MANIFOLDS ADMITTING CONTINUOUS FIELD OF FRAMES

YASURÔ TOMONAGA

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Introduction. Massey and Szczarba studied the continuous line element field on a differentiable manifold and obtained a necessary condition under which a manifold admits a continuous field of \( q \) independent line elements. ([1]) Meanwhile Adams investigated the continuous field of \( q \) frames on a sphere and clarified the relations between the number \( q \) and the dimension of the sphere. ([2]) A differentiable \( n \)-manifold admitting a continuous field of \( n \) frames is said to be parallelizable. It is well-known that a differentiable manifold admits a continuous non zero vector field if and only if its Euler characteristic is zero. The intermediate status between above two cases is in question. In this paper we shall mainly deal with the continuous field of \( n-3 \) frames on an \( n \)-manifold.

§ 1. A \( q \) frame means an ordered set of \( q \) linearly independent vectors while a \( q \) pseudo-frame means an ordered set of \( q \) vectors at least \( q-1 \) of which are linearly independent. The following facts are well-known:

I. Let \( M_n \) be a compact differentiable manifold. For each \( q \) there exists a continuous field of tangent \( n-q \) frames defined over the \( q \) dimensional skeleton \( K^q \) of \( M_n \). In order that there exist such a field on any \( K^{q+1} \), it is necessary that

\[
\omega_{q+1} = 0,
\]

where \( \omega_{q+1} \in H^{q+1}(M_n, \mathbb{Z}_2) \) denotes the Stiefel-Whitney class. ([7] p.199)

We have from I

COROLLARY. If \( M_n \) admits a continuous field of \( n-q \) frames \( (0 \leq q < n) \) it must be that

\[
\omega_{q+1} = \omega_{q+2} = \cdots = \omega_n = 0.
\]

Let \( M_n \) be a compact \( C^\infty \)-manifold with a Riemann metric and \( \Omega_{ij} \) be its curvature form. Chern introduced in [3] a characteristic class such that
(1.1) \[ P_{2q} = \alpha_q \sum_i \Omega_{i|\bar{s}_i} \cdots \Omega_{i|\bar{q}_i}, \quad P_{2q} \in H^{2q}(M_n, Z) \]

where \( \alpha_q \) denotes some constant. Meanwhile the Pontryagin class is expressed as follows:

(1.2) \[ P_{2q} = 1_q \sum_{i,j} \delta\left( i_1, \ldots, i_{2q}, j_1, \ldots, j_{2q} \right) \Omega_{i_{\bar{1}} \hdots i_{\bar{q}}} \cdots \Omega_{i_{\bar{1}} \hdots i_{\bar{q}}} \right), \quad P_{2q} \in H^{2q}(M_n, Z) \]

\[ 1_q = \left( (2\pi)^q q! \right)^{-1}, \]

where \( \delta \left( \ldots \right) \) denotes the generalized Kronecker symbol. It is well-known that \( P_{2q} = P_{2q}' = 0 \) (\( q = \text{odd} \)) and \( P_{2q} \) is a polynomial of \( P_{2q}' \) (\( t \leq q \)) and conversely \( P_{2q}' \) is a polynomial of \( P_{2q} \) (\( t \leq q \)). Chern proved in [3] the following theorem:

II. Let \( M_n \) be a manifold stated above. There exists a continuous field of \((n-2m+2)\) pseudo-frames over any \( 4m \) dimensional skeleton if and only if \( P_{4m} = 0 \) (\( 1 \leq m \leq \lfloor n/4 \rfloor \)). There exists a continuous field of \((n-2m+2)\) pseudo-frames over any skeleton whose dimension is less than \( 4m \).

We have from I and II the

THEOREM 1. Let \( M_n \) be a compact \( C^\infty \)-Riemannian manifold. In order that \( M_n \) admit a continuous field of \( n-1 \) frames, it is necessary that

\[ w_3 = \cdots = w_n = 0 \quad \text{and} \quad P_4 = P_5 = \cdots = P_{4(n/4)} = 0. \]

PROOF. The first part follows from I Corollary. We have from II

\[ P_{2k}' = 0, \quad k \geq 1 \]

because we can form a continuous field of \( n \) pseudo-frames from a continuous field of \( n-1 \) frames.

