In a previous paper [1], we studied the functions of Littlewood-Paley, Lusin and Marcinkiewicz. And now we wish to consider a proof for the function $g^*(f)$ independently of the decomposition theorem on Fourier series. The function $g^*(x, f)$ is defined for $f$ of $L(0,1)$ as follows:

$$g^*(x, f) = g^*(f) = \left( |\hat{f}_n|^2 + \sum_{n=1}^{\infty} \frac{|s_n(x, f) - \sigma_n(x, f)|^2}{n} \right)^{1/2},$$

where $\hat{f}_n$ is the $n$-th Fourier coefficient of $f$ and $s_n, \sigma_n$ denote the $n$-th partial sum of the Fourier series of $f$ and its $n$-th Cesàro mean respectively.

**Theorem.** Let $1 < p < \infty$, then

$$A_p \|f\|_p \leq \|g^*(f)\|_p \leq A_p \|f\|_p$$

for all $f$ of $L^p(0,1)$, $A_p, A'_p$ being positive constants independent of $f$.

Let $F_n$ be $n$-th Fejér kernel, that is,

$$F_n(x) = \sum_{v=-n}^{n} \left( 1 - \frac{|v|}{n+1} \right) e^{2\pi i vx} = \frac{1}{n+1} \left( \frac{\sin(2n+1)\pi x}{\sin \pi x} \right)^2$$

and let us denote $k_0(x) = 1$ and

$$k_n(x) = \frac{F_n(x) - F_n(x)}{c_n \sqrt{n}} \quad n = 1, 2, \cdots,$$

$c_n$ being constants bounded away both from 0 and from infinity which will be defined later. Our proof composes, as that of [1] on the decomposition theorem, of estimation of the vector valued kernel $K(x) = (k_0(x), k_1(x), \cdots)$.

Let us put $Kf = K*f$ for $f$ of $L(0,1)$ and $\mathbb{L}g = \int <g(\gamma), K(x-\gamma)> \, d\gamma$ for $g$ of
THE LITTLEWOOD-PALEY FUNCTION $g^*(f)$

$L(l^2)$, where $<\cdot,\cdot>$ means the inner product of the sequence space $l^2$. If $f \in L^2(0,1)$, then

$$\|\mathbb{F}f\|_2^2 = |\hat{f}_0|^2 + \sum_{\nu=1}^{\infty} |\hat{f}_\nu|^2 \left( \sum_{n=|\nu|}^{\infty} \frac{n\nu^2}{c_n^2(2n+1)^2(n+1)^2} + \sum_{n=|\nu|+1/2}^{\infty} \frac{(2n+1-|\nu|)^2}{c_n^2n(2n+1)^2} \right) \leq A\|f\|_2^2$$

for a constant $A$. Therefore $\mathbb{F}$ is of strong type $(2,2)$. On the other hand we have

$$\int <\mathbb{F}f, g> \, dx = \int \mathbb{F}f g \, dx$$

for $f$ of $L^2$ and $g$ of $L^2(l^2)$, so that $\mathbb{F}$ is also of strong type $(2,2)$.

**Lemma.** We have

$$\int_{|x+y| > 2^{-M}} |K(x+y) - K(x)| \, dx \leq B$$

for all $|y| \leq 2^{-M-1}$, $M = 1, 2, \ldots$, where $B$ is a constant.

**Proof.** By the definition of Fejér kernel, we have

$$|k_n(x+y) - k_n(x)| \leq |k_n(x+y)| + |k_n(x)| \leq \frac{A}{n\sqrt{n}} \frac{1}{x^2}$$

for all $2^{-1} > |x| > 2^{-M}$, $|y| < 2^{-M-1}$ and $M = 1, 2, \ldots$. On the other hand

$$|F_n(x+y) - F_n(x)| \leq \frac{1}{n} \left| \frac{\sin(n+1)\pi(x+y)}{\sin \pi(x+y)} \right| + \frac{\sin(n+1)\pi x}{\sin \pi x} \left| \frac{\sin(n+1)\pi(x+y)}{\sin \pi(x+y)} - \frac{\sin(n+1)\pi x}{\sin \pi x} \right|$$

$$\leq \frac{A\|y\|}{|x|}.$$

Therefore

$$|k_n(x+y) - k_n(x)| \leq A\sqrt{n} \frac{|y|}{|x|}.$$

Hence for arbitrary positive integer $N$, we have

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1) Constants $A$ may be different in each occasion.
for all \(2^{-1} > |x| > 2^{-M}\) and \(|y| < 2^{-M-1}\). Consequently

\[
\int_{2^{-1} > |x| > 2^{-M}} |K(x+y) - K(x)| \, dx 
\leq A \int_{2^{-1} > |x| > 2^{-M}} (2^{-M+N} |x|^{-1} + 2^{-N} x^{-2}) \, dx 
\leq A \sum_{v=1}^{M-1} \int_{2^{-v} > |x| > 2^{-v-1}} (2^{-M+N} x^{-1} + 2^{-N} x^{-2}) \, dx 
\leq A \sum_{v=1}^{M-1} (2^{-M+N} + 2^{-N+v}).
\]

If we choose \(N=\lfloor (v+M)/2 \rfloor\) in the \(v\)-th term, the last sum is not greater than

\[
A \sum_{v=1}^{M-1} (2^{-M/2} 2^{v/2} + 2^{-M/2} 2^{v/2}) \leq A,
\]

which proves our lemma.

**Proof of Theorem.** \(\mathcal{F}\) and \(\mathcal{G}\) are singular integral operators by the above lemma and therefore of strong type \((p, 2)\), \(1 < p \leq 2\) (cf. for example, the arguments in [1]), from which it results that \(\mathcal{F}\) and \(\mathcal{G}\) are of strong type \((p, p)\) for \(1 < p < \infty\) by conjugacy method. We remark that if \(|j| \leq n\), then \(j\)-th coefficient of \(n\)-th component of \(\mathcal{F}f\) is equal to that of \([s_n(x, f) - \sigma_n(x, f)] / c_n\sqrt{n}\). Therefore applying generalized M. Riesz theorem to \(|\mathcal{F}f| \leq A_p \|f\|_p\), we get the first half inequality of the theorem. If \(g = (g_0, g_1, \cdots) = (f_0, \cdots, [s_n(x, f) - \sigma_n(x, f)] / \sqrt{n}, \cdots)\) belongs to \(L^p(l^2), 1 < p < \infty\) and if we put \(\mathcal{G}_n g = k_0 * g_0 + \cdots + k_N * g_N\), then \(\|\mathcal{G}_n g\|_p\) is bounded. We denote one of its weak limit point by \(f'\), then we have \(\|f\|_p \leq \limsup_{N \to \infty} \|\mathcal{G}_n g\|_p \leq A_p \|g\|_p = A_p \|g^w(f)\|_p\). It remains only to show that \(f' = f\). For a suitable sequence \(\{N_j\}\), we have

\[
\hat{f}^*_n = \lim_{j \to \infty} \int \mathcal{G}_{N_j} g(x) e^{-2\pi i nx} \, dx 
= \lim_{j \to \infty} \sum_{m=n}^{N_j} \frac{n^2}{c_m(2m+1)(m+1)^2} \hat{f}^*_n.
\]
The last sum is equal to $\hat{f}_n$ if $c_m = m^2/(2m+1)^2, m \geq 1$. Therefore our proof is completed.

**REFERENCE**


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