ON A CONJECTURE OF BERBERIAN

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1. In [1], S. K. Berberian conjectured that the closure of the numerical range of a hyponormal operator coincides with the convex hull of its spectrum. The purpose of this note is to give an affirmative answer to his conjecture.

Throughout this paper, operator means a bounded linear operator on a Hilbert space. The spectrum of an operator $T$ is denoted by $\sigma(T)$, and its convex hull is denoted by $\sum(T)$. The numerical range of an operator $T$, denoted by $W(T)$, is the set $W(T) = \{(Tx, x) : \|x\| = 1\}$. We write $\overline{W(T)}$ for the closure of $W(T)$. An operator $T$ is called normaloid if $\|T\| = \sup \{ |\lambda| : \lambda \in W(T) \}$. For a compact convex subset $X$ of the plane, a point $\lambda \in X$ is bare if there is a circle through $\lambda$ such that no points of $X$ lie outside this circle. A closed subset $X$ of the plane is a spectral set for an operator $T$ if $\lambda(\sigma(T)) \leq \sup \{ |u(z)| : z \in X \}$ for every rational function $u(z)$ having no poles in $X$.

2. In this section, we shall prove the following theorem.

**Theorem.** Let $T$ be an operator such that $T - \lambda I$ is normaloid for every complex number $\lambda$, then we have $\overline{W(T)} = \sum(T)$.

A key of our proof is the following lemma.

**Lemma 1.** Let $T$ be an operator and $X \in W(T)$ a bare point of $W(T)$, then there exists a complex number $\lambda_0$ satisfying $|\lambda - \lambda_0| = \sup \{ |\mu - \lambda_0| : \mu \in W(T) \}$.

**Proof.** By the definition of bare point, there is a circle through $\lambda$ such that no points of $W(T)$ lie outside this circle. The center $\lambda_0$ of this circle satisfies our requirement.

For convenience we state the following known result as a lemma ([4: Corollary to Theorem 4]).

**Lemma 2.** For an operator $T$, $\lambda \in \overline{W(T)}$ and $|\lambda| = \|T\|$ imply $\lambda \in \sigma(T)$. 

PROOF OF THEOREM. It is sufficient to show that each bare point of \( W(T) \) belongs to \( \sigma(T) \) ([4: Lemma 3]). Let \( \lambda \) be a bare point of \( W(T) \), there is a \( \lambda_0 \) satisfying \( |\lambda - \lambda_0| = \sup \{ |\mu - \lambda_0| : \mu \in W(T) \} \) by Lemma 1. Thus, by the hypothesis on \( T \) and the fact \( W(T) - \lambda_0 = W(T - \lambda_0 I) \), we have \( \|T - \lambda_0 I\| = |\lambda - \lambda_0| \). Since \( \lambda - \lambda_0 \in W(T - \lambda_0 I), \lambda - \lambda_0 \in \sigma(T - \lambda_0 I) \) by Lemma 2 and so we have \( \lambda \in \sigma(T) \). Hence the proof is completed.

As a corollary, Berberian's conjecture is solved affirmatively.

**COROLLARY 1.** For a hyponormal operator \( T \), \( W(T) = \sum(T) \).

**PROOF.** If \( T \) is hyponormal, i.e. \( TT^* \leq T^*T \), \( T-\lambda I \) is also a hyponormal operator for every complex number \( \lambda \). Thus \( T-\lambda I \) is normaloid for every \( \lambda \) by [1: Corollary 4] and so \( W(T) = \sum(T) \) by our theorem.

M. Schreiber [5] has shown that if \( \sum(T) \) is a spectral set for \( T \), \( \sum(T) = W(T) \).

**COROLLARY 2.** If \( W(T) \) is a spectral set for a bounded operator \( T \), \( W(T) = \sum(T) \).

In fact, since \( W(T) \) is a spectral set for \( T \), we have
\[
\|u_\lambda(T)\| = \|T-\lambda I\| \leq \sup\{ |\mu - \lambda| : \mu \in W(T) \} \leq \|T-\lambda I\|
\]
for each rational function \( u_\lambda(z) = z - \lambda \), and the conclusion follows.

It is obvious that the spectrality of \( \sum(T) \) for \( T \) implies the spectrality of \( W(T) \) for \( T \), but by Corollary 2 the converse implication holds.

The following result is proved in [3] and [5].

**COROLLARY 3.** For a Toeplitz operator \( T_\phi \), \( W(T_\phi) = \sum(T_\phi) \).

**PROOF.** Let \( L_\phi \) be a Laurent operator corresponding to a Toeplitz operator \( T_\phi \), it is known \( \sigma(L_\phi) \subseteq \sigma(T_\phi) \). Thus we have
\[
\|T_\phi - \lambda I\| = \|L_{(\phi - \lambda)}\| \leq \|L_{(\phi - \lambda)}\| = \sup \{ |\mu - \lambda| : \mu \in \sigma(L_\phi) \}
\]
\[
\leq \sup \{ |\mu - \lambda| : \mu \in W(T_\phi) \} \leq \|T_\phi - \lambda I\|
\]
and the assertion is true by our theorem.
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REFERENCES


TôHOKU University and
Hachinoe Technical College.