to \sigma^2, \quad \text{as } n \to +\infty. \]

Further by (2.1) and (3.2), we have

\[ \left| V_{k,n}(t) \right| \leq \sum_{n=1}^{N_{n,n}} \sum_{t=1}^{N_{n,n}} \sum_{0 < m \leq \sigma(t) < 1} |a_m b_t| \]

\[ \leq A n^\beta \sum_{r=1}^{\infty} \left\{ \sum_{s=1}^{\infty} \alpha_s^{1/2} \left\{ \sum_{m > r-1} a_m^2 \right\}^{1/2} \leq A n^\beta. \]

Hence we have, by (3.1) and Lemma 2 and the above relations,

\[ \frac{T_{k,n}(t)}{\sqrt{N_{n,n}}} \leq \frac{A}{n} + U_{k,n}(t) N_{n,n}. \]

This implies, by (3.5),

\[ |P_n(t, \lambda)|^2 \leq \prod_{k=0}^{n-1} \left( 1 + \frac{\lambda A}{n} + \frac{\lambda^2 U_{k,n}(t)}{N_{n,n}} \right). \]

Now let $d_j \cos 2\pi u_j t$ be a term of $U_{k,n}(t)$, then $q^{n^{1/2}} \leq u_j \leq 2n^{\beta/2} q^{n^{1/2}}$. Therefore by (2.2), it follows that for any $k < n$,

\[ u_k - \sum_{j=0}^{k-1} u_j \geq q^{n^{1/2}} - 2n^{\beta/2} \sum_{j=0}^{k-1} q^{n^{1/2}} \geq q^{\frac{n}{2}} \left( 1 - 2n^{\beta/2} q^{-[\log q]} \sum_{j=0}^{k-1} q^{-[\log q]} \right) > 0, \quad \text{if } n > n_0. \]

This implies that for any $0 \leq j_0 < j_1 < \ldots < j_t < n$, we have

\[ M\left\{ \prod_{m=0}^{t} \cos 2\pi u_j t \right\} = 0, \quad \text{for } n > n_0. \]

Thus we have
\[ M\{ |P_n(t, \lambda)|^4 \} \leq M \left[ \prod_{k=0}^{n-1} \left( 1 + \frac{\lambda^2 A}{n} + \frac{\lambda^2 U_{kn}(t)}{N_{nn}} \right) \right] \]

\[ = \left( 1 + \frac{\lambda^2 A}{n} \right)^n \leq e^{\lambda^2}, \quad \text{for } n > n_0. \]

In the same way we can prove the second assertion of the lemma.

4. LEMMA 5. If \( \sigma^2 > 0 \), then we have for any fixed \( \lambda \),

\[ \lim_{n \to \infty} M \left[ \exp \left\{ \frac{i \lambda}{\sigma \sqrt{N_{nn}} \sum_{k=0}^{n-1} T_{kn}(t)} \right\} \right] = e^{-\lambda^2/2}. \]

PROOF. If we put

\[ E_n = \left\{ t; \left| \frac{1}{N_{nn}} \sum_{k=0}^{n-1} T_{kn}(t) - \sigma^2 \right| < 1 \right\}, \]

then by Lemma 3 and the Tchebyschev's inequality, it follows that

\[ \lim_{n \to \infty} \mu_k(E_n) = 1. \]

By (2.3), (2.5) and (2.6), we have

\[ \max_{0 \leq k < n} \frac{T_{kn}(t)}{\sqrt{N_{nn}}} \leq A n^{-1/2} + \frac{3}{4} = o(1), \quad \text{as } n \to \infty. \]

Therefore if \( t \in E_n \), then it is seen that

\[ \sum_{k=0}^{n-1} \frac{T_{kn}(t)}{\sqrt{N_{nn}}} \leq A \max_{0 \leq k < n} \frac{T_{kn}(t)}{\sqrt{N_{nn}}} = C_n = o(1), \quad \text{as } n \to \infty, \]

and

\[ P_n \left( t, \frac{\lambda}{\sigma} \right)^2 \leq \prod_{k=0}^{n-1} \left( 1 + \frac{\lambda^2 T_{kn}(t)}{\sigma^2} \right) \leq e^{\lambda^2 (1 + \sigma^2) t^2}. \]