COROLLARY. Let \( M_{4k} \) (\( k \geq 1 \)) be a compact orientable \( C^\infty \)-Riemannian manifold. In order that \( M_{4k} \) admit a continuous field of \( 4k-1 \) frames, it is necessary that \( 2M_{4k} \) is “bord”, i.e. \( 2M_{4k} \sim 0 \).

PROOF. By Theorem 1 every Pontryagin numbers are zero. Hence the free part of cobordism components of \( M_{4k} \) is zero. Since every torsions of the cobordism ring are of order 2, the statement holds. ([4])
REMARK. When $1 \leq k \leq 3$, $M_{4k}$ becomes “bord”, because in such a case the torsion doesn’t exist.

§ 2.

THEOREM 2. Let $M_n$ $(n \geq 4)$ be a compact $C^r$-Riemannian manifold. In order that $M_n$ admit a continuous field of $n-3$ frames, it is necessary that

(i) $w_4 = w_5 = \cdots = w_n = 0$,

(ii) $P_8 = P_{12} = \cdots = P_{4\left[\frac{n}{4}\right]} = 0$ and

(iii) $P_{4k} = \frac{1}{k!} (P_4)^k \quad 1 \leq k \leq \left\lfloor \frac{n}{4} \right\rfloor$.

If moreover $n=4m$ and $M_{4m}$ is orientable, then it is necessary that

(iv) $\tau(M_{4m}) = \frac{1}{3^m m!} (P_4)^m [M_{4m}] = \frac{1}{3^m} P_{4m}[M_{4m}]$ and

$A(M_{4m}) = \left( -\frac{1}{3^m m!} \right)^m 2^m (P_4)^m [M_{4m}] = \left( -\frac{2}{3} \right)^m P_{4m}[M_{4m}]$,

where $\tau$ or $A$ denotes the index or the $A$-genus respectively.

PROOF. (i) and (ii) follow from I and II as in the case of Theorem 1. We have from (1.2)

(2.1) $P_4 = \frac{1}{2 \pi^2} \sum_{i,j} \Omega_{ij} \Omega_{ji}$.

Meanwhile we have from (1.2) and (ii)

(2.2) $P_{4k} = \frac{(2k-1)(2k-3) \cdots 1}{(2\pi)^{2k} (2k)!} \sum_{i} \delta^{\left( i_1 i_2 \cdots i_{2k-1} i_{2k} \right)} \Omega_{ii} \Omega_{i1} \cdots \Omega_{i_{2k-1} i_{2k}} = \frac{(2k-1)(2k-3) \cdots 1}{(2\pi)^{2k} (2k)!} \left( -\sum_{i,j} \Omega_{ij} \Omega_{ji} \right)^k$.

Thus (iii) holds. Let us prove (iv). We put

(2.3) $\sum_{k \geq 0} P_{4k} = \prod_i (1 + \gamma_i)$.
Then the index is expressed as follows:

\[(2.4) \quad \tau = \left( \prod_i \frac{\sqrt{\gamma_i}}{\tgh \sqrt{\gamma_i}} \right) [M_{4m}]. \quad ([5])\]

We have from \((2.3)\)

\[(2.5) \quad P_4 = \sum_i \gamma_i \quad \text{and} \quad P_8 = \sum_{i,j} \gamma_i \gamma_j\]

which lead to

\[(2.6) \quad \sum_i \gamma_i^2 = \left( \sum_i \gamma_i \right)^2 - 2 \sum_{i,j} \gamma_i \gamma_j = P_4^2 - 2P_8.\]

Meanwhile we have from \((iii)\)

\[(2.7) \quad P_8 = \frac{1}{2} P_4^2.\]

From \((2.6)\) and \((2.7)\) we have

\[(2.8) \quad \sum_i \gamma_i^2 = 0.\]

