A THEOREM ON REGULAR VECTOR FIELDS AND ITS APPLICATIONS TO ALMOST CONTACT STRUCTURES

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Introduction. In the paper [1], Boothby-Wang dealt with the period function \( \lambda \) of the associated vector field of the regular contact form on a compact contact manifold and proved that \( \lambda \) is differentiable and constant ([6]).

In this note we prove a theorem on the proper and regular vector field, by this we can give a simple proof to one of their result. Moreover, as a natural consequence, this procedure enables us to generalize Morimoto’s theorem (Theorem 5, [4]), concerning the period function on a normal almost contact manifold.

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1. Regular vector fields. Let \( M \) be a connected differentiable manifold and \( X \) be a differentiable vector field on \( M \) such that \( X \) does not vanish everywhere. We assume that the distribution defined by \( X \) is regular and \( X \) is proper, i.e., \( X \) generates the global 1-parameter group \( \exp tX(-\infty < t < \infty) \) of transformations of \( M \). For the terminologies we refer to [5]. We can find always a 1-form \( \omega \) satisfying \( \omega(X)=1 \). Now the next assumption is that there exists a 1-form \( \omega \) such that \( \omega(X)=1 \) and \( L(X)\omega = i(X) d\omega = 0 \), where \( L(X) \) or \( i(X) \) is the operator of the Lie derivative or interior product operator by \( X \).

First we see that the quotient space \( M/X \) is a Hausdorff space, because \( X \) is proper and regular. Hence by Palais’ theorem ([5], Chap. I, § 5), \( M/X \) is a differentiable manifold and the projection \( \pi : M \to M/X \) is a differentiable map.

Let \( h \) be an arbitrary Riemannian metric in \( M/X \). The tensor \( g \) in \( M \) defined by \( g = \pi^*h + \omega \otimes \omega \) is easily seen to be a Riemannian metric in \( M \), \( \pi^* \) and \( \otimes \) denoting the dual of \( \pi \) and tensor product respectively. Clearly we have \( g(X,X) = 1 \), and we see that the relation \( L(X)g = 0 \) holds good. Namely \( X \) is a unit and Killing vector field with respect to \( g \). Thus each trajectory of \( X \) is a geodesic and the parameter \( t \) in \( \exp tX \) is nothing but the arc length of it.

Suppose that there is a point \( p \) and a positive number \( \lambda \) such that
exp \lambda X \cdot p = p, \text{ and } \exp tX \cdot p \neq p \text{ for } 0 < t < \lambda. \text{ Denote by } l(q) \text{ the leaf passing through a point } q \text{ of } M \text{ and let } U \text{ be a sufficiently small coordinate neighborhood of } p \text{ which is regular with respect to } X. \text{ If we take an arbitrary point } x \text{ in } U \text{ such that } x \text{ does not belong to } l(p), \text{ then we can draw the shortest geodesic } c(x) \text{ from } x \text{ to } l(p). \text{ Clearly at the intersecting point } \bar{x} \text{ of } c(x) \text{ and } l(p), \text{ both geodesics are orthogonal.}

Now as } \exp \lambda X \text{ is an isometry, the image of the geodesic } c(x) \text{ is the geodesic passing through } \bar{x}. \text{ On the other hand, by the regularity, for any point } q \text{ of } c(x) \exp \lambda X \cdot q \text{ belongs to } l(q). \text{ Hence we have } \exp \lambda X \cdot x = x, \text{ and so the period function } \lambda \text{ is constant on } U, \text{ and on } M \text{ as } M \text{ is connected.}

If there exists a point } p \text{ such that } \exp tX \cdot p = p \text{ for any } t, \text{ then this is the same for all point in } M.

**Theorem.** For a proper and regular vector field } X \text{ on } M, \text{ the following three conditions are equivalent.}

(i) The period function } \lambda \text{ of } X \text{ is constant (finite or infinite).
(ii) There exists a 1-form } w \text{ such that } w(X) = 1 \text{ and } L(X)w = 0.
(iii) There exists a Riemannian metric } g \text{ such that } g(X, X) = 1 \text{ and } L(X)g = 0.