We have by (4.1) and the fact that the integrand is less than one,

\[ \left| M \left[ \exp \left\{ \frac{i \lambda}{\sigma \sqrt{N_{nn}} \sum_{k=0}^{n-1} T_{kn}(t)} \right\} \right] \right| \leq \frac{1}{2T} \int_{(-T, T) \cap \mathbb{R}_n} \exp \left\{ \frac{i \lambda}{\sigma \sqrt{N_{nn}} \sum_{k=0}^{n-1} T_{kn}(t)} \right\} dt \leq \mu_k(E_n), \]

where \( E_n = (-\infty, \infty) - E_n \) and \( \mu_k(E_n) \to 0 \), as \( n \to +\infty \).

Using the relation \( \exp z = (1 + z) \exp \{ z^2/2 + O(|z|^3) \} \) as \( |z| \to 0 \), and (4.2), (4.3) and (4.4), we have

\[ \lim_{T \to \infty} \frac{1}{2T} \int_{(-T, T) \cap \mathbb{R}_n} \exp \left\{ \frac{i \lambda}{\sigma \sqrt{N_{nn}} \sum_{k=0}^{n-1} T_{kn}(t)} \right\} dt \]
ON THE DISTRIBUTION OF VALUES OF THE TYPE $\mathfrak{Z}(q^t)$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{(-T, T) \cap E_n} P_n\left(t, \frac{\lambda}{\sigma}\right) \exp \left\{ -\frac{\lambda^2}{2\sigma^2 N_{n,n}} \sum_{k=0}^{n-1} T_{k,n}(t) \right\} \, dt + o(1),$$

as $n \to +\infty$.

By (4.4) and (4.1), it is seen that if $t \in E_n$, then

$$P_n\left(t, \frac{\lambda}{\sigma}\right) \left[ \exp \left\{ -\frac{\lambda^2}{2\sigma^2 N_{n,n}} \sum_{k=0}^{n-1} T_{k,n}(t) \right\} - e^{-\lambda^2/2} \right] \leq B_\lambda \left[ \frac{1}{\sigma^2 N_{n,n}} \sum_{k=0}^{n-1} T_{k,n}(t) - 1 \right]$$

where $B_\lambda$ is a constant depending on $\lambda$.

By Lemma 3, the relative mean of the right hand side of the above formula tends to zero as $n \to +\infty$. Hence for the proof of lemma it is sufficient to show that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{(-T, T) \cap E_n} P_n\left(t, \frac{\lambda}{\sigma}\right) \, dt = 1 + o(1), \quad \text{as } n \to +\infty,$$

and by the second assertion of Lemma 4, the above relation reduces to

$$\lim_{T \to \infty} \frac{1}{2T} \int_{(-T, T) \cap E_n} P_n\left(t, \frac{\lambda}{\sigma}\right) \, dt = o(1), \quad \text{as } n \to +\infty.$$

By (4.1) and the first part part of Lemma 4, we have

$$\lim_{n \to \infty} \frac{1}{2T} \int_{(-T, T) \cap E_n} P_n\left(t, \frac{\lambda}{\sigma}\right) \, dt \leq \left[ M \left\{ P_n\left(t, \frac{\lambda}{\sigma}\right)^{\frac{1}{2}} \right\} \mu_n\{E_n\} \right]^{\frac{1}{2}} = o(1),$$

as $n \to +\infty$.

**Lemma 6.** If $\sigma^2 > 0$, then we have for any $\omega$,

$$\lim_{n \to \infty} \mu_n\left\{ t \in \frac{1}{\sigma \sqrt{N_{n,n}}} \sum_{k=0}^{n-1} T_{k,n}(t) \leq \omega \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2} \, du.$$