In such a way we can prove from \((iii)\) and \((2.3)\) that every symmetric function of \(\gamma_i\)'s are zero except for the elementary ones. Therefore we can regard \(\gamma_i (t \geq 2)\) as zero in the following computations. Since

\[(2.9) \quad \frac{\sqrt{\gamma_i}}{\tgh \sqrt{\gamma_i}} = 1 + \frac{1}{3} \gamma_i + \cdots\]

we have from \((2.3)\) and \((2.4)\)

\[(2.10) \quad \tau = \left[ \prod_i \left( 1 + \frac{1}{3} \gamma_i + \cdots \right) \right] [M_{4m}] = \frac{1}{3^m} P_{4m} [M_{4m}].\]

In such a way we have

\[(2.11) \quad A(M_{4m}) = \left[ \prod_i \frac{2\sqrt{\gamma_i}}{\sinh 2\sqrt{\gamma_i}} \right] [M_{4m}] = \left[ \prod_i \left( 1 - \frac{2}{3} \gamma_i + \cdots \right) \right] [M_{4m}]
\]

\[= \left( -\frac{2}{3} \right)^m P_{4m} [M_{4m}]. \quad \text{Q. E. D.}\]

**Remark.** We have from \((iv)\)

\[(2.12) \quad A/\tau = (-1)^m 2^m.\]
Thus in such a case the $A$-genus is divisible by $2^n$.

**Corollary 1.** Let $M_n$ be a compact orientable $C^\infty$-Riemannian manifold. If $M_n$ admits a continuous field of $n-3$ frames and $H^{4k}(M_n, \mathbb{Z})=0$ for some $k$ ($1 \leq k \leq [n/4]$), then

$$P_{4l} = 0 \quad (l \geq k) \quad \text{and} \quad 2M_n \sim 0.$$ 

**Proof.** We have from Theorem 2 (iii)

\begin{equation}
(2.13) \quad P_{4l} = \frac{1}{l!} P_4 = 0 \quad (l \geq k)
\end{equation}

and hence if $n = 4m$, every Pontryagin numbers become zero. Hence we have $2M_n \sim 0$ as in the case of Theorem 1 Corollary.

**Corollary 2.** Let $M_{4m}$ $(m > 1)$ be a compact orientable $C^\infty$-Riemannian manifold admitting a continuous field of $4m-3$ frames. If moreover either $\tau(M_{4m})$ or $A(M_{4m})$ is zero, then $2M_{4m} \sim 0$.

**Proof.** From Theorem 2 (iii) and (iv) we see that every Pontryagin numbers of $M_{4m}$ are zero.

**Corollary 3.** Let $M_{4n}$ $(n \geq 1)$ be a compact orientable $C^\infty$-Riemannian manifold admitting a continuous field of $4n-3$ frames. If moreover $M_{4n}$ is differentiably imbedded in an Euclidean space $E_{6n}$, then $2M_{4n} \sim 0$.

**Proof.** The dual Pontryagin classes are defined by

\begin{equation}
(2.14) \quad \sum_{k \geq 0} \overline{P}_{4k} \sum_{l \geq 0} (-1)^l P_{4l} = 1, \quad \overline{P}_{4k} \in H^{4k}(M_n, \mathbb{Z}).
\end{equation}

We have from Theorem 2 (iii)

\begin{equation}
(2.15) \quad \sum_{l \geq 0} (-1)^l P_{4l} = e^{-P_4}.
\end{equation}

Hence we have from (2.14)

\begin{equation}
(2.16) \quad \sum_{k \geq 0} \overline{P}_{4k} = e^{P_4}, \quad \text{i.e.}
\end{equation}

\begin{equation}
(2.17) \quad \overline{P}_{4k} = \frac{1}{k!} P_4^k.
\end{equation}

If $M_{4n} \subset E_{6n}$ differentiably, then we have
(2.18) \[ \overline{P}_{4n} = 0, \quad ([6]) \]
i.e.
(2.19) \[ P_4^n = 0. \]
Hence every Pontryagin numbers become zero. Q.E.D.

REFERENCES


UTSUNOMIYA UNIVERSITY, JAPAN.