Since we have proved (ii) \rightarrow (iii) \rightarrow (i), the next process is (i) \rightarrow (ii). By virtue of (i), } M \text{ can be considered as a principal fibre bundle whose structural group is a toroidal group } S^1 \text{ or a real additive group } R. \text{ If we take an infinitesimal connection } w \text{ on } M \text{ such that } X \text{ is a fundamental vector field, then (ii) holds goods.}

2. Applications. We denote by } \phi, \xi \text{ and } \eta \text{ the structure tensors of an almost contact structure. As the first application we have the following (see [4])}

**Theorem A.** Let } M \text{ be an almost contact manifold such that } L(\xi) \eta = 0 \text{ and } \xi \text{ is a proper and regular vector field.}

(i) If } \exp t\xi \cdot p \neq p \text{ for some point } p \text{ in } M \text{ and for any } t, \text{ then } M \text{ is a principal fibre bundle with the group } R.
(ii) If we have } \exp \lambda \xi \cdot p = p \text{ for some point } p \text{ in } M \text{ and a real number } \lambda, \text{ then } M \text{ is a principal fibre bundle with the group } S^1.

In both cases, } \eta \text{ defines an infinitesimal connection on } M.

If } \xi \text{ is an associated vector field of the contact form } \eta, \text{ we have } L(\xi) \eta = 0, \text{ consequently we have}
Corollary 1. In a contact manifold with the contact from $\eta$, if an associated vector field is proper and regular, then $M$ is a principal fibre bundle with the group $R$ or $S^1$ according as (i) or (ii) in Theorem A is satisfied. On $M$ the contact form defines an infinitesimal connection and $M/\xi$ is a symplectic manifold.

In this Corollary, as $L(\xi)d\eta=0$, the symplectic structure $W$ on $M/\xi$ is defined by the relation $\pi^*W = d\eta$.

If the manifold is compact, every vector field is proper. Thus we get the following (see [1])

**Corollary 2.** Let $M$ be a compact contact manifold with a regular contact form $\eta$. Then $M$ is a principal $S^1$-bundle over the symplectic manifold $M/\xi$ with $\eta$ as a connection form of an infinitesimal connection.

**Theorem B.** In the contact manifold, if the associated vector field $\xi$ is proper and regular. Then we can find an almost contact metric structure $(\xi, g)$ associated to the contact form $\eta$ such that $L(\xi)\phi = 0$, equivalently $\xi$ is a Killing vector field.

**Proof.** Let $\pi: M \to M/\xi$ be a fibering given in Corollary 1 and let $W$ be the symplectic structure on $M/\xi$ such that $d\eta = \pi^*W$. Further we can define an almost kählerian structure $F$ and the metric $h$ on $M/\xi$ satisfying $W(u, v) = h(u, Fv)$ and $h(Fu, Fv) = h(u, v)$ for any vector fields $u, v$ on $M/\xi$ ([2]). Therefore by the result in [3] $M$ has $(\phi, \xi, \eta, g)$-structure associated to $\eta$ such that $L(\xi)\phi = 0 = L(\xi)g$ and $g = \pi^*h + \eta \otimes \eta$. Therefore $M$ is a $K$-contact manifold.

For brevity, we say that an almost contact structure $(\phi, \xi, \eta)$ is of type $P$ if it is constructed in a principal fibre bundle $M$ with the structural group $S^1$ or $R$ and with an almost complex manifold $B$ as its base, using an almost complex structure of $B$ and an infinitesimal connection as in [3].

**Theorem C.** Suppose that in an almost contact manifold $\xi$ is proper and regular, then $L(\xi)\phi = 0$ if and only if the almost contact structure is of type $P$.

**Proof.** From $L(\xi)\phi = 0$ it follows that $L(\xi)\eta = 0$. So, $M$ is a principal fibre bundle with the group $S^1$ or $R$ and on $M$ $\eta$ defines an infinitesimal connection. We denote by $u^*$ the horizontal lift of a vector field $u$ on $M/\xi$ with respect to $\eta$. It is natural to define an almost complex structure $F$ on $M/\xi$ by
where $\phi u^*$ is considered at the point $q$ contained in the leaf over the origin of $u$, and $\pi \phi u^*$ does not depend upon the choice of the point $q$ in the leaf as $L(\xi)\phi=0$. Then Theorem C follows immediately.

**Remark.** The tensor $N_2=(N_j)$ is defined (7) by $N_j(X)=L(\xi)X$ for a vector field $X$. Theorem C gives a geometrical meaning of $N_2=0$ in the case when $\xi$ is proper and regular. In contact manifold $N_2=0$ is equivalent to the fact that $\xi$ is a Killing vector field.

**Corollary 3.** Let $M$ be an almost contact manifold such that $\xi$ is proper and regular. If $L(\xi)\phi=0$, then we can find an associated Riemannian metric to the almost contact structure so that $\xi$ is a Killing vector field.

This follows from Theorem C and the similar argument in the proof of Theorem B.

**Bibliography**


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