**Proof.** Let us put

$$Q_n(t) = \frac{1}{\sigma \sqrt{N_{n,n}}} \sum_{k=0}^{n-1} T_{k,n}(t).$$

Further let $\varphi_\varepsilon(t)$ (or $\varphi_\varepsilon(t)$) be the familiar trapezoidal function equal to 1 in the interval $(\omega_1, \omega_2)$ (or $(\omega_1 + \varepsilon, \omega_2 - \varepsilon)$) vanishing outside the interval $(\omega_1 - \varepsilon, \omega_2 + \varepsilon)$ (or $(\omega_1, \omega_2)$) and linear elsewhere, where $\varepsilon$ is a real number such that $0 < 2\varepsilon < \omega_2 - \omega_1$. Then we have

$$(4.5) \quad M[\varphi_\varepsilon(Q_n(t))] \leq \mu_r\{t; \omega_1 \leq Q_n(t) \leq \omega_2\} \leq M[\varphi_\varepsilon(Q_n(t))] \quad \text{as } n \to +\infty.$$

If we put

$$(*) \quad \text{Since } \varphi_\varepsilon(Q_n(t)) \text{ are uniformly almost periodic, } M[\varphi_\varepsilon(Q_n(t))] \text{ exist.}$$
then $\Phi^+_e(\xi)$ are absolutely integrable on $(-\infty, \infty)$. Therefore we have

\begin{equation}
M[\phi_e^+(Q_n(t))] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi^+_e(\xi) M[\exp i\xi Q_n(t)] \, d\xi.
\end{equation}

Since $\Phi^+_e(\xi)$ are absolutely integrable and $M[\exp i\xi Q_n(t)]$ converges boundedly to $e^{-\xi^2}$ as $n \to \infty$, we have by (4.5), (4.6) and the Prancherel's relation,

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi^+_e(t) e^{-\xi^2} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi^+_e(\xi) e^{-\xi^2} \, d\xi
$$

$$
\leq \lim_{n \to \infty} \mu_R \{ t; \omega_1 \leq Q_n(t) \leq \omega_2 \} \leq \lim_{n \to \infty} \mu_R \{ t; \omega_1 \leq Q_n(t) \leq \omega_2 \}
$$

$$
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi^+_e(\xi) e^{-\xi^2} \, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi^+_e(t) e^{-\xi^2} \, dt.
$$

Since $\xi$ is arbitrary we can prove the lemma.

5. Proof of the Theorem. By Lemma 1, we can prove the first part of the theorem. By the first assertion of Lemma 2 and Lemma 6, we obtain

\begin{equation}
\lim_{n \to \infty} \mu_R \left\{ t; \frac{1}{\sigma \sqrt{N_{m,n}}} \sum_{k=0}^{N_{m,n}-1} f(q^k t) \leq \omega \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \, du.
\end{equation}

On the other hand we have, by (2.2),

$$
\lim_{n \to \infty} \frac{N_{n+1,n+1}}{N_{m,n}} = 1.
$$

By the above relation and Lemma 1, we have for any $m$ such that $N_{m,n} < m \leq N_{n+1,n+1}$

$$
M \left\{ \left| \frac{1}{\sqrt{N_{m,n}}} \sum_{k=0}^{m} f(q^k t) \right|^2 \right\} = M \left\{ \left| \frac{1}{\sqrt{N_{m,n}}} \sum_{k=0}^{m-N_{m,n}} f(q^k t) \right|^2 \right\}
$$

$$
\leq A \frac{N_{n+1,n+1} - N_{m,n}}{N_{m,n}} \to 0, \quad \text{as } m \to +\infty,
$$

and

$$
M \left\{ \left| \left( \frac{1}{\sqrt{N_{m,n}}} - \frac{1}{m} \right) \sum_{k=0}^{m-1} f(q^k t) \right|^2 \right\} \leq A \left| \frac{1}{\sqrt{N_{m,n}}} - \frac{1}{m} \right|^2 m = o(1),
$$

as $m \to +\infty$.

Hence we have
ON THE DISTRIBUTION OF VALUES OF THE TYPE $\sum f(q^k t)$

$$M \left\{ \left| \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} f(q^k t) - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f(q^k t) \right| \right\}^2 = o(1), \quad \text{as } m \to +\infty.$$  

By the above relation and (5.1), we can prove the theorem.

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REFERENCES


DEPARTMENT OF MATHEMATICS, KANAZAWA UNIVERSITY.